1. Introduction. Let \( \{p_n\}_{n=0}^{\infty} \) be a sequence of non-negative constants, \( p_0 > 0 \) and \( P_n = \sum_{k=0}^{n} p_k \). A sequence \( \{U_n\}_{n=0}^{\infty} \) will be said to be absolutely summable by the Nörlund method defined by the sequence \( \{p_n\} \), or summable \( |N, p_n| \), if \( t_n = \sum_{k=0}^{\infty} \frac{(p_{n-k}U_k)}{P_n} \) and

\[
\sum_{n=1}^{\infty} |t_n - t_{n-1}| \leq c < \infty.
\]

Varshney [10] showed that if \( f(x) \) is a real-valued, \( 2\pi \)-periodic function and of bounded variation over \([0, 2\pi]\) and if

\[
|f(x + h) - f(x)| \leq A \log^{-1-\varepsilon}(\frac{1}{h}) (\varepsilon > 0, 0 \leq x \leq 2\pi, h > 0)
\]

then \( S(f) \), the Fourier series of \( f \), is summable \( |N, 1/(n+1)| \). The author [8] later proved this result under the following weaker hypothesis:

\[
\sum_{n=1}^{\infty} \frac{1}{n} \omega\left(1 \frac{n}{n}\right) < \infty,
\]

where \( \omega(t, f) = \omega(t) \) denotes, as usual, the modulus of continuity of \( f \). Recently Izumi and Izumi [3], Lal [5] and others have studied the conditions for \( |N, p_n| \) summability of \( S(f) \) for general \( \{p_n\} \). Lal has shown that, if (i) \( p_0 > 0 \), (ii) \( \{p_n\} \) is non-negative and non-increasing, (iii) \( \lim_{n \to \infty} p_n = 0 \), (iv) \( \{p_n - p_{n+1}\} \) is non-increasing, and if

\[
\sum_{n=1}^{\infty} n^{-r} < \infty \quad (1 < r \leq 2),
\]

and

\[
\sum_{n=1}^{\infty} \omega(n^{-1})P^{-1/n} < \infty, \left( \frac{1}{r} + \frac{1}{s} = 1 \right),
\]

then \( S(f) \) is summable \( |N, p_n| \). In this paper we obtain conditions for \( |N, p_n| \) summability of \( S(f) \) when the series in (1.4) may fail to con-
verge. Thus our results supplement those of Lal.

In what follows we will suppose that $\gamma$ is a fixed constant, $0 \leq \gamma < 1/2$, $c_1$ and $c_2$ are fixed positive constants and $\psi(x)$ is positive on $[0, \infty)$ and slowly oscillating in the sense of Karamata (see [2], [4]). Let \( \{p_n\} \) satisfy conditions (i)--(iv) and suppose that for $n \geq 1$,

\[(v) \quad c_1 n^\gamma \psi(n) \leq p_n \leq c_2 n^\gamma \psi(n),\]

These conditions are all satisfied if, for instance, we take $p_n = (n + 1)^{-1 + \gamma}$, $0 \leq \gamma < 1/2$. Some further examples are given in Section 4. We prove the following

**THEOREM 1.** Let $f(x)$ be a $2\pi$-periodic function of bounded variation over $[0, 2\pi]$ and suppose that the modulus of continuity $\omega(t, f)$ satisfies (1.3) and

\[(1.5) \quad \sum_{n=1}^{\infty} \frac{1}{nP_n} \omega^{1/2}(\frac{1}{n}) < \infty.\]

Then under the assumptions (i)--(v), $S(f)$ is summable $|N, p_n|$.

2. **Lemmas.** We shall denote by $A$ a positive constant (possibly depending on $\gamma$, $c_1$, $c_2$) not necessarily the same at each occurrence.

**LEMMA 1** [6]. If $\{p_n\}$ is non-negative and non-increasing, then for $0 \leq a \leq b \leq \infty$, $0 \leq t \leq \pi$ and any $n$, we have

\[(2.1) \quad \left| \sum_{k=a}^{b} p_k e^{i(n-k)t} \right| \leq \begin{cases} P(t^{-1}) & \text{for any } a, \\ At^{-1} p_{[a]} & \text{for } a \geq [t^{-1}]. \end{cases}\]

Here $[x]$ denotes the integer part of $x$, and $P(x) = P_{[x]}$.

**LEMMA 2** [6]. If $\{p_n\}$ is non-negative and non-increasing and $\{p_n - p_{n+1}\}$ is non-increasing, then

\[(2.2) \quad \frac{n^2(p_n - p_{n+1})}{P(n-1)} \leq \frac{n^2(p_{n-1} - p_n)}{P(n-1)} \leq A.\]

**LEMMA 3.** If $P(x)$ satisfies (v) then

\[(2.3) \quad \frac{n}{P(n-1)} \int_0^\infty \frac{P(u)du}{u^2} < A.\]

This follows from the properties of slowly oscillating functions [2]. We have $\int_0^\infty w^{-\gamma} \psi(u)du \sim \psi(n)(n^\gamma/1-\gamma)$, and $\psi(n) \sim \psi(n-1)$ and (2.3) follows.
LEMMA 4. Let

\[ I_n = \int_{1/\pi}^{2\pi} \omega^i(1/t) dt. \]

The series in (1.3) and the series

\[ \sum_{n=1}^{\infty} 2^{-n/2} I_n^{1/2} \]

are both convergent or both divergent.

PROOF. Since

\[ I_n > \frac{1}{4} 2^n \omega^i\left(\frac{1}{2^n}\right) \]

the convergence of (2.5) implies the convergence of \( \sum_{n=1}^{\infty} \omega(1/2^n) \) and hence that of the series in (1.3). Suppose now that the series in (1.3) is convergent. Then

\[ I_n < \omega^i(\pi) + \omega^i(1) + 2\omega^i\left(\frac{1}{2}\right) + \cdots + 2^{1-1} \omega^i\left(\frac{1}{2^{n-1}}\right), \]

\[ \sum_{n=1}^{\infty} 2^{-n/2} I_n^{1/2} < \omega(\pi) \sum_{1}^{\infty} 2^{-n/2} + \sum_{n=1}^{\infty} 2^{-n/2} \sum_{p=1}^{n-1} 2^{p/2} \omega\left(\frac{1}{2^p}\right) \]

\[ < A + \sum_{p=1}^{\infty} 2^{p/2} \omega\left(\frac{1}{2^p}\right) \sum_{n=p+1}^{\infty} 2^{-n/2} \]

\[ < A + A \sum_{p=1}^{\infty} \omega\left(\frac{1}{2^p}\right) < A. \]

3. Proof of Theorem 1. Let

\[ f(t) \sim \frac{1}{2} a_n + \sum_{i} (a_i \cos nt + b_i \sin nt) = \sum_{x} u_x, \]

\[ s_n = \sum_{y=0}^{n} u_y, \quad t_n = \sum_{y=0}^{n} \frac{p_y s_{n-y}}{P_y}, \]

\[ \phi(t) = f(x + t) + f(x - t) - 2f(x), \]

\[ \alpha(t) + i\beta(t) = \sum_{k=0}^{\infty} p_k e^{ikt}, \]

\[ \alpha_n = \int_{0}^{\pi} \phi(t)\alpha(t) \cos nt dt, \quad \beta_n = \int_{0}^{\pi} \phi(t)\beta(t) \sin nt dt. \]

We have (cf: [6], [8])
\[ \pi | t_n - t_{n-1} | = \left| \int_0^\pi \phi(t) \sum_{k=0}^{n-1} \left( \frac{P_k}{P_n} - \frac{P_{k-1}}{P_{n-1}} \right) \cos(n - k)t \, dt \right| \]

\[ \leq \frac{1}{P_{n-1}} \left| \int_0^{1/n} \phi(t) \sum_{k=0}^{n-1} p_k \cos(n - k)t \, dt \right| \]

\[ + \frac{1}{P_{n-1}} \left| \int_{1/n}^\pi \phi(t) \sum_{k=n}^{n-1} p_k \cos(n - k)t \, dt \right| \]

\[ + \frac{P_n}{P_n P_{n-1}} \left| \int_0^{1/n} \phi(t) \sum_{k=0}^{n-1} P_k \cos(n - k)t \, dt \right| \]

\[ + \frac{1}{P_{n-1}} \left| \int_{1/n}^\pi \phi(t) \left( \sum_{k=0}^{n-1} p_k \cos(n - k)t + \sum_{k=n}^{n-1} \frac{P_n}{P_k} \cos(n - k)t \right) \, dt \right| \]

\[ = T_1(n) + T_2(n) + T_3(n) + T_4(n) \text{ say} . \]

We have to prove that \( \sum | t_n - t_{n-1} | < \infty \). By Lemmas 1 and 3

\[ T_2(n) < \frac{2}{P_{n-1}} \int_0^{1/n} \omega(t) P \left( \frac{1}{t} \right) \, dt \leq 2 \frac{\omega(1/n)}{P_{n-1}} \int_0^\infty P(u) \, du \leq A n \frac{\omega(1/n)}{n} \]

and by (1.3), \( \sum_{s=2}^{n-1} T_s(n) < \infty \).

Further, since \( p_n \downarrow \),

\[ T_3(n) < \frac{2p_n}{P_n P_{n-1}} \omega \left( \frac{1}{n} \right) \frac{P_0 + \cdots + P_{n-1}}{n} \]

\[ < \frac{2p_n}{P_n P_{n-1}} \omega \left( \frac{1}{n} \right) P_{n-1} < \frac{2}{n} \omega \left( \frac{1}{n} \right) \]

and so \( \sum_{s=2}^{n-1} T_s(n) < \infty \).

Further

\[ T_4(n) = \frac{1}{P_{n-1}} \left| \int_0^\pi \phi(t) \left[ \frac{P_n}{2} + \sum_{k=n}^{n-1} (p_n - p_{k+1}) \sin(n - k + (1/2)t) \right] \frac{2 \sin(t/2)}{2 \sin(t/2)} \right| \]

\[ + \frac{p_n}{P_n} \left( \sum_{k=n}^{n-1} p_k \sin(n - k + (1/2)t) \right) \frac{1}{2 P_{n-1}} \right| \]

\[ \leq \frac{1}{P_{n-1}} \left| \int_0^{1/n} \phi(t) \frac{2}{2 \sin(t/2)} \left( \sum_{k=n}^{n-1} (p_n - p_{k+1}) \sin(n - k + (1/2)t) \right) \right| \]

\[ + \frac{p_n}{P_n P_{n-1}} \left| \int_{1/n}^\pi \phi(t) \frac{2}{2 \sin(t/2)} \left( \sum_{k=n}^{n-1} p_k \sin(n - k + (1/2)t) \right) \right| \]

\[ + \frac{p_n}{2P_{n-1}} \left( 1 - \frac{P_{n-1}}{P_n} \right) \left| \int_{1/n}^\pi \phi(t) \right| \]

\[ = T_4(n) + T_{4a}(n) + T_{4b}(n) \].

By Lemma 1
Lemma 2 now shows that
\[
\sum_{n=1}^{\infty} T_n(n) \leq A \sum_{n=2}^{\infty} \frac{1}{n^2} \left( A + \sum_{k=1}^{n} \omega \left( \frac{1}{k} \right) \right) < A + A \sum_{k=1}^{\infty} \omega \left( \frac{1}{k} \right) \sum_{n=2}^{\infty} \frac{1}{n^2} < A + A \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty.
\]

Further \( \int_{1/n}^{\pi} |\phi(t)| \, dt < A \) and so
\[
\sum_{n=1}^{\infty} T_n(n) < A \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} < A \sum_{n=1}^{\infty} \frac{p_n^2}{(n+1)p_n p_{n-1}} < A \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.
\]

By Lemma 1
\[
T_\alpha(n) \leq \frac{Ap_n}{P_n p_{n-1}} \int_{1/n}^{\pi} \frac{\phi(t)}{t} P \left( \frac{1}{t} \right) \, dt \leq \frac{Ap_n}{P_n p_{n-1}} \int_{1/n}^{\pi} \omega \left( \frac{1}{t} \right) P(t) \, dt,
\]
and
\[
\sum_{n=2}^{\infty} T_\alpha(n) \leq A \sum_{n=2}^{\infty} \frac{p_n}{P_n p_{n-1}} \left( A + \sum_{k=1}^{n} \omega \left( \frac{1}{k} \right) \frac{P(k)}{k} \right) < A + A \sum_{n=2}^{\infty} \frac{p_n}{P_n p_{n-1}} \sum_{k=1}^{n} \omega \left( \frac{1}{k} \right) \frac{P(k)}{k} < A + A \sum_{k=1}^{\infty} \omega \left( \frac{1}{k} \right) \frac{P(k)}{k} \frac{1}{P(k-1)} < A + A \sum_{k=1}^{\infty} \omega \left( \frac{1}{k} \right) \frac{1}{k} < \infty.
\]

We now consider \( T_\alpha(n) \leq (|\alpha_n| + |\beta_n|)/P_{n-1} \).

Let \( \psi(t) = \phi(t + h)\alpha(t + h) - \phi(t - h)\alpha(t - h) \). Then \( \psi(t) \) is even and \( \alpha(t) \in L^2 \), for by Lemma 1,
\[
\int_{0}^{\pi} \psi^2(t) \, dt \leq \int_{0}^{\pi} \omega^2 \left( \frac{1}{t} \right) t^{-2} \, dt < \infty.
\]
By Bessel’s inequality we have for $0 < h \leq \pi/4$,

$$\sum_{k} |\alpha_k^2 \sin^2 nh| < A \int_{0}^{\pi} |F^2(t)| dt$$

$$< A \left[ \int_{0}^{\pi} \alpha^2(t + h) \{\phi(t + h) - \phi(t - h)\}^2 dt + \int_{-h}^{h} |\phi^2(t)| \alpha^2(t) dt \right.$$  

$$+ \int_{-h}^{h} |\phi^2(t)| \alpha^2(t + 2h) dt + \int_{-h}^{h} |\phi^2(t)| \alpha^2(t) dt$$

$$\left. + \int_{h}^{\pi} |\phi^2(t)| (\alpha(t + 2h) - \alpha(t))^2 dt \right]$$

$$\equiv A[I_1(h) + I_2(h) + I_3(h) + I_4(h)].$$

By (v) and Lemma 1,

$$I_2(h) < A \int_{-h}^{h} \omega^2(t)P^2\left(\frac{1}{t + 2h}\right) dt$$

$$\leq A\omega^2(h) \int_{-h}^{h} P^2\left(\frac{1}{t}\right) dt < Ah\omega^2(h)P^2\left(\frac{1}{h}\right),$$

and

$$I_3(h) = \int_{-h}^{h} \omega^2(t) |\phi^2(t)| |\alpha^2(t)| dt < A\omega^2(h) \int_{-h}^{h} \alpha^2(t) dt$$

$$< A\omega^2(h) \int_{-h}^{h} P^2\left(\frac{1}{t}\right) dt < A\omega^2(h) hP^2\left(\frac{1}{h}\right).$$

Since [6]

$$|\alpha(t + 2h) - \alpha(t)| \leq Ah^{-1}P(h^{-1}),$$

$$I_4(h) < Ah^2P^2\left(\frac{1}{h}\right) \int_{-h}^{h} \omega^2(t) \frac{dt}{t^2} < Ah^2P^2\left(\frac{1}{h}\right) \int_{1/h}^{1} \frac{\omega(t)}{t} dt.$$

We now estimate $I_1$. Since $f$ is of bounded variation over $[0, 2\pi]$ we have

$$\sum_{\ell = 1}^{\infty} \alpha^2(t + \frac{k\pi}{N}) \left| \left\{ \phi(t + \frac{k\pi}{N}) - \phi\left(t + \left(k - 1\right)\frac{\pi}{N}\right) \right\} \right|^2 < A\omega\left(\frac{\pi}{N}\right) P^2\left(\frac{1}{t}\right).$$

Integrating from $0$ to $\pi$ (cf. [8; p. 241-2]) we get

$$2NI_1\left(\frac{\pi}{2N}\right) < A\omega\left(\frac{\pi}{N}\right) \int_{0}^{\pi} P^2\left(\frac{1}{t}\right) dt < A\omega\left(\frac{\pi}{N}\right) \int_{1/\pi}^{\infty} \frac{P^2(t) dt}{t^2} < A\omega\left(\frac{\pi}{N}\right).$$

Taking $h = \pi/(2N)$ we get
\begin{align*}
\sum_{1}^{\infty} \left| \alpha_n \sin^2 \left( \frac{n\pi}{2N} \right) \right| &< A \left\{ \frac{1}{N} \omega \left( \frac{\pi}{N} \right) + \frac{1}{N^2} \omega^{\prime} \left( \frac{2N}{\pi} \right) \right\} + \frac{1}{N^2} P^2 \left( \frac{2N}{\pi} \right) \int_{1/\pi}^{(2N)/\pi} \omega^{\prime} \left( \frac{1}{t} \right) dt \\
&< A \left\{ \frac{1}{N} \omega \left( \frac{\pi}{N} \right) + \frac{1}{N^2} P^2 \left( \frac{2N}{\pi} \right) \right\} \int_{1/\pi}^{(2N)/\pi} \omega^{\prime} \left( \frac{1}{t} \right) dt \\
\end{align*}

Letting \( N = 2^v \) we have

\begin{align*}
\left\{ \sum_{2^{v-1}+1}^{2^v} | \alpha_n \right\}^{1/2} &< A \left\{ \sum_{1}^{\infty} | \alpha_n | \sin^2 \left( \frac{n\pi}{2^{v+1}} \right) \right\}^{1/2} \\
&< A \left\{ \frac{1}{2^v} \omega \left( \frac{\pi}{2^v} \right) + \frac{1}{2^v} P^2 \left( \frac{2^{v+1}}{\pi} \right) \right\} \int_{1/\pi}^{(2^{v+1})/\pi} \omega^{\prime} \left( \frac{1}{t} \right) dt \\
&< A \left\{ \frac{1}{2^v} \omega \left( \frac{\pi}{2^v} \right) + \frac{1}{2^v} P \left( \frac{2^{v+1}}{\pi} \right) \int_{1/\pi}^{(2^{v+1})/\pi} \omega^{\prime} \left( \frac{1}{t} \right) dt \right\}.
\end{align*}

By (v) we have

\[ \sum_{2^{v-1}+1}^{2^v} \frac{1}{P_{n-1}^2} < \frac{2^v}{P(2^v)} \]

and an application of Schwarz inequality gives

\[ \sum_{2^{v-1}+1}^{2^v} \frac{| \alpha_n |}{P_{n-1}} \leq A \frac{2^v}{P(2^v)} \left\{ \frac{1}{2^{v/2}} \omega^{1/2} \left( \frac{\pi}{2^v} \right) + \frac{1}{2^v} P \left( \frac{2^{v+1}}{\pi} \right) \int_{1/\pi}^{(2^{v+1})/\pi} \omega^{\prime} \left( \frac{1}{t} \right) dt \right\}.
\]

By (1.5),

\[ \sum \frac{1}{P(2^v)} \omega^{1/2} \left( \frac{\pi}{2^v} \right) < A, \]

and by Lemma 4,

\[ \sum \frac{P(2^{v+1}/\pi)}{P(2^v)} \frac{1}{2^{v/2}} \int_{1/\pi}^{(2^{v+1})/\pi} \omega^{\prime} \left( \frac{1}{t} \right) dt < A. \]

Hence \( \sum_{n=1}^{\infty} | \alpha_n |/P_{n-1} < \infty \). Similarly \( \sum_{n=1}^{\infty} | \beta_n |/P_{n-1} < \infty \) and so \( \sum_{n=1}^{\infty} | t_n - t_{n-1} | < A < \infty \) and the proof is complete.

4. Remarks and Examples.

(a) If (1.3) holds and \( \sum 1/(nP^2(n)) < \infty \), then an application of Schwarz inequality shows that (1.5) holds.

(b) The condition (1.5) implies that

\[ \omega \left( \frac{1}{t} \right) < A(P^2(t))/\log^2 t, \quad t \geq 2. \]

Consequently
Hence if (1.5) holds and

\[ \sum_{n} o\left(\frac{1}{2^n}\right) < A \sum (P(2^n))/n^2. \]

then the series in (1.3) is convergent.

If we take, for instance, \( p_n = (n + a)^{-(\log (n + a))^{-1}}, \ a \geq 3, \) then by considering \( y(x) = (x + a)^{-(\log (x + a))^{-1}} \) we see that \( p_n \) satisfies the conditions (i)-(iv). Further \( P_n \sim \log \log n \) and so (4.1) and (v) are satisfied (with \( \gamma = 0 \)).

(c) Zygmund [11; 241-2] proved that if \( f(x) \) is of bounded variation and

\[ \sum n^{-\gamma} \omega^{1/2}(n^{-\gamma}) < \infty, \]

then \( S(f) \) is absolutely convergent. Our theorem gives the following analogue of Zygmund's result:

If \( f(x) \) is of bounded variation and if (4.1) holds, the then convergence of the series in (1.5) implies the absolute summability \( |N, p_n| \) of \( S(f) \).

Note that if we take \( p_n = 1 \) and \( p_n = 0 \) \( (n > 0) \) then (1.5) is the same as (4.2) and the summability \( |N, p_n| \) is the same as the absolute convergence.

Example. Let

\[ p_n = \frac{c \log(n + c)}{(n + c) \log c}, \quad \log c \geq 2. \]

Then \( p_n > 0, \ \{p_n\} \downarrow, \ \{p_n - p_{n+1}\} \downarrow \) (cf: [6]). \( P_n \sim A(\log n)^\gamma. \) Hence condition (v) is satisfied (with \( \gamma = 0 \)) and \( \sum 1/(nP_n) < \infty. \) (This implies that (1.5) is satisfied if (1.3) is.) By considering \( y'(x) \) where

\[ y(x) = \frac{(x + c)}{(x + 1 + c)} \frac{\log(x + 1 + c)}{\log(x + c)}, \]

we see that \( p_{n+1}/p_n \uparrow \) and so by a known inclusion theorem [6], \( |N, p_n| \subset |C, 1|. \)

5. Weighted Arithmetic Means. We now consider the weighted arithmetic mean ([7; pp. 16-17], [9; p. 32]) of the series \( \sum_{n} u_n. \) Let \( S_n = \sum_{n} u_n. \) Let \( p_n \geq 0, \ P_n > 0 \) and \( \sigma_n = 1/P_n \sum_{n} p_n S_n. \) To avoid trivial cases we shall suppose that \( p_n > 0 \) for an infinity of \( n. \) The sequence \( \{S_n\} \) is said to be absolutely summable by the weighted arithmetic mean method, defined by the sequence \( \{p_n\}, \) or briefly summable \( |M, p_n|, \) if
Let $f \in C_2$ (continuous and $2\pi$-periodic) and let
\[
\omega_l(\delta, f) = \sup_{x, y \in \mathbb{R}} |f(x + h) + f(x - h) - 2f(x)| (x \in [0, 2\pi])
\]
denote the modulus of smoothness of $f$.

**THEOREM 2.** Let $p_n \geq 0$, $P_n = \sum_{i=0}^{n} p_i > 0$, $P_n \to \infty$ and $f \in C_2$. If
\[
\sum_{n=1}^{\infty} \frac{p_n}{P_n} \log n \omega_l\left(\frac{1}{n}\right) < \infty,
\]
then $S(f)$ is summable $|M, p_n|$.

**PROOF.** We have [1; p. 300, p. 533]
\[
|S_n(t) - f(t)| < C \omega_l((n + 1)^{-1}) \max(1, \log n)
\]
where $C$ is an absolute constant. Hence for $n \geq 1$,
\[
|\sigma_n(t) - \sigma_{n-1}(t)| = \left| \frac{1}{P_n} \sum_{k=0}^{n-1} p_k (S_k(t) - f(t)) - \frac{1}{P_{n-1}} \sum_{k=0}^{n-1} p_k (S_k(t) - f(t)) \right|
\]
\[
= \left| \left( \frac{1}{P_n} - \frac{1}{P_{n-1}} \right) \sum_{k=0}^{n-1} p_k (S_k(t) - f(t)) + \left( \frac{1}{P_n} \right) p_n (S_n(t) - f(t)) \right|.
\]
Thus for $0 \leq t \leq 2\pi$,
\[
\sum_{n=1}^{\infty} |\sigma_n(t) - \sigma_{n-1}(t)| \leq C \sum_{n=1}^{\infty} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \sum_{k=0}^{n-1} p_k \omega_l\left(\frac{1}{k + 1}\right) \max(1, \log k)
\]
\[+ C \sum_{n=1}^{\infty} \frac{p_n}{P_n} \omega_l\left(\frac{1}{n + 1}\right) \max(1, \log n)
\]
\[\leq 2C \left\{ \sum_{k=0}^{\infty} \frac{p_k}{P_k} \omega_l\left(\frac{1}{k + 1}\right) \max(1, \log k) \right\},
\]
and our hypothesis shows that the last series is convergent. The proof is complete.

**COROLLARY.** If $f \in C_2$ and $\sum_{n=1}^{\infty} (\log n/(n + 1)) \omega_l(1/n) < \infty$, then $S(f)$ is summable $|C, 1|$.

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