A NOTE ON THE VON NEUMANN ALGEBRA WITH A CYCLIC AND SEPARATING VECTOR

YASUHIDE MIURA

(Received November 11, 1975)

Let $M$ be a von Neumann algebra on a Hilbert space $H$ with a cyclic and separating vector. If, for some cyclic and separating vector $\xi_0$ in $H$ for $M$,

$$M\xi_0 = M^'*\xi_0,$$

(*)

then we shall call that $M$ has the property (J).

Note that $M$ satisfying the equality (*) for $\xi_0$ does not always imply for the other cyclic and separating vector.

In [Theorem 1] we show that $M$ has the property (J) if and only if $M$ is a finite von Neumann algebra. In terms of the Hilbert algebra, we can consider $M\xi_0$ as an achieved left Hilbert algebra $\mathcal{A}$ with the product: $(x\xi_0)(y\xi_0) = xy\xi_0$, and the involution: $S(x\xi_0) = x^*\xi_0$, $x, y \in M$, and $M^'*\xi_0$ as the right Hilbert algebra $\mathcal{A}'$ of $\mathcal{A}$. (see [2], [3]) The analysis in this paper may be the special case of the characterization of the type of the left von Neumann algebra $\mathcal{A}(\mathcal{H})$ associated to the achieved left Hilbert algebra $\mathcal{A}$ under the condition $\mathcal{A} = \mathcal{A}'$ as a set. Without difficulty, we can prove that an achieved left Hilbert algebra $\mathcal{A}$ is equal to $\mathcal{A}'$ as a set if and only if $\mathcal{A}$ is a Tomita algebra.

In [Theorem 2] we shall give a characterization of a finite von Neumann algebra via the Radon-Nikodym theorem for the state. We mainly refer [1] and [2].

Now, we state here that if $M$ is finite, then $M$ has the property (J).

In fact, let $\xi_0$ be a cyclic and separating trace vector in $H$ for $M$. Then we have

$$||S(x\xi_0)||^2 = ||x^*\xi_0||^2 = (xx^*\xi_0|\xi_0) = (x^*x\xi_0|\xi_0) = ||x\xi_0||^2,$$

for all $x$ in $M$. Therefore $M\xi_0$ is a uni-modular Hilbert algebra. From [2] Cor. 10.1, we have $M\xi_0 = M^'*\xi_0$.

Now we need the following lemma to prove [Theorem 1].

**Lemma (cf. [1] Chap. I §1 ex. 5).** Suppose that $M$ is a von Neumann algebra on a Hilbert space $H$ such that $M\xi_0 = M^'*\xi_0$ for a cyclic and separating vector $\xi_0$ in $H$, that is, for any element $x$ in $M$, there exists
a unique element $x'$ in $M'$ such that $x \xi = x' \xi_0$. Then the mapping $\Phi: x \mapsto x'$ is a norm bi-continuous anti-isomorphism of $M$ onto $M'$.

**Proof.** It is clear that $\Phi$ is anti-isomorphic. Let $\{x_n\}$ be a sequence in $M$ such that $x_n \to x$ and $\Phi(x_n) \to y'$, $x \in M$, $y' \in M'$. Then $x_n \xi_0 \to x_0 \xi_0$ and $\Phi(x_n) \xi_0 \to y' \xi_0$. Thus we have $x_0 \xi_0 = y' \xi_0$, i.e., $\Phi(x) = y'$. Therefore $\Phi$ is norm continuous by the closed graph theorem. We see the continuity of $\Phi^{-1}$ from the symmetrical argument. q.e.d.

**Theorem 1.** Let $M$ be a von Neumann algebra on a Hilbert space $H$ with a cyclic and separating vector. Then $M$ is finite if and only if $M$ has the property $(J)$.

**Proof.** We must prove that if $M$ is not finite, then $M$ does not have the property $(J)$. As any von Neumann algebra is uniquely decomposed into direct sum of a finite and a properly infinite algebra, we may assume that $M$ is properly infinite. Then $M$ is spatially isomorphic to $M \otimes \mathcal{B}(K)$ where $\mathcal{B}(K)$ is the algebra of all bounded operators on an infinite dimensional separable Hilbert space $K$. If $M$ has a cyclic and separating vector, then $M \otimes \mathcal{B}(K)$ has also a cyclic and separating vector. We see that $M$ has not the property $(J)$ if and only if $M \otimes \mathcal{B}(K)$ has not the property $(J)$.

In fact, we assume that $M_\xi \neq M'_\xi$ for any cyclic and separating vector $\xi$ in $H$ for $M$. For any cyclic and separating vector $\eta$ in $H \otimes K$ for $M \otimes \mathcal{B}(K)$, there exists a cyclic and separating vector $\xi$ in $H$ for $M$ such that $\eta = U_\xi$ where $U$ is an isometry of $H$ onto $H \otimes K$ with $UMU^{-1} = M \otimes \mathcal{B}(K)$. Then,

$$(M \otimes \mathcal{B}(K))\eta = (UMU^{-1})U_\xi = UM_\xi \neq UM'_\xi$$

$$= UM'U^{-1}U_\xi = (UMU^{-1})'U_\xi = (M \otimes \mathcal{B}(K))'\eta .$$

Now, we will prove that $M \otimes \mathcal{B}(K)$ has not the property $(J)$. Suppose that

$$(M \otimes \mathcal{B}(K))\eta = (M \otimes \mathcal{B}(K))'\eta ,$$

for some cyclic and separating vector $\eta$ in $H \otimes K$. Let $\eta$ be a form $\sum_{i=1}^n \xi_i \otimes \varepsilon_i$ where $\xi_i \in H$, and $\{\varepsilon_i\}$ is a completely orthonormal system in $K$. Let $v_j$, $j = 1, 2, \ldots$, be partial isometries in $\mathcal{B}(K)$ such that $v_j \varepsilon_i = \varepsilon_i$, $v_j \varepsilon_i = 0$ ($i = 2, 3, \ldots$). Then there exists an element $y_j$ in $M'$ for each $j$ such that

$$(1 \otimes v_j)\eta = (y_j \otimes 1)\eta$$

because of $(M \otimes \mathcal{B}(K))' = M' \otimes I_K$. We have, for each $j$,
VON NEUMANN ALGEBRA WITH A CYCLIC AND SEPARATING VECTOR

\[(1 \otimes v_j)\gamma = (1 \otimes v_j)(\sum_{i=1}^{\infty} \xi_i \otimes \varepsilon_i) = \xi_j \otimes \varepsilon_j,\]

and,

\[(y_j \otimes 1)\gamma = (y_j \otimes 1)(\sum_{i=1}^{\infty} \xi_i \otimes \varepsilon_i) = \sum_{i=1}^{\infty} y_j \xi_i \otimes \varepsilon_i.\]

Hence we have

\[\xi_j \otimes \varepsilon_j = (y_j \otimes 1)(\xi_j \otimes \varepsilon_j),\]

and,

\[||\xi_i|| = ||\xi_i \otimes \varepsilon_j|| \leq ||y_j \otimes 1|| ||\xi_j \otimes \varepsilon_j|| = ||y_j \otimes 1|| ||\xi_j||,\]

for each \(j = 1, 2, \ldots\). Then we have \(\xi_i = 0\), because the sequence \(\{\xi_j\}\) is convergent to 0 and \(\{y_j \otimes 1\}\) is bounded from the lemma.

Applying this argument to the other elements \(\xi_i, i = 2, 3, \ldots\), we obtain \(\xi_i = 0\) for each \(i\). Thus we have \(\gamma = 0\). Therefore,

\[(M \otimes \mathcal{B}(K))\gamma = (M \otimes \mathcal{B}(K))'\gamma,\]

for any cyclic and separating vector \(\gamma\) in \(H \otimes K\). This completes the proof, q.e.d.

Next, we state the following theorem.

**THEOREM 2.** Let \(M\) be a von Neumann algebra on a Hilbert space \(H\) with a cyclic and separating vector. Then the following statements are equivalent;

i) \(M\) is finite.

ii) We can find a cyclic and separating vector \(\xi_0\) in \(H\) satisfying the following condition:

For any element \(a\) in \(M\), there exists a positive number \(r\) such that

\[aw_{\xi_0}a^* \leq r\omega_{\xi_0},\]

where \(\omega_{\xi_0}\) is a vector state on \(M\) for \(\xi_0\).

**PROOF.** We immediately see that i) implies ii). In fact, if \(M\) is finite, then there exists a cyclic and separating vector \(\xi_0\) such that \(M\xi_0 = M'\xi_0\) from [Theorem 1]. Then, if \(a\xi_0 = a'\xi_0, a \in M, a' \in M'\), then we have

\[(a^*xa^*x_0|\xi_0) = ||xa\xi_0||^2 = ||xa'\xi_0||^2 = ||a'x\xi_0||^2 \leq ||a'||^2||x\xi_0||^2 = ||a'||^2(x^*x_0|\xi_0),\]

for all \(x\) in \(M\).

Conversely, suppose that we choose the element \(\xi_0\) satisfying the condition in ii). Then, for each element \(a\) in \(M\), there exists a positive element \(\delta\) in \(M'\) such that

\[\omega_{a\xi_0} = \omega_{\xi_0}.\]
Then we have $||x h' \xi_0|| = ||xa\xi_0||$ for all $x$ in $M$. Put $u_o(xh'\xi_0) = xo\xi_0$, $x \in M$, then $u_o$ can be extended to a partial isometry $u'$ in $M'$. Therefore, $a\xi_0 = u' h' \xi_0$, that is, an element $a\xi_0$ falls in $M' \xi_0$, i.e., $M\xi_0 \subset M' \xi_0$. Then we have

$$M' \xi_0 = JM\xi_0 \subset JM' \xi_0 = M\xi_0,$$

where $J$ is a modular conjugation operator of an achieved left Hilbert algebra $M\xi_0$. (see [2] Cor. 10.1) Hence we have $M\xi_0 = M' \xi_0$, and then $M$ is finite from [Theorem 1]. q.e.d.

Finally, the author would like to express his gratitude to Prof. M. Fukamiya and Prof. J. Tomiyama for their useful suggestions.

**REFERENCES**

