ON BOREL'S DIRECTIONS OF MEROMORPHIC FUNCTIONS
OF FINITE ORDER*)

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1. Introduction.
Let \( w(\zeta) \) be meromorphic for \( |\zeta| < \infty \) and
\[
T(r) = \int_0^r \frac{S(r)}{r} \, dr,
\]
where
\[
S(r) = \frac{1}{\pi} \int_0^{2\pi} \left( \frac{|w'(re^{i\theta})|}{1 + |w(re^{i\theta})|^2} \right)^2 \, d\theta
\]
be its Nevanlinna's characteristic function and
\[
\lim_{r \to \infty} \frac{\log T(r)}{\log r} = \rho
\]
be its order. If \( \rho < \infty \), then by Borel's theorem, for any \( \varepsilon > 0 \),
\[
\sum \frac{1}{|\zeta_\alpha(a)|^{\rho+\varepsilon}} < \infty
\]
for any \( a \) and if \( 0 < \rho < \infty \),
\[
\sum \frac{1}{|\zeta_\alpha(a)|^{\rho}} = \infty
\]
for any \( a \), with two possible exceptions, where \( \zeta_\alpha(a) \) are zero points of \( w(\zeta) - a \).

Varion\(^{1)} \) proved that there exists a direction \( \Delta \), which is called a Borel's direction, such that
\[
\sum \frac{1}{|\zeta_\alpha(a, \Delta)|^{\rho}} = \infty,
\]

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for any \( a \), with two possible exceptions, where \( \Delta \) is any angular domain, which contains \( f \) and \( \varphi_a(z, \Delta) \) are zero points of \( w(z) - a \) in \( \Delta \).

In §3, we will prove this Valiron's theorem simply by means of Theorem 2 of §2. In §5, we consider meromorphic functions in a half-plane \( \Re z \geq 0 \) and establish theorems, which are analogous to Nevanlinna's fundamental theorems for meromorphic functions for \(|z| < R (\leq \infty)\) and by means of which we prove theorems of Valiron and Nevanlinna in §6.

2. Main theorems.

**Theorem 1.** Let \( w = w(z) \) be meromorphic in \( |z| < 1 \) and the number of zero points of \( (w(z) - a_1)(w(z) - a_2)(w(z) - a_3) \) in \( |z| < 1 \) be \( \leq n \), where multiple zeros are counted only once, then

\[
S(r) \leq n + A/(1 - r), \quad (0 \leq r < 1),
\]

where \( A \) is a constant, which depends on \( a_1, a_2, a_3 \) only.

**Proof.** Let \( z_1, \ldots, z_n (n \leq n) \) be zero points of \( \prod_{i=1}^{3} (w(z) - a_i) \) in \( |z| < 1 \) and we take off these points from \( |z| < 1 \) and \( D_0 \) be the remaining domain and \( D_r \) be the part of \( D_0 \) which lies in \( |z| \leq r (\leq 1) \). Let \( F_r \) be the Riemann surface spread upon the \( w \)-sphere, which is generated by \( w = w(z) \), when \( z \) varies in \( D_0 \), then \( F_r \) is a covering surface of the basic domain \( F_0 \), which is obtained from the \( w \)-sphere by taking off three points \( a_1, a_2, a_3 \). Let \( \rho (r) \) be the Euler's characteristic of \( F_r \), then by Ahlfors' fundamental theorem on covering surfaces,\(^2\)

\[
\rho (r) \geq S(r) - hL(r), \quad \rho (r) = \max (\rho (r), 0),
\]

(1)

where

\[
L(r) = \int_0^{2\pi} \frac{|w'(re^{i\theta})|}{1 + |w(re^{i\theta})|^2} r \, d\theta
\]

(2)

and \( h \) is a constant, which depends on \( a_1, a_2, a_3 \) only.

By Schwarz's inequality, we have

\[
[L(r)]^2 \leq 2\pi r \sum_{r} \frac{dS(r)}{dr}.
\]

(3)

Since by the hypothesis, \( \rho (r) \leq n \), we have by (1),

\[
S(r) \leq n + hL(r), \quad (0 \leq r < 1).
\]

(4)

Hence if \( S(r) - n > 0 \) for all \( r' \) \( (r \leq r' < 1) \), then by (3), (4),

\[
S(r) \leq 2\pi h \int_{r'}^{1} \frac{dS(r')}{dr'} \leq 2\pi h \int_{r}^{1} \frac{1}{r'} \, dr'
\]

\[
= 2\pi h (S(r') - n), \quad \text{or}
\]

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\[ S(r) \leq n + 2\pi^2 \frac{k^2}{(1-r)}. \]  

If \( S(r) - n \leq 0 \) for some \( r' (r \leq r' < 1) \), then \( S(r) \leq S(r') \leq n \), so that (5) holds. Hence (5) holds for \( 0 \leq r < 1 \), which proves the theorem.

Let \( w(z) \) be meromorphic in an angular domain \( \Delta : |\arg z| \leq \alpha \) and put

\[
S(r; \Delta) = \frac{1}{\pi} \int_{\rho}^{r} \int_{-\alpha}^{\alpha} \left( \frac{|w'(te^{i\theta})|}{1 + |w(te^{i\theta})|^2} \right)^2 t \, dt \, d\theta,
\]

\[
T(r; \Delta) = \int_{\rho}^{r} \frac{S(t; \Delta)}{t} \, dt,
\]

\[
N(r, a; \Delta) = \int_{\rho}^{r} n(t, a; \Delta) \frac{1}{t} \, dt,
\]

where \( n(r, a; \Delta) \) is the number of zero points of \( w(z) - a \) in a sector: \( |\arg z| \leq \alpha \), \( 0 \leq |z| \leq r \), where multiple zeros are counted only once.

**Theorem 2.** Let \( w(z) \) be meromorphic in an angular domain \( \Delta_0 : |\arg z| \leq \alpha_0 \) and \( \Delta : |\arg z| \leq \alpha < \alpha_0 \) be an angular domain contained in \( \Delta_0 \). Then for any \( \lambda > 1 \)

\[ T(r; \Delta) \leq 3 \sum_{i=1}^{3} N(\lambda r, a_1; \Delta_i) + A(\log \rho^2), \]

where \( A \) is a constant, which depends on \( a_1, a_2, a_3, \alpha, \alpha_0, \lambda \) only.

**Proof.** We put \( k = \lambda^{1/3} > 1 \) and for \( r > 1 \), let

\[ N = [\log r/\log k], \quad k^N \leq r < k^{N+1}, \]

so that

\[ k^{N+1} = \lambda k^N \leq \lambda r. \]

Let \( Q^0_1, Q_2 \) be curvilinear quadrilaterals:

\[
Q^0_1 : |\arg z| \leq \alpha_1, \quad k^{N-1} \leq |z| \leq k^{N+1},
\]

\[
Q_2 : |\arg z| \leq \alpha, \quad k^N \leq |z| \leq k^N,
\]

\[ S_1 = \int_{Q_1} \int_{Q_2} \left( \frac{w'(te^{i\theta})}{1 + |w(te^{i\theta})|^2} \right)^2 t \, dt \, d\theta \]

and \( n_0^2 \) be the number of zero points of \( \bigcup_{i=1}^{3} (w(z) - a_i) \) in \( Q^0_1 \). If we map \( Q^0_1 \) on \( |\zeta| < 1 \) conformally, such that the center of \( Q^0_1 \) becomes \( \zeta = 0 \), then \( Q_1 \) is mapped on a domain, which lies in \( |\zeta| \leq \rho < 1 \).

Since \( Q^0_1 \) is similar to \( Q^0_1 \), we have by Theorem 1,

\[ S_1 \leq n_0^2 + A. \]
where $A$ is a constant, which depends on $a_1, a_2, a_3, \alpha, \alpha_0, \lambda$ only.

In the following, we denote such constants by the same letter $A$.

We put

$$n(r; \Delta_0) = \sum_{i=1}^{3} n(r, a_i; \Delta_i). \quad (6)$$

Since $Q^0_n$ overlap only twice,

$$\int_{k^{N+1}}^{k'} S(t; \Delta) \leq S(k'; \Delta) \log k = (S_1 + \cdots + S_v) \log k \leq (n_1 + \cdots + n_v) \log k + Av \leq 3n(k^{N+2}; \Delta_0) \log k + AN,$$

so that

$$\int_{1}^{k^{N+1}} \frac{S(t; \Delta)}{t} \, dt = \sum_{v=1}^{N} \int_{k^{v-1}}^{k^v} \frac{S(t; \Delta)}{t} \, dt \leq 3 (n(k^v; \Delta_0) + \cdots + n(k^{N+1}; \Delta_0)) \log k + AN^2. \quad (7)$$

Since

$$\int_{k^v}^{k^{N+1}} \frac{n(t; \Delta)}{t} \, dt \geq n(k^v; \Delta_0) \log k,$$

we have from (7), (3),

$$T(r; \Delta) = \int_{1}^{r} \frac{S(t; \Delta)}{t} \, dt \leq 3 \int_{1}^{k^{N+3}} \frac{n(t; \Delta_0)}{t} \, dt + AN^2 \leq 3 \int_{1}^{r} \frac{n(t; \Delta_0)}{t} \, dt + A(\log r)^2 = 3 \sum_{i=1}^{3} N(\kappa r, a_i; \Delta_0) + A(\log r)^2.$$


1. Now we will prove Valiron’s theorem:

**Theorem 3.** Let $w(z)$ be a meromorphic function of finite order $\rho > 0$, then there exists a direction $J$: arg $z = \alpha$, such that for any $\epsilon > 0$.

(i)\[
\sum_{|z_0|} \frac{1}{|z_0 (a; \Delta)|^{\rho - \epsilon}} = \infty
\]

for any $a$, with two possible exceptions, and if

$$\int_{r^{-\epsilon}}^{r} T(r) \, dr = \infty$$

then
for any \( a \), with two possible exceptions, where \( \Delta \) is any angular domain, which contains \( J \), and \( \chi_\alpha(a, \Delta) \) are zero points of \( w(z) - a \) in \( \Delta \), multiple zeros being counted only once.

**Proof.** Suppose that for some \( k > 0 \),

\[
\int_0^\infty \frac{T(r)}{r^{k+1}} \, dr = \infty.
\]

Then dividing \((0, 2\pi)\) into \( 2^n \) equal parts, we see that there exists an angular domain \( \Delta_n \) of magnitude \( 2\pi/2^n \), such that \( \Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_n \),

\[
\int_0^\infty \frac{T(r; \Delta_n)}{r^{k+1}} \, dr = \infty, \quad (n=1, 2, \ldots).
\]

Let \( \Delta_n \) converge to a direction \( J : \arg z = \alpha \), then for any angular domain \( \Delta : |\arg z - \alpha| \leq \delta \), which contains \( J \), \( \Delta_n \subset \Delta \) for \( n \geq n_0 \), so that

\[
\int_0^\infty \frac{T(r; \Delta)}{r^{k+1}} \, dr = \infty.
\]

Let \( \Delta_n : |\arg z - \alpha| \leq 2\delta \), then by Theorem 2,

\[
\int_0^\infty \frac{T(r; \Delta)}{r^{k+1}} \, dr \leq 3 \sum_{i=1}^3 \int_0^\infty \frac{N(r, \alpha; \Delta_i)}{r^{k+1}} \, dr + O(1) \quad (\lambda > 0),
\]

so that from (3),

\[
\int_0^\infty \frac{N(r, \alpha; \Delta_n)}{r^{k+1}} \, dr = \infty, \quad \text{or} \quad \sum |\chi_\alpha(a, \Delta)|^k = \infty
\]

for any \( a \), with two possible exceptions.

Since \( \int_0^\infty \frac{T(r)}{r^{k+1}} \, dr = \infty \), if we take \( k = \rho - \epsilon \), then we have (i) and for \( k = \rho \), we have (ii). q. e. d.

2. Theorem 3 can be extended as follows.

**Theorem 4.** Let \( C : \zeta = \zeta(t) \) \((0 \leq t < \infty)\) \((\zeta(0) = 0, \zeta(\infty) = \infty)\) be a simple curve, which connects \( \zeta = 0 \) to \( \zeta = \infty \) and for any \( \delta > 0 \), let \( \Delta(\delta) \) be the set of points, which is covered by all discs \( |\zeta - \xi(t)| \leq |\zeta(t)| \delta \) \((0 \leq t < \infty)\) and \( \Delta_\delta \) be the set obtained from \( \Delta(\delta) \) by rotating an angle \( \theta \). Let \( w = w(z) \) be a meromorphic function of finite order \( \rho > 0 \) for \( |z| < \infty \). Then there exists a certain \( \theta_0 \), such that for any \( \delta > 0 \), \( \epsilon > 0 \),
(i) \[ \sum_{i=1}^{N} |1/|z_{0}(a; \Delta_{0}(\delta))|^{p-1} = \infty, \text{ for any } a, \text{ with two possible exceptions and} \]

if \( w(\zeta) \) is of order \( \rho \) of divergence type, then

(ii) \[ \sum_{i=1}^{N} |1/|z_{0}(a; \Delta_{0}(\delta))|^{p} = \infty, \text{ for any } a, \text{ with two possible exceptions, where} \]

\( z_{0}(a; \Delta_{0}(\delta)) \) are zero points of \( w(\zeta) - a \) in \( \Delta_{0}(\delta) \).

First we prove a lemma.

**Lemma.** Let \( E \) be a closed set contained in \( |\zeta| \leq 1 \) and \( 0 < \rho < 1 \). Then we can cover \( E \) by \( N \) circles \( C_{i} \) (\( i = 1, 2, \ldots, N \)) of radius \( \rho \) with its center \( (\zeta \in E) \), such that \( N < \frac{16\pi}{\sqrt{3} \rho^{1/4}} \) and circles \( C_{i} \) (\( i = 1, 2, \ldots, N \)) of radius \( 2\rho \) with center \( \zeta \) overlap at most \( 54 \)-times.

**Proof.** We cover the \( \zeta \)-plane by a net of regular triangles, whose vertices are \( \zeta_{n} = m\pi/n + np \) (\( m, n = 0, \pm 1, \pm 2, \ldots \)). Let \( \Delta_{1}, \Delta_{2}, \ldots, \Delta_{N} \) be the triangles, which contain points of \( E \), then since \( \Delta_{1} \) is contained in \( |\zeta| \leq 1 + \rho \) and the area of \( \Delta_{1} \) is \( \sqrt{3} \rho^{1/4} \) and \( 0 < \rho < 1 \),

\[
N \leq \pi \left(1 + \rho\right)^{2} / \frac{\sqrt{3} \rho^{1/4}}{4} = \frac{4\pi}{\sqrt{3} \rho^{1/4}} \left(1 + \rho\right)^{2} \leq \frac{4\pi}{\sqrt{3} \rho^{1/4}} \left(1 + \frac{1}{\rho}\right)^{2} = \frac{16\pi}{\sqrt{3} \rho^{1/4}}.
\]

We take a point \( \zeta_{0} (\epsilon E) \) in \( \Delta_{1} \) and draw a circle \( C_{i} \) of radius \( \rho \) with \( \zeta_{0} \) as its center, then \( C_{i} \) contains \( \Delta_{1} \), so that \( C_{i}, \ldots, C_{N} \) cover \( E \). Let \( C_{i}^{0} \) be a circle of radius \( 2\rho \) with \( \zeta_{0} \) as its center, then it is easily seen that \( C_{i}^{0} \) overlap at most \( 54 \)-times.

**Proof of Theorem 4.** Let \( k > 1 \) and \( \Delta_{i}(\delta) \) be the part of \( \Delta(\delta) \) contained in \( k\delta^{-1} \leq |\zeta| \leq k^{\nu}(\nu = 0, 1, 2, \ldots) \) and \( \Delta^{(3)}(\delta) \) be the part of \( \Delta(\delta) \) contained in \( k^{\nu-1} \leq |\zeta| \leq k^{\nu}+1 \), so that \( \Delta_{i}(\delta) \subset \Delta^{(3)}(\delta) \). By transforming \( \Delta_{i}(\delta) \) into a closed set in \( |\zeta| \leq 1 \) by \( \zeta = \zeta_{0}^{k_{i}} \) and applying the lemma, with \( \rho = \delta^{k_{i}} \) we see easily that \( \Delta_{i}(\delta) \) can be covered by \( N \) circles \( C_{i}^{0}(\epsilon \Delta_{i}(\delta)) \) (\( i = 1, 2, \ldots, N \)) of radius \( k^{\nu-1} \delta^{k_{i}} \) and center \( \zeta_{0}^{k_{i}} \) (\( \epsilon \Delta_{i}(\delta) \)), such that

\[
N \leq \frac{16\pi}{\sqrt{3} \delta^{k_{i}}}
\]

and circles \( C_{i}^{0}(\epsilon \Delta_{i}(\delta)) \) of radius \( \gamma k^{\nu-1} \delta^{k_{i}} \) with center \( \zeta_{0}^{k_{i}} \) overlap at most \( 54 \)-times.

Let \( a_{1}, a_{2}, a_{3} \) be any three values and \( S_{0}, S_{1}, S_{2} \) be the area on the \( w \)-sphere generated by \( w = w(\zeta) \), when \( \zeta \) varies in \( \Delta_{i}(\delta) \). \( C_{i}^{0} \) and \( n_{0}^{(i)} \) be the number of zero points of \( \Pi(w(\zeta) - a_{0}) \in \Delta_{i}(\delta) \), \( n_{0}^{(i)} \) respectively, then by Theorem 1,

\[
S_{1}^{(i)} \leq n_{0}^{(i)} + A_{i},
\]
where $A$ depends on $a_1, a_2, a_3, \delta$ only.

Since $C_i^{(1)}$ is contained in $\Delta_i (3\delta)$ and overlap at most $5\delta$-times and

$S_i \leq \sum_{i=1}^{N} S_i^{(1)}$, we have

$$S_i \leq 7A_1^6 + NA.$$  

From this we have the similar theorem as Theorem 2, where $\Delta = \Delta(3\delta)$,

$\Delta_3 = \Delta (3\delta)$ and from this we can prove Therem 4 as Theorem 3.

Remark. From (3) in the proof of Theorem 3, we see that there exists

$$r_1 < r_2 < \cdots < r_n \to \infty,$$

such that

$$\lim_{n \to \infty} S_r(r_n; \Delta) \log r_n = \infty. \quad (4)$$

Let

$$N = [\log r_n / \log k], \quad k^n \leq r < k^{n+1},$$

then from (1), there exists a certain curvilinear quadrilateral $Q_n : |arg z - \alpha| \leq \delta, k^{n-1} \leq |z| \leq k^n (r_n \leq N)$, such that

$$S_n = \frac{1}{\pi} \int_{[0, \pi]} \left( \frac{|w'(te^{i \theta})|}{1 + |w(te^{i \theta})|^2} \right)^2 \, dt \, d\theta \to \infty \quad (n \to \infty).$$

Let $Q_n : |arg z - \alpha| \leq 2\delta, k^{n-1} \leq |z| \leq k^n+1$. We map $Q_n$ conformally on $|z| < 1$ by $w = w(z)$, such that the center of $Q_n$ becomes $z = 0$, then the image of $Q_n$ lies in $|z| \leq \eta < 1$, where $\eta$ depends on $k, \delta$ only. We put

$w(z) = v(\xi)$ and put

$$S(r) = \frac{1}{\pi} \int_{[0, \pi]} \left( \frac{|v'(e^{i \theta})|}{1 + |v(e^{i \theta})|^2} \right)^2 \, dt \, d\theta \quad (0 \leq r \leq 1),$$

$$L(r) = \int_{0}^{r} \frac{|v'(r\xi)|}{1 + |v(r\xi)|^2} \, rd\xi,$$

then $S_n \leq S(r)$ and

$$(L(r))^n \leq 2\pi^2 r - \frac{dS(r)}{dr}.$$  

Suppose that

$$L(r) \geq (S(r))^{8/5} \quad \text{for } \eta \leq r \leq 1,$$

then
\[
(S'(r))^{1/2} \leq 2\pi^2 r \frac{dS(r)}{dr}, \\
1 - \eta \leq \int_0^1 \frac{dr}{r} \leq 2\pi^2 \left( \int_0^1 \frac{dS(r)}{S'(r)} \right)^{1/2} \leq \frac{4\pi^2}{\eta^{1/2}}, \text{ or} \\
S_n \leq S(\eta) \leq \left( \frac{4\pi^2}{1 - \eta} \right)^{1/2}.
\]

Hence if \( S_n > \left( \frac{4\pi^2}{1 - \eta} \right)^2 \), then there exists a certain \( r_n (\eta \leq r_n \leq 1) \), such that
\[
L(r_n) < (S(r_n))^4, \text{ or} \\
L(r_n) / S(r_n) < 1 / S(r_n)^{1/4} \leq 1 / S_n^{1/4} \to 0 \ (n \to \infty).
\]

From this we conclude by Ahlfors' theorem on covering surfaces, the following theorem:

**Let \( J: \arg \zeta = \alpha \) be a Borel's direction, then for any \( \delta > 0 \), the image of \( \Delta; \arg \zeta = \alpha \), \( \delta \) by \( w = w(\zeta) \) on the \( w \)-sphere covers schlicht infinitely often one of any five disjoint simply connected domains on the \( w \)-sphere.**

4. **Borel's directions of meromorphic functions of zero order.**

We consider meromorphic functions of zero order, such that
\[
\lim \log T(r) / \log r = 0, \quad \lim T(r) / (\log r)^2 = \infty.
\]

First we will prove a lemma.

**Lemma.** Let \( T(r) > 0 \) be an increasing function, such that
\[
\lim_{r \to 0} \log T(r) / \log r = 0, \quad \lim_{r \to \infty} T(r) / (\log r)^2 = \infty,
\]
then for any \( \lambda > 1, k > 1 \), there exists \( r_1 < r_2 \cdots < r_n \to \infty \), such that
\[
\lim_{n \to \infty} T(r_n) / (\log r_n)^2 = \infty, \quad T(\lambda r_n) \leq kT(r_n) \ (n = 1, 2, \ldots).
\]

**Proof.** First we will prove that for any \( M > 0 \), there exists \( r_1 < r_2 < \cdots < r_n \to \infty \), such that
\[
T(\lambda^\nu) \geq M (\log \lambda)^\nu \quad (1)
\]
holds for \( \nu = r_n \ (n = 1, 2, \ldots) \).

For, if for \( \nu \geq r_0 \), \( T(\lambda^\nu) < M (\log \lambda)^\nu \), then for \( \lambda^\nu \leq r < \lambda^{\nu+1} \), \( T(r) \leq T(\lambda^{\nu+1}) = M (\log \lambda)^{\nu+1} = M ((\nu + 1) / \nu)^\nu (\log \lambda)^\nu \leq M ((\nu + 1) / \nu)^\nu (\log \lambda)^\nu \), so that
\[
\lim_{r \to \infty} T(r) / (\log r)^2 \leq M < \infty.
\]
which contradicts the hypothesis, hence (1) holds for an infinite number of $v$.

Next we will prove that there exists an infinite number of $v$, for which (1) and

$$ T(\lambda^{v+1}) \leq kT(\lambda^v) $$

hold simultaneously.

For, suppose that for all $v \geq v_0$, for which (1) holds,

$$ T(\lambda^{v+1}) > kT(\lambda^v), $$

then since $k > 1$,

$$ T(\lambda^{v+1}) > kT(\lambda^v) \geq k M \left( \log \lambda \right)^2 \geq M \left( \log \lambda^{v+1} \right)^2, $$

so that $\lambda^{v+1}$ satisfies (1), hence by the hypothesis,

$$ T(\lambda^{v+1}) > kT(\lambda^v). $$

Hence (3) holds for all sufficiently large $v$, so that

$$ \lim_{r \to \infty} \log T(r)/\log r \geq \log k/\log \lambda > 0, $$

which contradicts the hypothesis, hence there exists an infinite number of $v$, which satisfy (1) and (2) simultaneously. If we take $M_1 < M_2 < \cdots < M_n \to \infty$ for $M$, then we have the lemma.

**Theorem 5.** Let $w(z)$ be a meromorphic function of order zero, such that

$$ \lim_{r \to \infty} T(r)/(\log r)^2 = 0, $$

then there exists a direction $J$: $\arg z = \alpha$, such that for any angular domain $\Delta$: $|\arg z - \alpha | \leq \delta$, which contains $J$,

$$ \lim_{n \to \infty} N(r_n, a; \Delta)/T(r_n) \geq |\Delta|/(\pi n), \quad (|\Delta| = 2\delta) $$

for any $a$, with two possible exceptions, where the sequence $(r_n)$ is independent of $a$ and $\Delta$, such that

$$ \lim_{n \to \infty} T(r_n)/(\log r_n)^2 = \infty. $$

**Proof.** By the lemma, for any $\lambda > 1$, $k > 1$, there exists $(r_n)$, such that

$$ \lim_{n \to \infty} T(r_n)/(\log r_n)^2 = \infty, \ T(\lambda r_n) \leq kT(r_n), \ (n = 1, 2, \cdots). \quad (1) $$

By dividing \((0, 2\pi)\) into \(2^m\) equal parts, we see that there exists an angular domain \(\Delta_m\) of magnitude \(2\pi/2^m\), such that 
\[
T(\varphi_n; \Delta_m) \leq T(\varphi_n)/2^m
\]
holds for an infinite number of \(n\).

Let \(\Delta_m\) converge to a direction \(J\): \(\arg \zeta = \alpha\) and \(\Delta: \|\arg \zeta - \alpha\| \leq \delta (1 - \varepsilon)\) \((\varepsilon > 0)\) be any angular domain, which contains \(J\).

Let \(m\) be such that \(2\pi/2^m \leq \delta (1 - \varepsilon) < 2\pi/2^{m-1}\), then \(\Delta \supset \Delta_m\), so that by (2), (1),
\[
T(\varphi_n; \Delta) \geq T(\varphi_n; \Delta_m) \geq 2^{-m} T(\varphi_n) \geq k^{-1} 2^{-m} T(\lambda r_n)
\]
holds for an infinite number of \(n\).

Let \(\Delta_0: \|\arg \zeta - \alpha\| \leq 8\), then
\[
|\Delta_0| = 28 < 8\pi (2^m (1 - \varepsilon)).
\]

We apply Theorem 2 for \(\Delta_0, \Delta\) and \(r_n\), then
\[
T(\lambda r_n) k2^m \leq T(\varphi_n; \Delta) \leq 3 \sum_{i=1}^{3} N(\lambda r_n, \sigma_i; \Delta_0) + A (\log r_n)^2,
\]
hence by (1), (4),
\[
|\Delta_0| (1 - \varepsilon)/(24k\pi) \leq \sum_{i=1}^{3} \lim_{n \to \infty} N(\lambda r_n, \sigma_i; \Delta_0) T(\lambda r_n).
\]
If we make \(\varepsilon \to 0, k \to 1\), we have
\[
|\Delta_0| (24\pi) \leq \sum_{i=1}^{3} \lim_{n \to \infty} N(\lambda r_n, \sigma_i; \Delta_0) T(\lambda r_n).
\]
Hence
\[
\lim_{n \to \infty} N(\lambda r_n, \sigma; \Delta_0)/(T(\lambda r_n)) \geq |\Delta_0|/(2\pi),
\]
with two possible exceptions. If we write \(r_n, \Delta\) instead of \(\lambda r_n, \Delta_0\), then we have the theorem.

5. Meromorphic functions in a half-plane.

1. First fundamental theorem.

Let \(w(\zeta)\) be meromorphic in \(\Re \zeta \geq 0\) and let \(\zeta = \rho e^{i\theta}\) \((|\theta| \leq \pi/2)\),
\[
\zeta = -1/\zeta = \sigma + ii,
\]
\[
\sigma = -\cos \theta/\rho, \quad t = \sin \theta/\rho,
\]
then the niveau curve \(\Re (1/\zeta) = \text{const.} = 1/r\), or
\[
\sigma = \text{const.} = -1/r \quad (r > 0)
\]
is a circle: \(r \cos \theta = \rho\), whose diameter is \(r\) and which touches the imaginary
axis at the origin and the niveau curve

\[ t = \text{const.} = 1/\tau_0 \]  

is a circle, whose diameter is \(|\tau_0|\) and which touches the real axis at the origin. Hence to a rectangle \( Q_\tau \) on the \( \zeta \)-plane, which is bounded by four lines: \( t = \pm \pi, \ \sigma = \sigma_0 = -1/\tau_0, \ \sigma = -1/r(r > \tau_0) \), there corresponds on the \( \zeta \)-plane a domain \( \Delta_\tau \), which is bounded by four circles.

We put \( w(\zeta) = w(\xi) \) and let \( n(\sigma, a) \) be the number of zero points of \( w(\zeta) - a \) in \( Q_\tau \) and

\[
m(\sigma, a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \frac{1}{|w(\sigma + it), a|} \, dt,
\]

\[
\Re(\sigma, a) = \int_{\sigma_0}^{\sigma} n(\sigma, a) \, d\sigma,
\]

where

\[
[a, b] = |a - b|/[1 + |a|^2] (1 + |b|^2)]^{1/2}.
\]

Since \( w(\xi) \) is meromorphic on three circles, which correspond to three lines; \( \sigma = \sigma_0, t = \pm \pi, \), we have by the argument principle, if \( w(\xi) \neq a, \pm b \) on \( \Re \xi = \sigma, \)

\[
\frac{\partial m(\sigma, a)}{\partial \sigma} - \frac{\partial m(\sigma, b)}{\partial \sigma} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial \sigma} \log \left| \frac{w - b}{w - a} \right| \, dt
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d \arg \left( \frac{w - b}{w - a} \right) = n(\sigma, b) - n(\sigma, a) + O(1),
\]

so that

\[
m(\sigma, a) + \Re(\sigma, a) = m(\sigma, b) + \Re(\sigma, b) + O(1).
\]

Returning to the \( \xi \)-plane, if we write

\[
m(\sigma, a) = m(r, a), \ \Re(\sigma, a) = \Re(r, a), \ \Re(\sigma, a) = N(r, a),
\]

then we have easily

\[
m(r, a) = \frac{1}{2\pi} \int_{\tan^{-1} r}^{\tan^{-1} \infty} \log (1/|w(\xi)|, a) \sec^2 \theta \, d\theta,
\]

\[
N(r, a) = \int_{\tau_0}^{r} n(r, a) \, dr,
\]

where the right hand side of (8) is integrated on a circle \( \Re(1\zeta) = 1/r \) and \( n(r, a) \) is the number of zero points of \( w(\zeta) - a \) in \( \Delta_r \). If we put

\[
T(r, a) = m(r, a) + N(r, a),
\]
then (7) becomes
\[ T(r, a) = T(r, b) + O(1). \] (11)

From this we have easily the following

**Theorem 6. (First fundamental theorem).**
\[ T(r, a) = T(r) + O(1), \]
where
\[ T(r) = \int_0^r \frac{S(r)}{r^2} \, dr, \]
\[ S(r) = \frac{1}{\pi} \int_0^\pi \left( \frac{1}{1 + |w(\rho e^{i\theta})|^2} \right)^2 \rho \, d\rho \, d\theta. \]

Hence \( T(r) \) is an increasing convex function of \( -\frac{1}{r} \). We call \( T(r) \) the characteristic function of \( w(z) \) for \( \Re z \geq 0 \).

2. It can easily be proved:

**Theorem 7.** \( \int_0^\infty \frac{T(r)}{r^{\lambda+1}} \, dr \) and \( \int_0^\infty \frac{S(r)}{r^{\lambda+2}} \, dr \) \( (\lambda > 0) \) converge simultaneously and
\[ \int_0^\infty \frac{N(r, a)}{r^{\lambda+1}} \, dr, \int_0^\infty \frac{n(r, a)}{r^{\lambda+2}} \, dr, \sum \frac{[\Re (1/z_a)]^{k+1}}{\lambda} (\lambda > 0) \]
converge simultaneously, where \( z_a \) are zero points of \( w(z) - a \).

**Theorem 8.** Let \( w(z) \) be regular for \( \Re z \geq 0 \) and \( \Delta : |\arg z| \leq \alpha < \pi/2, \)
\[ M(r; \Delta) = \max_{|z| \leq a} |w(re^{i\theta})|, \]
then
\[ \log M(r; \Delta) \leq Ar (T(\alpha r) + O(1)). \]

where
\[ A = 2 \left( 1 + \sin\alpha \right) / \left( \cos\alpha (1 + \sin\alpha) \right), \quad \lambda = 2 / \cos\alpha. \]

**Proof.** Let \( M(r, \Delta) = \max_{|z| \leq a} |w(re^{i\theta})| \) be attained at \( z_0 = re^{i\theta_0} (|\theta_0| \leq \alpha) \), which lies in a circle \( |z - \rho| = \rho \sin\alpha (\rho = r / \cos\alpha) \), which touches two lines \( \arg z = \pm \alpha \), so that
\[ z_0 = re^{i\theta_0} = \rho + i\rho \sin\alpha, \quad |\rho| \leq \rho \sin\alpha. \]
Since \( \log |w(z)| \) is subharmonic, we have by means of Poisson integral on \( |z - \rho| = \rho, \)
\[
\log^+ M(r; \Delta) = \log^+ |w(z_0)| \leq \frac{\rho + \rho_0}{\rho - \rho_0} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |w(\rho e^{i\theta})| \, d\theta \\
\leq \frac{1 + \sin \alpha}{1 - \sin \alpha} \frac{1}{2\pi} \left[ \log \left( 1 + |w(\rho + \rho_0 e^{i\theta})|^2 \right)^{\frac{1}{2}} + \log\left( 1 + |w(\rho + \rho_0 e^{i\theta})| \right) \right] \, d\theta \\
= \frac{1 + \sin \alpha}{1 - \sin \alpha} \frac{1}{2\pi} \left[ \log \left[ 1, |w(\rho + \rho_0 e^{i\theta}), \infty] \right] \right] \, d\theta \\
\leq \frac{1 + \sin \alpha}{1 - \sin \alpha} 2\rho \left( T(2\rho, \infty) + O(1) \right) = \frac{1 + \sin \alpha}{1 - \sin \alpha} 2\rho \left( T(2\rho, \infty) + O(1) \right) \\
\leq \frac{1 + \sin \alpha}{1 - \sin \alpha} 2\rho \left( T(2\rho, \infty) + O(1) \right) = A r(T(xr) + O(1)),
\]

where

\[ A = 2(1 + \sin \alpha)/(\cos \alpha (1 - \sin \alpha)), \quad \lambda = 2/\cos \alpha. \]

**Theorem 9.** Let \( w(z) \) be meromorphic in \( \Re(z) \geq 0 \) and \( T(r) = O(1) \), then \( w(z) = g(z)/h(z) \), where \( g(z), h(z) \) are regular and \( |g(z)| \leq 1, |h(z)| \leq 1 \) for \( \Re(z) > 0 \).

**Proof.** By \( x = (\zeta - 1)/(\zeta + 1) \), we map \( \Re(z) \geq 0 \) on \( |x| < 1 \) and put \( w(z) = w_1(x) \) and \( T_1(\rho) \) be the Nevanlinna's characteristic function of \( w_1(x) \) in \( |x| < 1 \),

\[
T_1(\rho) = \int_0^\rho \frac{S_1(\rho)}{\rho} - \frac{d \rho}{d \rho} \quad (0 \leq \rho < 1),
\]

\[
S_1(\rho) = \frac{1}{\pi} \int_0^{2\pi} \left( \frac{|w_1'(re^{i\theta})|}{1 + |w_1(re^{i\theta})|^2} \right)^2 r \, d\theta.
\]

Since the circle \( \Re(1/z) = 1/r (r > 1) \) is mapped on a circle, which contains a circle \( |x| = (r-1)/(r+1) = \rho \),

\[
S_1(\rho) \leq S(r) + O(1) \quad (\rho = (r-1)/(r+1),
\]

and since \( d\rho \rho = 2(r^2 - 1) \, dr \leq 4/r^2 \, dr \) \( (r \geq \sqrt{2}) \), we have

\[
\int_1^\rho \frac{S_1(\rho)}{\rho} \, d\rho \leq 4 \int_1^\rho \frac{S(r)}{r^2} \, dr + O(1) = O(1).
\]

Hence \( T_1(\rho) = O(1) \), so that by Nevanlinna's theorem, \( w_1(x) = g_1(x)/h_1(x) \), where \( g_1(x), h_1(x) \) are regular and \( |g_1(x)| \leq 1, |h_1(x)| \leq 1 \) in \( |x| < 1 \). Returning to the \( z \)-plane, we have the theorem.

**3. Second Fundamental Theorem.**

In Ahlfors' proof of Nevanlinna's second fundamental theorem,\(^4\) if we

\[ 4) \text{L. Ahlfors: } \text{""""UBER EINE METHODE IN DER THEORIE DER MEROMORPHEN FUNKTIONEN, """" SOC. SCI. PENN. COMMENT. PHYS-MATH.} \text{8, NO. 10 (1932).} \]
replace \( \log \zeta = \log r + i\theta \) by \( \zeta = \frac{1}{r} \zeta = \sigma + it \), we have the following

**Theorem 10. (Second fundamental theorem).**

\[
(q - 2) T(r) \leq \sum_{i=1}^{q} N(\alpha, a_i) - N_i(r) + O(\log r + \log T(r)),
\]

outside certain intervals \( \{J_v\} \), such that

\[
\sum_{v} \int_{J_v} r^{\lambda-1} \, dr < \infty \quad (0 \leq \lambda < 1),
\]

where \( N_i(r) \) is formed similarly as \( N(r, a) \) with respect to all multiple values, a ple value being counted \( (a - 1) \)-times.

Especially if we take \( q = 3, \lambda = 0 \),

\[
T(r) \leq \sum_{i=1}^{3} N(r, a_i) + O(\log r + \log T(r)),
\]

outside intervals \( \{J_v\} \), such that

\[
\sum_{v} \int_{J_v} d \log r < \infty.
\]

From this we have

**Theorem 11.** If \( \lim_{r \to \infty} T(r) \log r = \infty \), then \( w(z) \) takes any value infinitely often with two possible exceptions.

**6. Theorems of Valiron and Nevanlinna.**

As an application of the theorems proved in § 5, we will prove theorems of Valiron and Nevanlinna as follows.

**Theorem 12 (Valiron)**. Let \( w(z) \) be meromorphic in \( \Delta \); \( |\arg z| \leq \alpha_0 \), \( (|\Delta_0| = 2\alpha_0) \) and \( \Delta; |\arg z| \leq \alpha < \alpha_0 \) be an angular domain contained in \( \Delta_0 \). If for a certain value \( a \) and \( \rho > \pi/|\Delta_0| \),

\[
\sum_{v} 1/|w_0(a, \Delta)|^\rho = \infty,
\]

then

\[
\sum_{v} 1/|w_0(a, \Delta)|^\rho = \infty.
\]

5) G. Valiron : Sur les directions de Borel des fonctions méromorphes d'ordre fini, Journ. de Math. 9 série 10 (1931).
for any \( a \), with two possible exceptions: and \( \Delta_0 \) contains a Borel's direction of order \( \rho \) of divergence type.

**Proof.** We choose

\[
\Delta_1: \quad |\arg z| \leq \alpha_1 \quad (\alpha < \alpha_1 < \alpha_0),
\]

such that \( \rho > \kappa_1 = \pi/|\Delta_1| \).

By \( \Delta_{h_1} = \infty \), we map \( \Delta_1 \) on \( \Re (\infty) \geq 0 \), then \( \Delta \) is mapped on \( \omega : |\arg x| \leq \beta < \pi/2 \). We put \( w(z) = w_1(x), |z| = r, |x| = R (-e^{h_1}) \),

\[
(z, \Delta)^{h_1} = x_\omega (a, \omega) = R e^{i\varphi_\omega}, \quad (|\varphi_\omega| \leq \beta),
\]

so that

\[
\Re \left( \frac{1}{|x^2|} (a, \omega) \right) = \cos \varphi_\omega / R \omega \geq \cos \beta R \omega = \cos \beta / |\varphi_\omega| (a, \Delta)^{h_1}.
\]

Hence \( \sum \Re (1/x^2) (a, \omega) \) is a fortiori \( \sum \Re (1/x^2) (a, \omega) \) where \( x_\omega (a) \) are zero points of \( w_1(z) - a \) in \( \Re (x) > 0 \).

Let \( T_1 (R), N_1 (R, a) \) be the functions defined in \( \S \) 5 for \( w_1(x) \), then since \( \rho > \kappa_1 > 1 \), we have by Theorem 7,

\[
\int_{R}^{\infty} \frac{S_1 (R)}{R^{h_{1}+1}} \, dR = \infty. \tag{1}
\]

If \( S (r, \Delta) \) is defined as in \( \S \) 2, then \( S_1 (R) \leq S (r, \Delta) (R = e^{h_1}) \), so that from (1),

\[
\int_{0}^{\infty} \frac{S (r, \Delta)}{r^{h_{1}+1}} \, dr = \infty.
\]

Since \( T (r, \Delta) \geq S (r, \Delta) \log 2 \), we have

\[
\int_{0}^{\infty} \frac{T (r, \Delta)}{r^{h_{1}+1}} \, dr = \infty. \tag{2}
\]

Hence by Theorem 2,

\[
\int_{0}^{\infty} \frac{N (r, a; \Delta_0)}{r^{h_{1}+1}} \, dr = \infty, \quad \text{or} \sum_{\gamma} |\zeta_\omega (a, \Delta_0)|^{-p} = \infty,
\]

with two possible exceptions. From (2) we conclude as Theorem 3 that \( \Delta_0 \) contains a Borel's direction of order \( \rho \) of divergence type.

**Theorem 13 (Nevanlinna-Valiron).** Let \( w(z) \) be regular in \( \Delta_0 : |\arg z| \leq \alpha_0 \) and \( \Delta : |\arg z| \leq \alpha < \alpha_0 \) be an angular domain contained in \( \Delta_0 \). If for some \( \rho > \pi/|\Delta_0| \) \((\geq 1/2)\)
\[ \int \frac{\log^+ M(r, \Delta)}{r^{\rho+1}} \, dr = \infty, \]
then
\[ \sum_{r} \frac{1}{|z_{\alpha}(a, \Delta_0)|^\rho} = \infty \]
for any \( a \), with two possible exceptions\(^6\) and \( \Delta_0 \) contains a Borel's direction of order \( \rho \) of divergence type\(^7\).

**Proof.** Let \( \Delta_1 : |\arg z_1| \leq \alpha_1 (\alpha < \alpha_1 < \alpha_0) \) be so chosen that \( \rho > \kappa_1 = \pi/|\Delta_1| \) and by \( \zeta_1 = \infty \), we map \( \Delta_1 \) on \( \Re z \geq 0 \), then \( \Delta \) is mapped on \( \omega : |\arg x| \leq \beta < \pi/2 \). We put \( w(z) = w_1(x) \), then
\[ M_1(R, \omega) = \max_{\rho \leq \beta} |w_1(Re^{i\omega})| = M(r, \Delta) \quad (R = r^{k_1}), \]
so that
\[ \int \frac{\log^+ M_1(R, \omega)}{R^{\kappa_1+1}} \, dR = k_1 \int \frac{\log^+ M(r, \Delta)}{r^{\rho+1}} \, dr = \infty. \quad (1) \]

Let \( T_1(R) \) be the characteristic function of \( w_1(x) \) defined in §5, then by Theorem 8
\[ \log^+ M_1(R, \omega) \leq A R (T_1(\lambda R) + O(1)), (\lambda > 1), \]
so that from (1),
\[ \int \frac{T_1(R)}{R^{\kappa_1}} \, dR = \infty, \text{ hence } \int \frac{S_1(R)}{R^{\kappa_1+1}} \, dR = \infty. \]
From this we proceed similarly as Theorem 12 and have the theorem.

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7) G. Valiron, I. c. (7)