SOME QUASI-HAUSDORFF TRANSFORMATIONS

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1. Let \( \{c_n\} \) be any given sequence of complex numbers. The quasi-Hausdorff transformation \((H^*, c_n)\) is defined by

\[
(1) \quad t_n = \sum_{k=n}^{\infty} \binom{k}{n} (c_k - c_n) s_k
\]

whenever this series converges. We will use \((H^*, c_n)\) also to denote the matrix of the transformation (1), and write \(s, t\) for the sequences \(\{s_k\}, \{t_n\}\); thus (1) may be written

\[
t = (H^*, c_n)s.
\]

We say that the \((H^*, c_n)\) method is applicable to \(s\) if (1) converges for all \(n\), so that \(t\) is defined; we say that \(s\) is summable \((H^*, c_n)\) to \(l\) if, further \(t_n \rightarrow l\) as \(n \rightarrow \infty\). We use a similar terminology for other transformations.

The matrix \((H^*, c_n)\) is the transpose of the matrix of the Hausdorff transformation \((H, c_n)\). It is familiar that, given two sequences \(\{c_n\}, \{\omega_n\}\) (say), we have

\[
(H, c_n)(H, \omega_n) = (H, c_n\omega_n).
\]

Taking the transpose of this result (with \(c, \omega\) interchanged) we have, as is familiar

\[
(2) \quad (H^*, c_n)(H^*, \omega_n) = (H^*, c_n\omega_n).
\]

But the matrices considered are not, in general, row finite, so that their multiplication is not necessarily associative; thus we cannot assert that

\[
(3) \quad (H^*, c_n)(H^*, \omega_n)s = [(H^*, c_n)(H^*, \omega_n)]s.
\]

Thus the situation differs from that which applies for the corresponding Hausdorff transformations in that, notwithstanding (2), we cannot assert that the result of applying first the \((H^*, \omega_n)\) and then the \((H^*, c_n)\) transformation is the same as that of applying the \((H^*, c_n\omega_n)\) transformation.

It has been shown by Ramanujan [4] that there is a close connection between Hausdorff summability \((H, \mu_n)\) and quasi-Hausdorff summability

\[\text{for those properties of Hausdorff transformations to which reference is made, see, e.g. [1, Chapter XI].}\]
(H*, μn+1); in particular, whenever (H, μn) is regular then so is (H*, μn+1).

When

\[ \mu_n = \frac{1}{n + r}, \]

(H, μn) reduces to the Cesàro transformation (C, r); thus it is natural to describe the quasi-Hausdorff transformation (H*, μn+1) with μn given by (4) as the quasi-Cesàro transformation (C*, r). The properties of (C*, r) have been investigated by me [2], [3]; a more general transformation was investigated independently by A. J. White [5].

When

\[ \omega_n = \frac{1}{(n + 1)^r}, \]

(H, ωn) reduces to the Hölder transformation (H, r); we will therefore describe the (H*, ωn+1) transformation with ωn given by (5) as the quasi-Hölder transformation (H*, r).

It is known (e.g. [1]) that Cesàro and Hölder summabilities (C, r), (H, r) are equivalent. Thus if for a given r, μn, ωn are given by (4), (5) we have μn = νnωn where (H, νn) is regular. Hence, by what has already been said

\[ (H^*, \mu_{n+1}) = (H^*, \nu_{n+1})(H^*, \omega_{n+1}), \]

and (H*, νn+1) is regular. But, since we cannot assert (3), we cannot deduce from this that summability (C*, r) is implied by summability (H*, r).

Similar remarks apply with the roles of (C*, r), (H*, r) interchanged.

When r is an integer, the Hölder transformation (H, r) is the same as the transformation obtained by r iterations of the (C, 1) transformation; and we can deduce that

\[ (H^*, \nu_r) = [(C^*, 1)]^r. \]

But although (6) holds as a relation between matrices, we cannot deduce that the result of r iterations of the (C*, 1) transformation is the same as (H*, r).

We will restrict consideration to integer values of r; accordingly, it will be assumed throughout from now on that r is a positive integer. On this understanding, we investigate the relations between (C*, r), (C*, 1)r, (H*, r). Here (C*, 1)r is used to denote the result of r iterations of the (C*, 1) transformation.

The results to be proved are as follows.
THEOREM 1. \((C^*, r)\) and \((C^*, 1)^r\) are equivalent.

THEOREM 2. If \(s\) is summable \((H^*, r)\) to \(l\), then it is summable \((C^*, r)\) to \(l\). If \(s\) is summable \((C^*, r)\) to \(l\), and if \((H^*, r)\) is applicable, then \(s\) is summable \((H^*, r)\) to \(l\). However, except in the trivial case \(r = 1\), the applicability of \((H^*, r)\) is not implied by \((C^*, r)\) summability.

Let now \(r_1 > r\) (where \(r_1\) is also an integer). It is known [3, Theorem 1; 5, Theorems 2, 3] that, if \(s\) is summable \((C^*, r)\) to \(l\) then it is summable \((C^*, r_1)\) to \(l\). It therefore follows at once from Theorem 2 that, if \(s\) is summable \((H^*, r)\) to \(l\) and if \((H^*, r_1)\) is applicable, then \(s\) is summable \((H^*, r_1)\) to \(l\). However, the hypothesis that \((H^*, r_1)\) is applicable cannot in general be omitted.

THEOREM 3. Let \(r_1 > r\) (\(r_1\) an integer). Let \(s\) be summable \((H^*, r)\) to \(l\). If \(r = 1\), then \((H^*, r_1)\) is applicable. This result becomes false if \(r > 1\).

It follows at once from Theorem 3 and the remarks made above that summability \((H^*, r)\) implies summability \((H^*, r_1)\) without any supplementary “applicability condition” when \(r = 1\), but not when \(r > 1\).

2. We require some lemmas.

LEMMA 1. Let

\[
F(k, x) = \sum_{\rho=0}^{r} (-1)^\rho P_\rho(k)x^\rho ;
\]

\[
G(k, x) = \sum_{\rho=0}^{r} (1-\rho)P_\rho(k - \rho)x^\rho ,
\]

where, for each \(\rho\), \(P_\rho(k)\) is a polynomial in \(k\) of degree not exceeding \(r\). Suppose that \(F(k, x)\) has the property that, when expressed as a polynomial in \(k\), the coefficient of \(k^q\) is divisible by \((1-x)^q\) \((q = 1, 2, \cdots, r)\). Then \(G(k, x)\) also has this property.

Write

\[ F(k, x) = \sum_{q=1}^{r} \phi_q(x)k^q . \]

It is enough to consider the contribution to \(G(k, x)\) of one term in the sum (7), since the general result can then be obtained by addition. Taking, then, \(q\) as fixed, let \(a_\rho\) be the coefficient of \(k^\rho\) in \((-1)^\rho P_\rho(k)\); thus

\[ \phi_q(x) = \sum_{\rho=0}^{r} a_\rho x^\rho . \]

The contribution of this term to \(G(k, x)\) is
We can write (8) as $L^q \phi_q(x)$, where the operator $L$ is defined by

$$L f(x) = k f(x) - x f'(x).$$

Since $\phi_q(x)$ is divisible by $(1 - x)^q$, it follows by induction on $t$ that $L^t \phi_q(x)$ is a polynomial in $k$ of degree $t$, the coefficient of $k^t$ being divisible by $(1 - x)^{q + s - t}$. Applying this result with $t = q$, the lemma follows.

**Lemma 2.** Suppose that

$$f(x) = \sum_{\rho=0}^{r} (-1)^{\rho} a_{\rho} x^{\rho}$$

is divisible by $(1 - x)^r$. Let $Q(x)$ be a polynomial in $x$ of degree $\nu$. Then

$$\sum_{\rho=0}^{r} (-1)^{\rho} a_{\rho} Q(k - \rho)$$

is a polynomial in $k$ of degree at most $\nu - q$. In the case $q = \nu$, the conclusion is to be interpreted as meaning that (9) is constant; in the case $q > \nu$, it is to be interpreted as meaning that (9) is identically zero.

It is slightly more convenient to prove a similar result, but with (9) replaced by

$$\sum_{\rho=0}^{r} (-1)^{\rho} a_{\rho} Q(k + \rho);$$

this will give the conclusion, for we can apply this result with $Q(x)$ replaced by $Q(-x)$ and with $k$ replaced by $-k$.

Write

$$\psi(x) = (1 - x)^s \psi_1(x),$$

and write $E$ for the "shift operator" defined by $EQ(k) = Q(k + 1)$. Then we can write (10) as

$$\left(\sum_{\rho=0}^{r} (-1)^{\rho} a_{\rho} E^\rho\right) Q(k) = \left((1 - E)^s \psi_1(E)\right) Q(k) = \Delta^s(\psi_1(E) Q(k)).$$

The operator $\psi_1(E)$ operating on a polynomial cannot increase its degree; the operator $\Delta^s$ decreases its degree by $q$ (with the same conventions as in the statement of the lemma). Hence the conclusion.

**Lemma 3.** Let $F(k, x), P_{\rho}(k)$ satisfy the conditions of Lemma 1. Let $Q(k, n)$ be a polynomial in $k, n$ of degree $\nu$. Then

$$\sum_{\rho=0}^{r} (-1)^{\rho} Q(k - \rho, n) P_{\rho}(k - \rho)$$

QUASI-HAUSDORFF TRANSFORMATIONS

is a polynomial in $k, n$ of degree at most $\nu$.

Write

$$Q(k, n) = \sum_{\rho=0}^{r} n^\rho Q_\rho(k)$$

thus, for each $\mu$, $Q_\mu(k)$ is a polynomial of degree at most $\nu - \mu$. By Lemma 1, we can write

$$P_\rho(k - \rho) = \sum_{q=0}^{r} a_{q,\rho} k^q$$

where, for each $q$,

$$\sum_{\rho=0}^{r} (-1)^\rho a_{q,\rho} x^\rho$$

is divisible by $(1 - x)^q$. Hence, by Lemma 2

$$\sum_{\rho=0}^{r} (-1)^\rho a_{q,\rho} Q_\rho(k - \rho)$$

is a polynomial in $k$ of degree at most $\nu - \mu - q$. Multiplying by $k^q n^\mu$ and summing with respect to $q, \mu$, we obtain the conclusion.

**L E M M A 4.** Suppose that the $(C^*, 1)^r$ transformation is applicable to $s$; let the $(C^*, 1)^r$ transform be denoted by $\{t^{(r)}_n\}$. Then

$$s_n = \sum_{\rho=0}^{r} (-1)^\rho P^{(r)}_\rho(k) t^{(r)}_{k+\rho}$$

where, for each $\rho$, $P^{(r)}_\rho(k)$ is a polynomial in $k$ of degree $r$, and where

(i) For $\rho = 1, 2, \ldots, r$, $P^{(r)}_\rho(k)$ is divisible by $(k + 1)(k + 2) \cdots (k + \rho)$;

(ii) The coefficient of $k^q$ in

$$f^{(r)}(k, x) = \sum_{\rho=0}^{r} (-1)^\rho P^{(r)}_\rho(k)x^\rho$$

is divisible by $(1 - x)^q$.

Since the $(C^*, 1)$ transformation is defined by

$$t^{(1)}_n = (n + 1) \sum_{k=n}^{\infty} \frac{s_k}{(k + 1)(k + 2)}$$

it is clear that, whenever (13) converges,

$$s_k = (k + 2)t^{(1)}_k - (k + 1)t^{(1)}_{k+1}$$

thus the conclusion of the lemma holds when $r = 1$. Assume now that the result is true for $r - 1$ (where $r \geq 2$). Since
it follows that

\[ s_k = \sum_{\rho=0}^{r-1} (-1)^\rho P_\rho^{(r-1)}(k) [(k + \rho + 2) t_{k+\rho}^{(r)} - (k + \rho + 1) t_{k+\rho+1}^{(r)}] \]

\[ = \sum_{\rho=0}^{r} (-1)^\rho P_\rho^{(r)}(k) t_{k+\rho}^{(r)} , \]

where

\[ P_\rho^{(r)}(k) = (k + \rho + 2) P_\rho^{(r-1)}(k) + (k + \rho) P_{\rho-1}^{(r-1)}(k) . \]

Here we adopt the convention that \( P_\rho^{(r-1)}(k), P_{\rho-1}^{(r-1)}(k) \) are taken to mean 0.

It follows at once from (15) and the induction hypothesis that \( P_\rho^{(r)}(k) \) is a polynomial of degree \( r \), and that (i) holds. To prove (ii), we deduce from (15) that

\[ f^{(r)}(k, x) = x(1 - x) \frac{d}{dx} f^{(r-1)}(k, x) + k(1 - x) f^{(r-1)}(k, x) + (2 - x) f^{(r-1)}(k, x) , \]

and (ii) now follows from the induction hypothesis.

It may be remarked that the transformation (14), giving \( s \) in terms of \( \{t_k^{(1)}\} \), is the \((H^*, n + 2)\) transformation. The transformation (12) is obtained by \( r \) iterations of this and thus (since we are now considering row finite matrices) it is the \((H^*, (n + 2)^r)\) transformation. Hence

\[ P_\rho^{(r)}(k) = (-1)^\rho \binom{k + \rho}{k} A^r(k + 2)^r . \]

But this result does not appear to be of any help in proving (ii).

We now define \( S_n^{(r)} \) inductively by

\[ S_n^{(0)} = s_n; \quad S_n^{(r)} = S_n^{(r-1)} + S_{n-1}^{(r-1)} + \cdots + S_1^{(r-1)} (r \geq 1) . \]

As is familiar, this is equivalent to the definition

\[ S_n^{(r)} = \sum_{k=0}^{n} \binom{n - k + r - 1}{n - k} s_k . \]

**Lemma 5.** If \( \lambda > 0 \), and if

\[ \sum_{n=1}^{\infty} \frac{s_n}{n^\lambda} \]

converges, then

\[ \sum_{n=1}^{\infty} \frac{S_n^{(1)}}{n^{\lambda+1}} \]

converges.
QUASI-HAUSDORFF TRANSFORMATIONS

We take the hypothesis and conclusion in the equivalent forms that

\[ \sum_{n=0}^{\infty} \frac{s_n}{(n + \lambda) n}, \quad \sum_{n=0}^{\infty} \frac{S_n^{(1)}}{(n + \lambda + 1) n} \]

converge respectively. Write

\[ T_n = \sum_{n=0}^{\infty} \frac{s_n}{(\nu + \lambda) \nu}, \]

so that \( T_n \to 0 \) as \( n \to \infty \). Then

\[
\begin{align*}
\sum_{n=0}^{N} \frac{S_n^{(1)}}{(n + \lambda + 1) n} \\
= \sum_{n=0}^{N} \frac{1}{(n + \lambda + 1) n} \sum_{\nu=0}^{\infty} \left( \frac{\nu + \lambda}{\nu} \right) (T_\nu - T_{\nu+1}) \\
= \sum_{\nu=0}^{\infty} \left( \frac{\nu + \lambda}{\nu} \right) (T_\nu - T_{\nu+1}) \sum_{n=0}^{N} \frac{1}{(n + \lambda + 1) n} \\
= \frac{\lambda + 1}{\lambda} \left\{ \sum_{\nu=0}^{\infty} (T_\nu - T_{\nu+1}) - \frac{1}{(N + \lambda + 1) (N + 1)} \sum_{\nu=0}^{N} \left( \frac{\nu + \lambda}{\nu} \right) (T_\nu - T_{\nu+1}) \right\}.
\end{align*}
\]

Applying a straightforward partial summation to the second sum inside the curly brackets, we can now easily prove that this expression tends to a limit as \( N \to \infty \).

**COROLLARY.** If \( \rho \) is a positive integer, and if

(16) \[ \sum_{n=0}^{\infty} \frac{s_n}{n^2} \]

converges, then

\[ \sum_{n=0}^{\infty} \frac{S_n^{(\rho)}}{n^{2+\rho}} \]

converges.

3. We can now prove Theorem 1. Suppose first that \( s \) is summable \((C^*, 1)^*\); there is no loss of generality in supposing that it is summable
to 0, so that, with the notation of Lemma 4, \( t^{(r)}_n = o(1) \). It will be
enough to prove that \( s \) is summable \((C, r)\) to 0; in other words, that
\[ S^{(r)}_n = o(n^r). \]
For the applicability of \((C^*, 1)^r\), and thus, a fortiori, the \((C^*, 1)^r\)
summability of \( s \) requires, in particular, that \( t^{(1)}_n \) should be defined; and this is
equivalent to the convergence of (16). But it follows from [3, Theorem 3]
or [5, Theorem 4] that, if \( s \) is summable \((C, r)\), and if (16) converges,
then \( s \) is summable \((C^*, r)\).

Now, by Lemma 4, and with the notation used there,
\[ S^{(r)}_n = \sum_{\nu=0}^{s} \binom{n - \nu + r - 1}{n - \nu} s_{\nu} \]
\[ = \sum_{\nu=0}^{s} \binom{n - \nu + r - 1}{n - \nu} \sum_{\rho=0}^{r} (-1)^\rho P^{(r)}_\rho(\nu) t^{(r)}_{n+\rho} \]
\[ = \sum_{\rho=0}^{r} (-1)^\rho \sum_{\nu=0}^{s} \binom{n - \nu + r - 1}{n - \nu} P^{(r)}_\rho(\nu) t^{(r)}_{n+\rho} \]
\[ = \sum_{\rho=0}^{r} (-1)^\rho \sum_{k=0}^{n+\rho} \binom{n - k + \rho + r - 1}{n - k + \rho} P^{(r)}_\rho(k - \rho) t^{(r)}_k. \]
We may replace the lower limit of summation in the inner sum in (18)
by \( k = 0 \), since, by Lemma 4(i) \( P^{(r)}_\rho(k - \rho) \) vanishes for the extra terms.
Similarly, since the polynomial
\[ \left( \begin{array}{c} n - k + \rho + r - 1 \\ n - k + \rho \end{array} \right) \]
vanishes for \( k = n + \rho + 1, \ldots, n + r - 1 \), we may, except in the case
\( \rho = r \), replace the upper limit of summation in the inner sum by \( n + r - 1 \).
If we then invert the order of summation, we obtain
\[ S^{(r)}_n = \sum_{k=0}^{n+r-1} t^{(r)}_k \sum_{\rho=0}^{r} (-1)^\rho \left( \begin{array}{c} n - k + \rho + r - 1 \\ n - k + \rho \end{array} \right) P^{(r)}_\rho(k - \rho) \]
\[ + (-1)^r P^{(r)}_\rho(n) t^{(r)}_{n+\rho} = \sum_{k=0}^{n+r} \alpha^{(r)}_{n,k} t^{(r)}_k, \]
say. But since \( \left( \begin{array}{c} n - k + r - 1 \\ n - k \end{array} \right) \) is a polynomial in \( n, k \) of degree \( r - 1 \),
it follows from Lemmas 3, 4(ii) that, for \( 0 \leq k \leq n + r - 1 \), \( \alpha^{(r)}_{n,k} \) is a
polynomial in \( n, k \) of degree not exceeding \( r - 1 \). Further, \( \alpha^{(r)}_{n,n+r} \) is a
polynomial in \( n \) of degree \( r \); and, since \( t^{(r)}_k = o(1) \), (17) now follows, as
required.
We now consider the converse implication. Suppose, then, that \( s \) is summable \((C^*, r)\); we may again suppose that it is summable to 0. It follows that (16) converges; also, by [3, Theorem 4] or [5, Theorem 5], \( s \) is summable \((C, r)\), so that (17) holds. Now let \( R^{(\nu)}(n) \) denote a rational function of \( n \) (possibly different at each occurrence), the degree of the denominator exceeding that of the numerator by \( \nu \), and the denominator being a product of factors of the form \((n + p)\), with \( p \) a positive integer (repetitions being allowed). With this notation, we will prove that, for \( \rho = 1, 2, \cdots, r \), \( t_n^{(\rho)} \) exists, and that

\[
(19) \quad t_n^{(\rho)} = \sum_{\nu=0}^{\rho-1} S_n^{(\nu)} R^{(\nu)}(n) + o(1) .
\]

When \( \rho = r \), the sum in (19) is empty, so that (19) reduces to \( t_n^{(r)} = o(1) \). Thus, once (19) has been proved, the proof of the theorem will be completed. We prove (19) by an induction argument. Consider first the case \( \rho = 1 \). It follows by partial summation from the convergence of (16) that

\[
S_n^{(1)} = o(n^2) .
\]

Hence, for \( \nu \geq 1 \),

\[
(20) \quad S_n^{(\nu)} = o(n^{\nu+1}) .
\]

Using (20), we deduce from (13), by repeated partial summations, that

\[

t_n^{(1)} = (n + 1) \left\{ - \frac{S_{n-1}^{(1)}}{(n + 1)(n + 2)} + 2 \sum_{k=n}^{\infty} \frac{S_k^{(1)}}{(k + 1)(k + 2)(k + 3)} \right\} \\
= (n + 1) \left\{ - \sum_{\nu=1}^{\infty} \frac{S_{n-1}^{(\nu)}}{(n + 1)(n + 2) \cdots (n + \nu + 1)} \right\} \\
+ (r + 1)! \sum_{k=n}^{\infty} \frac{S_k^{(r)}}{(k + 1)(k + 2) \cdots (k + r + 2)} \\
= - \sum_{\nu=1}^{\infty} \frac{\nu! S_{n-1}^{(\nu)}}{(n + 2) \cdots (n + \nu + 1)} + o(1)
\]

since, when \( \nu = r \), we can replace (20) by the stronger result (17). Hence (19) holds when \( \rho = 1 \).

We now assume that (19) holds for \( \rho \), where \( 1 \leq \rho < r \), and prove that it holds for \( \rho + 1 \). By definition, \( \{t_n^{(\rho+1)}\} \) is the \((C^*, 1)\) transform of \( \{t_n^{(\rho)}\} \). The \((C^*, 1)\) transform of the term \( o(1) \) in (19) exists and is \( o(1) \), by the regularity of \((C^*, 1)\). It is therefore enough to consider the \((C^*, 1)\) transform of a typical term in the sum (19); that is to say, to consider

\[
(21) \quad (n + 1) \sum_{k=n}^{\infty} \frac{S_{k-1}^{(\nu)} R^{(\nu)}(k)}{(k + 1)(k + 2)} ,
\]
where \( \rho \leq \nu < r \). This series converges, by Lemma 5, Corollary. Also, by repeated partial summation, again using (20), the expression (21) is equal to

\[
(n + 1)\left\{ - \sum_{p=r+1}^{\nu} S_{n-p-1}^{(\nu)} A^{n-p} \left( \frac{R^{(\nu)}(n)}{(n+1)(n+2)} \right) + \sum_{k=n}^{\infty} S_{k-p}^{(\nu)} A^{k} \left( \frac{R^{(\nu)}(k)}{(k+1)(k+2)} \right) \right\}
\]

\[
= \sum_{p=r+1}^{\nu} S_{n-p-1}^{(\nu)} R^{(\nu)}(n) + o(1).
\]

Here, again, we use (17) to deal with the second sum, and the term \( \mu = r \) of the first sum, inside the curly brackets. Thus (19), if true for \( \rho \), is true for \( \rho + 1 \), and the proof of the theorem is completed.

4. In order to prove the remaining theorems, we require some further lemmas.

**Lemma 6.** Let \( r \) be a positive integer. Then

(i) For \( k \geq n \),

\[
A_{k-n} \left( \frac{1}{(n+2)^r} \right) = \frac{(k-n)! (n+1)!}{(k+2)!} K_{r}(n, k),
\]

where \( K_{r}(n, k) \) is defined by induction (on \( r \)) by

\[
K_{r}(n, k) = 1 ;
\]

\[
K_{r}(n, k) = \sum_{\nu=n}^{k} \frac{K_{r-1}(\nu, k)}{\nu + 2} \quad (r \geq 2).
\]

Alternatively, (23) may be replaced by

\[
K_{r}(n, k) = \sum_{\nu=n}^{k} \frac{K_{r-1}(n, \nu)}{\nu + 2} \quad (r \geq 2).
\]

(ii) For fixed \( n \),

\[
K_{r}(n, k) = \frac{(\log k)^{r-1}}{(r-1)!} + O((\log k)^{r-2})
\]

as \( k \to \infty \). Further,

\[
(\log k)^{-(r-1)} K_{r}(n, k)
\]

is of bounded variation in \( k \geq n \).

The result that (22) holds is familiar, and easily verified, when \( r = 1 \). Assume the result true for \( r - 1 \), where \( r \geq 2 \). Applying the familiar formula

\[
A^{q}(a_{n} b_{n}) = \sum_{p=0}^{q} \binom{q}{p} A^{p} a_{n} A^{q-p} b_{n+p}
\]
with
\[ a_n = \frac{1}{(n+2)} , \quad b_n = \frac{1}{(n+2)^{r-1}} , \quad q = k - n , \]
we obtain
\[ (27) \quad A^{k-n} \left( \frac{1}{(n+2)^r} \right) = \sum_{\nu=0}^{k-n} \binom{k-n}{\nu} \frac{\nu!}{(n+2)(n+3) \cdots (n+\nu+2)} \times \frac{(k-n-\nu)!}{(n+\nu+2) \cdots (k+2)} K_{r-1}(n+\nu, k) \]
\[ = \frac{(k-n)! (n+1)! k_{\nu-1} K_{r-1}(n+\nu, k)}{(k+2)!} \sum_{\nu=0}^{k-n} \frac{1}{n+\nu+2} . \]

On changing the notation by replacing \((n+\nu)\) by \(\nu\) in the sum in (27), we see that (22) holds for \(r\), with \(K_r(n, k)\) given by (23).

If we had applied (26) with
\[ a_n = \frac{1}{(n+2)^{r-1}} , \quad b_n = \frac{1}{n+2} , \]
a similar argument would have yielded (24). We remark that it may be verified directly that the two induction definitions are equivalent; for either gives, for \(r \geq 2\),
\[ K_r(n, k) = \sum \frac{1}{(\nu_1+2)(\nu_2+2) \cdots (\nu_{r-1}+2)} , \]
the sum being taken over all \(\nu_1, \nu_2, \ldots, \nu_{r-1}\) for which
\[ n \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_{r-1} \leq k . \]

Once (i) has been proved, (25) follows at once by induction on \(r\) (using (24)). Further, again using (24), we have, for \(r \geq 2\)
\[ A((\log k)^{-(r-1)} K_r(n, k)) \]
\[ = (\log (k+1))^{-(r-1)} A K_r(n, k) + K_r(n, k) A((\log k)^{-(r-1)}) \]
\[ = -(\log (k+1))^{-(r-1)} K_{r-1}(n, k+1) + \frac{(r-1)}{k} K_r(n, k)(\log k)^{-r}(1 + O\left(\frac{1}{k}\right)) \]
\[ = O\left(\frac{1}{k \log^2 k}\right) , \]
by (25). The result follows.

**Lemma 7.** For fixed \(n > 0\),
\[ \frac{K_r(n, k)}{K_r(0, k)} \]
is a non-decreasing function of $k$ for $k \geq n$.

The proof is by induction. The result is trivial when $r = 1$. Assume the result true for $r - 1$, where $r \geq 2$. Then, by (24),

$$\frac{K_r(n, k)}{K_r(0, k)} - \frac{K_r(n, k + 1)}{K_r(0, k + 1)} = \frac{L_r(n, k)}{K_r(0, k)K_r(0, k + 1)},$$

where

$$L_r(n, k) = \sum_{\nu=0}^{k+1} \frac{K_r(0, \nu)}{\nu + 2} - \sum_{\nu=0}^{k+1} \frac{K_r(n, \nu)}{\nu + 2} = \frac{1}{k + 3} \left\{ \frac{K_{r-1}(0, k + 1)}{\nu + 2} - \frac{K_{r-1}(n, k + 1)}{\nu + 2} \right\}. $$

But, by the induction hypothesis, we have

$$K_{r-1}(0, k + 1)K_{r-1}(n, \nu) \leq K_{r-1}(n, k + 1)K_{r-1}(0, \nu)$$

for $n \leq \nu \leq k$. Hence

$$K_{r-1}(0, k + 1) \sum_{\nu=0}^{k} \frac{K_{r-1}(n, \nu)}{\nu + 2} \leq K_{r-1}(n, k + 1) \sum_{\nu=0}^{k} \frac{K_{r-1}(0, \nu)}{\nu + 2} < K_{r-1}(n, k + 1) \sum_{\nu=0}^{k} \frac{K_{r-1}(0, \nu)}{\nu + 2}. $$

Thus $L_r(n, k) < 0$, which gives the conclusion.

We now note that, if the $(H^*, r)$ transform of $s$ is denoted by $\{h_n^{(r)}\}$, then it follows from (22) that $h_n^{(r)}$ is defined by

$$h_n^{(r)} = (n + 1)^r \sum_{k=n}^{\infty} \frac{K_r(n, k)}{(k + 1)(k + 2)^{r+1}}$$

whenever this series converges. Further, it follows from Lemma 6 (ii) that, if (28) converges for one value of $n$, then it converges for all $n$, and that a necessary and sufficient condition for this to happen is that

$$\sum_{k=n}^{\infty} (\log k)^{r-1} < \infty$$

should converge.

**Lemma 8.** If the $(H^*, r)$ transformation is applicable to $s$, then the $(C^*, 1)^r$ transformation is also applicable to $s$, and the $(C^*, 1)^r$ transform is equal to the $(H^*, r)$ transform.

We again prove the result by induction. The result is trivial when $r = 1$, since, in this case, the definitions of $(H^*, r), (C^*, 1)^r$ are the same.
Suppose, then, the result true for \( r - 1 \), where \( r \geq 2 \). Suppose the \((H^*, r)\) transformation is applicable. Then (29) converges; and hence the corresponding series with \( r \) replaced by \( r - 1 \) also converges, so that \((H^*, r - 1)\) is also applicable. By (23) and (28),

\[
(30) \quad h_{n}^{(r-1)} = (n + 1) \sum_{k=n}^{\infty} \frac{K_{r-1}(n, k)}{(k + 1)(k + 2)} s_k
\]

\[
= (n + 1)(n + 2) \sum_{k=n}^{\infty} \frac{(K_r(n, k) - K_r(n + 1, k))}{(k + 1)(k + 2)} s_k
\]

\[
= (n + 2) h_{n}^{(r)} - (n + 1) h_{n}^{(r-1)}.
\]

But, in view of Lemma 7, it follows easily from the convergence of (28) with \( n = 0 \) that

\[
h_{n}^{(r)} = o(n).
\]

We therefore deduce from (30) that

\[
(31) \quad h_{n}^{(r)} = (n + 1) \sum_{k=n}^{\infty} \frac{h_{k}^{(r-1)}}{(k + 1)(k + 2)}.
\]

By the induction hypothesis, and with the notation used in the proof of Theorem 1, \( t_{k}^{(r-1)} \) exists and equals \( h_{k}^{(r-1)} \). Hence, by (31) and the definition of \( t_{k}^{(r)} \), \( t_{n}^{(r)} \) exists and equals \( h_{n}^{(r)} \).

5. The positive part of Theorem 2 follows at once from Theorem 1 and Lemma 8. In order to prove the negative part of Theorem 2, and also of Theorem 3, we consider the example

\[
s_{k} = \begin{cases}
t^{-1}2^{2t} & (k = 2^t, t = 1, 2, \cdots) ; \\
-t^{-1}2^{2t} & (k = 2^t + 1, t = 1, 2, \cdots) ; \\
0 & \text{(otherwise)}.
\end{cases}
\]

where \( \lambda > 0 \). Then

\[
S_{k}^{(1)} = \begin{cases}
t^{-1}2^{2t} & (k = 2^t, t = 1, 2, \cdots) ; \\
0 & \text{(otherwise)}.
\end{cases}
\]

Since

\[
\sum_{t=1}^{r} t^{-1}2^{2t} = O(T^{-1}2^{2T}),
\]

we see that \( S_{k}^{(2)} = o(k^2) \), so that \( s \) is summable \((C, 2)\) to 0. The series (29) diverges if \( r \geq \lambda + 1 \), since the general term does not tend to 0; and it is easily proved that it converges if \( r < \lambda + 1 \). In particular, (29) converges when \( r = 1 \); in other words, (16) converges, so that \((C^*, r)\) is
applicable (for any $r$). Thus, by [3, Theorem 3] or [5, Theorem 4], $s$ is summable $(C^*, r)$ for $r \geq 2$. But, if $r \geq 2$ and we choose $\lambda \leq r - 1$, $(H^*, r)$ is not applicable. Further, if $2 \leq r < r_1$, we may choose $\lambda$ so that $r - 1 < \lambda \leq r_1 - 1$. Then $(H^*, r)$ is applicable, so that, since $s$ is summable $(C^*, r)$, it is summable also $(H^*, r)$; but $(H^*, r_1)$ is not applicable.

It remains only to consider the case $r = 1$ of Theorem 3. Summability $(H^*, 1)$ is the same as $(C^*, 1)$, and this is known to be equivalent to $(C, 1)$. It follows, a fortiori that if $s$ is summable $(H^*, 1)$ then the $(C, 1)$ means are bounded; that is to say

$S_k^{(1)} = O(k)$.

The convergence of (29) (with $r$ replaced by $r_1$) follows at once by partial summation; indeed, a weaker result that (32) would suffice for this. This gives the conclusion.

REFERENCES


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