AW*-ALGEBRAS WITH MONOTONE CONVERGENCE PROPERTY
AND EXAMPLES BY TAKENOUCHI AND DYER

KAZUYUKI SAITO

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In 1951, I. Kaplansky [6] introduced a class of C*-algebras called
AW*-algebras to separate the discussion of the internal structure of a
W*-algebra (or von Neumann algebra) from the action of its elements on
a Hilbert space and showed that much of the “non-spatial theory” of
W*-algebras can be extended to AW*-algebras.

Every W*-algebra is of course AW*, however, the converse is not
true as was shown by Dixmier [3] with an abelian example (the algebra
of all bounded Baire functions on the real line modulo the set of first
category is a non-W*, AW*-algebra). I. Kaplansky [7] proved that an
AW*-algebra of type I is a W*-algebra if and only if its center is a
W*-algebra and conjectured that the theorem is true without the assump-
tion of “type I”. In 1970, O. Takenouchi [12] and Dyer [2], independent-
ly, showed this to be false by counter examples (non-W*, AW*-factors).
In 1976, J. D. Maitland Wright [16, 13] defined a regular σ-completion
(some kind of Dedekind cut completion) of a separable C*-algebra and
proved that the regular σ-completion of an infinite dimensional simple
separable C*-algebra is a type III, non-W*, σ-finite AW*-factor with the
monotone convergence property (see the definition below).

In this paper, the author will give a modification of a J. D. M.
Wright’s theorem and using this, will show that the non-W*, AW*-factors
given by Takenouchi and Dyer are σ-finite, type III AW*-factors. The
key point of the proof is, roughly speaking, to construct a faithful
state on these factors. To do this, a J. D. Maitland Wright’s theorem
plays an essential role. He states that the pure state space of the regular
σ-completion C[0, 1] (which is essentially the same as ∅ in section 1) of
the C*-algebra C[0, 1] of continuous complex functions on [0, 1] is separa-
table ([18, p. 85]).

The AW*-factor given by Takenouchi is a “weakly closed” (in the sense
of [13]) AW*-subalgebra of type I AW*-algebra ∅(M) of all bounded
module endomorphisms of some AW*-module M over an abelian AW*-algebra. The author believes that it is natural to represent AW*-
algebras as “weakly closed” $AW^*$-subalgebra of some $B(\mathcal{M})$. The author then will show that the $AW^*$-factor constructed by Dyer can be represented faithfully as a “weakly closed” $AW^*$-subalgebra of some $B(\mathcal{M})$. Moreover, we shall remark that these factors are monotone closed (in the sense of [5]), simple and do not have any non-trivial separable representations.

1. $AW^*$-algebras with a monotone convergence property (M. C. P.).

An $AW^*$-algebra $M$ means that it is both a $C^*$-algebra and a Baer*-ring ([1], [6]). $M$ has a monotone convergence property (M. C. P.) if for every increasing sequence $\{x_n\}$ of self-adjoint elements in $M$ bounded above has the supremum $x$ in the self-adjoint part of $M$ (we simply denote $x_n \uparrow x$ or $\text{Sup}_n x_n = x$).

First of all, we shall show the following technical lemma.

**Lemma.** Let $M$ be an $AW^*$-algebra with M. C. P. For every increasing sequence $\{e_n\}$ of projections in $M$, let $\bigvee_{n=1}^{\infty} e_n$ be the supremum projection of $\{e_n\}$ in the projection of $M$. Then $\text{Sup}_n e_n = \bigvee_{n=1}^{\infty} e_n$.

Moreover, for any $a \in M$,

$$\text{Sup}_n a^* e_n a = a^* \left( \bigvee_{n=1}^{\infty} e_n \right) a.$$ 

**Proof.** Put $b = \text{Sup}_n e_n$, then $0 \leq e_n \leq b \leq \bigvee_{n=1}^{\infty} e_n$ in $M$ for each $n$. Thus $e_n \leq \text{LP} (b) \leq \bigvee_{n=1}^{\infty} e_n$ for all $n$ and hence $\text{LP} (b) = \bigvee_{n=1}^{\infty} e_n$ where $\text{LP} (b)$ is the left projection of $b$ in $M$ ([1, 6]). On the other hand, $e_n = b e_n$ for every $n$ implies by [6, Lemma 2.2] that $\text{LP} (b) = b \text{LP} (b) = b$ and $\text{Sup}_n e_n = \bigvee_{n=1}^{\infty} e_n$.

Now arguments used in [5] tells us that for any $a \in M$, $a^* e_n a \uparrow a^* \left( \bigvee_{n=1}^{\infty} e_n \right) a$ in $M$. This completes the proof.

Using this, we have the following theorem which is a modification of a J. D. M. Wright’s result ([17, Theorem 6]).

**Theorem 1.** Let $M$ be an $AW^*$-factor with M. C. P. Suppose that $M$ has a faithful state (not necessarily normal) $\phi$ and is semi-finite, then $M$ is a $\sigma$-finite $W^*$-algebra. The assumption of semi-finiteness cannot be dropped.

**Remark.** Maitland Wright proved, without the assumption of M. C. P., however under the condition that $M$ is finite, that the above proposition holds.

**Proof of Theorem 1.** For any non-zero finite projection $e$ (note that $M$ is semi-finite), put $N = e M e$, then $N$ is a finite $AW^*$-factor with the faithful
positive functional ψ where ψ(exe) = φ(exe) for x ∈ M. J. D. M. Wright's theorem [17, Theorem 6] tells us that N is a W*-algebra, that is, there exists a faithful W*-representation πe of N on some Hilbert space ℋπe (πe(N) is a weakly closed *-subalgebra with the identity of ℋ(ℋπe)) (the algebra of all bounded linear operators on ℋπe)). Next we shall show that for any ξ ∈ ℋπe, the positive functional φ(e, ξ) on M (where φ(e, ξ)(x) = (πe(exe)ξ, ξ), x ∈ M) is completely additive on projections. To prove this we have only to show that for any decreasing sequence {en} of projections in M with en ↓ 0, φ(e, ξ)(en) ↓ 0 (n → ∞), because M is σ-finite (note that M has a faithful state). Let {en} be any decreasing sequence of projections in M with en ↓ 0, then by the above lemma, Inf en ene = 0 in the self-adjoint part of N. Since {πe(eene)} is a decreasing sequence in the non-negative portion of ℋ(ℋπe), there is A ∈ ℋ(ℋπe) such that πe(eene) ↓ A (strongly). The strong closeness of πe(N) implies A ∈ πe(N). Hence there is a ∈ N(a ≥ 0) such that A = πe(a). The faithfulness of πe implies that a = 0, that is, πe(eene) ↓ 0 strongly. Thus φ(e, ξ) is completely additive on projections of M. The semi-finiteness of M tells us that {φ(e, ξ); e is any non-zero finite projection, e ∈ ℋπe} is a separating family of positive functionals on M which are completely additive on projections of M. Hence by ([10], Theorem 5.2, see also [9]) M is a semi-finite W*-algebra. Non W*, A W*-factors constructed by Takenouchi and Dyer have the M. C. P. and faithful states (see the next section), thus the assumption of semi-finiteness cannot be dropped. This completes the proof of Theorem 1.

Remark. In the above proof, we suppose that M has the M. C. P., however, Theorem 1 still holds under a nominally weaker assumption such that for any increasing sequence {en} of projections in M and for any projection e in M, Sup en ene exists in the self-adjoint part of M and Sup en ene = e(V n−1 en)e.

The above theorem implies that if non-W*, A W*-factor with M. C. P. has a faithful state, then it is of type III ([6, p. 241 Definition]).

In the rest of this section, we treat with examples of abelian A W*-algebras with groups of *-automorphisms of them which are needed in the later sections.

Let B°[0, 1) be the algebra of all bounded Baire functions on [0, 1) and let A be the algebra B°[0, 1) modulo the set of first category. Then one can easily check that A is a non-W*, abelian A W*-algebra which is *-isomorphic with the regular σ-completion of a separable abelian C*-algebra ([2], [18], p. 86). J. D. Maitland Wright proved also that A has
a faithful state because the pure state space of $\mathcal{A}$ is separable [18, Proposition A, Corollary D].

Let $G_\theta$ (resp. $G_\theta$) be the group of translations on $[0,1)$ by an irrational number $\theta (\operatorname{mod} 1)$ (resp. by all dyadic rationals in $[0,1)$ (mod 1)). Denote for each $\sigma \in G_\theta$ (resp. $G_\theta$), $\sigma(t) = t + \sigma (\operatorname{mod} 1)$, $f^\sigma(t) = f(\sigma(t))$ for all $t \in [0,1)$, $f \in B^\omega [0,1)$ and $a^\sigma = f^\sigma$ where $f$ belongs to a coset $a(a = f)$, $f \in B^\omega [0,1)$ for all $a \in \mathcal{A}$. Then both $G_\theta$ and $G_\theta$ naturally induce groups of $*$-automorphisms $(a \to a^\sigma, a \in \mathcal{A})$ of $\mathcal{A}$ (we denote them by the same notations $G_\theta$ and $G_\theta$ since any confusion does not occur). It is easy to check that $G_\theta$ and $G_\theta$ act freely and ergodically on $\mathcal{A}$.

2. Types of the $AW^*$-factors constructed by Takenouchi. First, we shall sketch briefly the construction of $AW^*$-factors of [12]. Let $Z$ be an abelian $AW^*$-algebra, $G$ be an abelian group of $*$-automorphisms of $Z$ with an action $a \to a^g (a \in Z, g \in G)$. One can construct a faithful $AW^*$-module ([8]) $\mathcal{M}$ over $Z$ as follows: Let $\mathcal{M}$ be the set $l^\omega (G, Z)$ of all sequences $\{x_g\}$ of elements in $Z$ with the indices $g \in G$ such that $\sum_{g \in G} x_g^* x_g$ is in $Z$ (the supremum of the family of finite sums). Then $\mathcal{M}$ is a faithful $AW^*$-module over $Z$ and the set $\mathcal{B}(\mathcal{M})$ of all bounded module endomorphisms (we simply call them “operators”) of $\mathcal{M}$ is a type I $AW^*$-algebra with center $Z$.

Define, for any $a \in Z$ and $g \in G$, the following types of “operators” on $\mathcal{M}$:

$$L_a: \{x_g\} \to \{a^g x_g\} \quad U_g: \{x_g\} \to \{y_g\} \quad \text{where} \quad y_g = x_{g \cdot h}$$

Then one can easily show that $a \to L_a$ is a $*$-isomorphism of $Z$ into $\mathcal{B}(\mathcal{M})$ and $h \to U_h$ is a unitary representation of $G$ into $\mathcal{B}(\mathcal{M})$ such that $U_h^* L_a U_h = L_{a \cdot h}$ for all $a \in Z$ and $h \in G$.

Next, for any $h \in G$, we introduce the following linear operator (note that this is not a module endomorphism of $\mathcal{M}$) on $\mathcal{M}$:

$$V_h: \{x_g\} \to \{y_g\} \quad \text{where} \quad y_g = (x_{g \cdot h})^{-h}$$

For every “operator” on $\mathcal{M}$ has a matrix representation $A \sim \langle a_{g,h} \rangle$ where $a_{g,h} = (A u_{g,h}, v_{g,h}) (g, h \in G)$ (where $u_{g,h} = \{\delta_{g,h}\}$ ($h \in G$) and $\delta_{g,h}$ is the Kronecker’s delta).

Let $\mathcal{M}(Z, G) = \{A \in \mathcal{B}(\mathcal{M}); A \sim \langle a_{g,h} \rangle \text{ where } a_{g,h} = (a_{g \cdot h,e})^h \text{ for any pair } g, h \in G(e \text{ is a unit of } G)\}$, then $A \in \mathcal{M}(Z, G)$ if and only if $AV_h = V_h A$ for all $h \in G$ and $\mathcal{M}(Z, G)$ is an $AW^*$-subalgebra of $\mathcal{B}(\mathcal{M})$ which contains all $U_h$ and $L_a$, where an $AW^*$-subalgebra means that the structure of an $AW^*$-algebra of $\mathcal{M}(Z, G)$ is compatible with that of $\mathcal{B}(\mathcal{M})$ in the
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sense of [1, 7].

Takenouchi showed under the condition that the action of $G$ on $Z$ is free and ergodic, $M(Z, G)$ is an $AW^*$-factor such that $\{L_a; a \in Z\} = \hat{Z}$ is a maximal abelian $*$-subalgebra (whose proof is analogous to that of Murray-von Neumann's) and gave an example of $(Z, G)$ as $(\mathcal{U}, G_\theta)$ in section 1. If $M(\mathcal{U}, G_\theta)$ is a $W^*$-algebra, then $\hat{Z}$ ($*$-isomorphic with $\mathcal{U}$) is a $W^*$-algebra. This is a contradiction and hence $M(\mathcal{U}, G_\theta)$ is a non-$W^*$, $AW^*$-factor.

The rest of this section is devoted to prove

THEOREM 2. $M(\mathcal{U}, G_\theta)$ is a $\sigma$-finite, type III, non-$W^*$, $AW^*$-factor with M. C. P. (more precisely, $M(\mathcal{U}, G_\theta)$ is “weakly closed” $*$-subalgebra of $\mathcal{B}(\mathcal{U})$ ($\mathcal{U} = l^2(G_\theta, \mathcal{U})$) in the sense of H. Widom [13], and that, it is monotone closed in the sense that in its self-adjoint part, every norm-bounded increasing net has a least upper bound).

PROOF. First of all, we shall show that $M(\mathcal{U}, G_\theta)$ is “weakly closed” subalgebra of $\mathcal{B}(\mathcal{U})$ where $\mathcal{U} = l^2(G_\theta, \mathcal{U})$ in the sense that for any net $\{A_n\}$ in $M(\mathcal{U}, G_\theta)$ such that $(A_n \xi, \gamma) \to (A \xi, \gamma)$ (order convergence in $\mathcal{U}$) [13], for some $A \in \mathcal{B}(\mathcal{U})$, $A \in M(\mathcal{U}, G_\theta)$. In fact, putting $A_n \sim \langle a^*_g, h \rangle A \sim \langle a^*_g, h \rangle$, then $a^*_g, h \to a_g, h$ (order convergence in $\mathcal{U}$) for each pair $g$ and $h$ in $G_\theta$. Thus $a_g, h = \langle a_{g, h} \rangle^h$ for $g, h \in G_\theta$ and $A \in M(\mathcal{U}, G_\theta)$. In particular, $M(\mathcal{U}, G_\theta)$ has M. C. P. In fact, let $\{A_n\}$ be an increasing sequence of self-adjoint elements of $M(\mathcal{U}, G_\theta)$ bounded above by $B \in M(\mathcal{U}, G_\theta)$, then $A_n \uparrow A$ “weakly” for some $A$ (where $A$ is the supremum of $\{A_n\}$ in the self-adjoint part of $\mathcal{B}(\mathcal{U})$), in $\mathcal{B}(\mathcal{U})$ ([13, Lemma 1.4]). It follows by the above argument that $A \in M(\mathcal{U}, G_\theta)$ and $A \leq B$ and hence $M(\mathcal{U}, G_\theta)$ has M. C. P. By the same way, we can easily show that $M(\mathcal{U}, G_\theta)$ is monotone closed.

Next, we shall show that $M(\mathcal{U}, G_\theta)$ has a faithful positive projection map onto $\mathcal{U} = \{L_a; a \in \mathcal{U}\}$. In fact, for any $A \in M(\mathcal{U}, G_\theta)$, let $\phi(A) = L_{a_g, h}$ where $A \sim \langle a_{g, h} \rangle$, then one can easily check that $\phi$ is a positive projection map of $M(\mathcal{U}, G_\theta)$ onto $\mathcal{U}$. To prove the faithfulness of $\phi$, we argue as follows. For any $A \in M(\mathcal{U}, G_\theta)$ with $A \sim \langle a_{g, h} \rangle$, noting that, $(A^* A)_{s, s} = \sum_{g, h} a_{g, s}^* a_{g, s}$, we have $\phi(A^* A) = 0$ implies $a_{g, s} = 0$ for all $g \in G$ and hence $A = 0$ because $a_{g, h} = (a_{g, h})_s = 0$ for all $g, h \in G$.

Let $\psi$ be a faithful state on $\mathcal{U}$ in section 1, and let $\phi = \psi \circ \phi$, then $\phi$ is a faithful state on $M(\mathcal{U}, G_\theta)$. Assume that $M(\mathcal{U}, G_\theta)$ is semi-finite, then by Theorem 1, $M(\mathcal{U}, G_\theta)$ is a $W^*$-algebra, however this is a contradiction because $M(\mathcal{U}, G_\theta)$ is non-$W^*$. Hence $M(\mathcal{U}, G_\theta)$ is of type III. Since $M(\mathcal{U}, G_\theta)$ has a faithful state $\phi$, we can easily show that $M(\mathcal{U}, G_\theta)$ is $\sigma$-finite. This completes the proof.
3. **Dyer's example.** In this section, we shall sketch briefly the construction by Dyer [3] and then show that the Dyer's example is a $\sigma$-finite, non-$W^*$, type III $AW^*$-factor with M. C. P. Moreover we shall prove that it can be represented faithfully as $M(\mathfrak{U}, G_0)$ in section 2. Thus Dyer's factor is also monotone closed.

Let $\mathfrak{H}$ be a Hilbert space with an orthonormal basis $\{e_x; 0 \leq x < 1, x: \text{a real number}\}$. Every bounded linear operator $A$ on $\mathfrak{H}$ has a matrix representation $A_{x,y} = (Ae_y, e_x)$ for $x$ and $y \in [0, 1)$. Let $\mathfrak{U}_i$ (respectively $\mathfrak{Z}_i$) denote the algebra of operators $A$ such that $A_{x,y} = \delta_{x,y}f(x)$ for any $x$, $y$ where $f \in B^*[0, 1)$ and $\delta_{x,y}$ is a Kronecker's delta (resp. $\{x; 0 \leq x < 1, f(x) \neq 0\}$ is contained in a set of 1st category in $[0, 1)$).

Let $\mathfrak{U}_0$ (resp. $\mathfrak{Z}_0$) be the set of operators $A$ on $\mathfrak{H}$ with matrices $A_{x,y}$ with

1. $A_{x,y} = 0$ except when $y - x = j2^{-k}$ for some $k \geq 1$ and $-2^k < j < 2^k$ (integer).
2. For $k \geq 1$ and $0 \leq i < j < 2^k$, the function defined for $x \in [0, 1)$ by $f(x) = A_{x-2^{-k+i}+x^{-k(i+j)}}$ is a bounded Baire function (resp. $\{x; 0 < x < 1, f(x) \neq 0\}$ is contained in a set of 1st category in $[0, 1)$).

Dyer [3] proved that $\mathfrak{U}_0$ (resp. $\mathfrak{U}_i$) is a $C^*$-algebra with a closed two-sided ideal $\mathfrak{Z}_0$ (resp. $\mathfrak{Z}_i$) and the quotient algebra $\mathfrak{U}_0/\mathfrak{Z}_0$ is a non-$W^*$, $AW^*$-factor of which $\mathfrak{U}_i/\mathfrak{Z}_i$ is a maximal abelian $*$-subalgebra (note that $\mathfrak{U}_i/\mathfrak{Z}_i$ is $*$-isomorphic with $\mathfrak{U}$ in section 2).

By the above construction, a straightforward verification tells us that $\mathfrak{U}_0$ has M. C. P. and $\mathfrak{Z}_0$ is a $\sigma$-ideal in the sense that for every increasing sequence $\{A_n\}$ of self-adjoint elements in $\mathfrak{Z}_0$ which converges strongly to some operator $A$, $A \in \mathfrak{Z}_0$. Now by the arguments of J. D. M. Wright [15] it follows that $\mathfrak{U}_0/\mathfrak{Z}_0$ has M. C. P. Moreover, $\mathfrak{U}_0/\mathfrak{Z}_0$ has a faithful positive projection $\mathcal{P}$ onto $\mathfrak{U}_i/\mathfrak{Z}_i$. In fact, for any $A \in \mathfrak{U}_0$ with $A \sim \langle A_{x,y} \rangle$, put $B \sim \langle \delta_{x,y}A_{x,y} \rangle (B \in \mathfrak{U}_i)$ and consider the following mapping $\mathcal{P}: \mathfrak{U}_0/\mathfrak{Z}_0 \rightarrow B + \mathfrak{Z}_i$ of $\mathfrak{U}_0/\mathfrak{Z}_0$ onto $\mathfrak{U}_i/\mathfrak{Z}_i$. Then it is easy to check that $\mathcal{P}$ is a projection map of $\mathfrak{U}_0/\mathfrak{Z}_0$ onto $\mathfrak{U}_i/\mathfrak{Z}_i$. For any $A \in \mathfrak{U}_0$ with $A \sim \langle A_{x,y} \rangle$, we have that $(A^*A)_{x,y} = \sum_{0 \leq x \leq 1} |A_{x,y}|^2$ for all $x$. This implies that $\mathcal{P}$ is positive and faithful. Since $\mathfrak{U}_i/\mathfrak{Z}_i$ is $*$-isomorphic with $\mathfrak{U}$ in section 1, $\mathfrak{U}_i/\mathfrak{Z}_i$ has a faithful state and then by the same reasoning as in Theorem 2, $\mathfrak{U}_0/\mathfrak{Z}_0$ has a faithful state and thus by Theorem 1 we have

**Theorem 3.** $\mathfrak{U}_0/\mathfrak{Z}_0$ is a $\sigma$-finite, non-$W^*$, type III $AW^*$-factor.

The rest of this section is devoted to prove the following:

**Theorem 4.** $\mathfrak{U}_0/\mathfrak{Z}_0$ is $*$-isomorphic with $M(\mathfrak{U}, G_0)$ in section 2.
PROOF. For any \( A + \mathcal{Y} \in \mathcal{A}/\mathcal{S}_0 \) (\( A \in \mathfrak{U}_0 \)), let \( A \sim \langle A_{x,y} \rangle \), then \( A_{x,y} = 0 \) except when \( y - x = j \cdot 2^{-k} \) for some \( k \geq 1 \), \(-2^k < j < 2^k\) and for \( k \geq 1 \), \( 0 \leq j \leq 2^k \) the function \( x \rightarrow f(x) = A_{x-k(2^k),x-k(2^{-k})} \) (\( 0 \leq x < 1 \)) is in \( B^a[0,1] \). Keeping the notations in the last paragraph of section 1, for any \( g \in \mathcal{G}_0 \), \( x \rightarrow A_{x;g(x),x} \) is a bounded Biare function on \([0,1]\). Let \( \phi \) be the canonical map of \( B^a[0,1] \) onto \( \mathfrak{U} \) and \( a_{g,h} = \phi(x \rightarrow A_{x;g(x),x}) \) for \( g \in \mathcal{G}_0 \). Note that \( a_{g,h} \) does not depend on the choice of \( A \in \mathcal{A} + \mathcal{Y} \).

In fact, if \( A, B \in \mathcal{A} + \mathcal{Y} \), then \( A - B \in \mathcal{S}_0 \) and hence \( \phi(x \rightarrow A_{x;g(x),x}) = \phi(x \rightarrow B_{x;g(x),x}) \) for all \( g \in \mathcal{G}_0 \). Let \( a_{g,h} = (a_{g,h})_k \) for any \( g, h \in \mathcal{G}_0 \), then \( (a_{g,h}) \) defines an “operator” \( \psi(A + \mathcal{Y}) \) on \( \mathfrak{U} = \psi(G_0, \mathfrak{U}) \) such that \( \psi(A + \mathcal{Y})g,h = a_{g,h} \) for all \( g, h \in \mathcal{G}_0 \).

Observe that \( a_{g,h} \in \mathfrak{U} \) is the canonical image of \( x \rightarrow A_{x;g(x),h(x)} \) for any \( g, h \in \mathcal{G}_0 \).

Since \( \sum_{g \in \mathcal{G}_0} |\langle A_{x;g(x),h(x)} \rangle|^2 = \sum_{g \in \mathcal{G}_0} |(A_{x;g(x)}, e_{g(x)})|^2 \leq \sum_{g \in \mathcal{G}_0} |(A_{x;g(x)}, e_{g(x)})|^2 = ||A||^2 \leq ||A|| \) for all \( x \in [0,1] \), we have that \( \sum_{g \in \mathcal{G}_0} |a_{g,h}|^2 \leq ||A|| \cdot 1 \) on \( \mathfrak{U} \). Thus, for any \( \xi = (x_g) \in \mathfrak{U} \), \( \sum_{g \in \mathcal{G}_0} x_g a_{g,h} \leq ||\xi|| \cdot ||A|| \). This implies that \( \sum_{g \in \mathcal{G}_0} |x_g a_{g,h}|^2 \in \mathfrak{U} \) (order convergent in \( \mathfrak{U} \)). Put \( \eta_g = \sum_{g \in \mathcal{G}_0} x_g a_{g,h} \in \mathfrak{U} \), we can show that \( \sum_{g \in \mathcal{G}_0} |\eta_g|^2 \in \mathfrak{U} \) (order convergent in \( \mathfrak{U} \)). In fact, let \( x_g \) be the inverse image of \( x_g \) by \( \phi \) in \( B^a[0,1] \), then \( \sum_{g \in \mathcal{G}_0} |x_g(x)|^2 \leq ||\xi||^2 \) except on a set of first category. Hence it follows that

\[
\sum_{h \in \mathcal{G}_0} \sum_{g \in \mathcal{G}_0} |\widehat{x}_g(x)A_{x;g(x),h(x)}|^2 = \sum_{h \in \mathcal{G}_0} \sum_{g \in \mathcal{G}_0} |\widehat{x}_g(x)(A_{x;g(x),h(x)})|^2
= \sum_{h \in \mathcal{G}_0} \sum_{g \in \mathcal{G}_0} |\phi_g(x)(A_{x;g(x),h(x)}e_{g(x)})|^2
\]

(\( \widehat{x}_g(x) = 0 \) if \( y \neq \sigma_g(x) \) for any \( g \in \mathcal{G}_0 \) and \( \widehat{x}_g(x) = \widehat{x}_g(x) \) if \( y = \sigma_g(x) \) \( g \in \mathcal{G}_0 \).

\[
= \sum_{h \in \mathcal{G}_0} |(A_{x;g(x),h(x)})|^2 \quad \text{(where \( \widehat{x}_g(x) \) is in \( \mathfrak{U} \))}
= \sum_{h \in \mathcal{G}_0} |(e_{g(x)}, A^*(x_{g(x)}))|^2 \leq ||A||^2 \cdot ||\widehat{x}_g(x)||^2
= ||A||^2 \cdot ||\xi||^2
\]

except on a set of first category. Thus \( \sum_{g \in \mathcal{G}_0} |\sum_{g \in \mathcal{G}_0} x_g a_{g,h}|^2 \leq ||A|| \cdot ||\xi||^2 \) and \( \sum_{h \in \mathcal{G}_0} |\eta_h|^2 \in \mathfrak{U} \). Hence let \( \psi(A + \mathcal{Y})\xi = (\sum_{g \in \mathcal{G}_0} x_g a_{g,h}) \in \mathfrak{U} \), then \( \psi(A + \mathcal{Y}) \in \mathcal{B}(\mathfrak{U}) \) and \( ||\psi(A + \mathcal{Y})|| \leq ||A + \mathcal{Y}|| \). \( \psi(A + \mathcal{Y})g,h = a_{g,h} \) for all \( g, h \) implies that \( \psi(A + \mathcal{Y}) \in \mathcal{M}(\mathfrak{U}, G_0) \). Thus \( \psi \) is a bounded \(*\)-linear map of \( \mathfrak{U}/\mathcal{S}_0 \) into \( \mathcal{M}(\mathfrak{U}, G_0) \). Next we shall show that \( \psi \) is a \(*\)-isomorphism.

For any \( A + \mathcal{Y}, B + \mathcal{Y} \in \mathfrak{U}/\mathcal{S}_0 \) (\( A, B \in \mathfrak{U}_0 \)),

\[
(AB)_{g,h} = \sum_{k \in \mathcal{G}_0} A_{x;g(x),h(x)}B_{x;g(x),h(x)}
\]

for all \( 0 \leq x < 1 \). Thus \( \sum_{k \in \mathcal{G}_0} a_{g,h} b_{g,h} \) is order convergent to \( \phi(x \rightarrow (AB)_{g,h}(x)) \) in \( \mathfrak{U} \). Hence \( \psi(AB + \mathcal{Y}) = \psi(A + \mathcal{Y}) \psi(B + \mathcal{Y}) \). If
\[ \varphi(A + \mathfrak{S}_0) = 0 \]  \( (A \in \mathfrak{N}_0) \), then \( \{ x; 0 \leq x < 1, A_{g}(x), h(x) \neq 0 \} \) is contained in a set of 1st category in \([0, 1)\). Thus for all \( k \geq 1, \ 0 \leq i, j < 2^k, \)
\[ x \rightarrow A_{2^{-k}(i+x), 2^{-k}(j+x)} \]  has a first category support, and hence \( A \in \mathfrak{S}_0 \), that is, \( A + \mathfrak{S}_0 = 0 \). This implies that \( \varphi \) is a \( * \)-isomorphism of \( \mathfrak{N}_0/\mathfrak{S}_0 \) into \( M(\mathfrak{A}, G_0) \).

Next, we shall show that the map \( \varphi \) is onto. To do this we argue as follows: Let \( A \in M(\mathfrak{A}, G_0) \) with \( A \sim \langle \sigma_{g,h} \rangle \). Then one can choose for any \( g \) and \( h \in G_0 \), a function \( a_{g,h}(x) \in L^\infty(0, 1) \) such that there is a Baire set contained in a set of 1st category \( I \) in \([0, 1)\) such that
\[ \left| \sum_{g,h \in G_0} a_{g,h}(x) \xi_g \eta_h \right| \leq \| A \| \left( \sum_{g,h \in G_0} \| \xi_g \| \right)^{1/2} \left( \sum_{g,h \in G_0} \| \eta_h \| \right)^{1/2} \]
for all \( \{ \xi_g \}, \{ \eta_h \} \in l^2(G_0) \) and for all \( x \in [0, 1) \setminus I \) where \( \bar{c} \) is the complex conjugate of a complex number \( c \). Replacing \( a_{g,h}(x) \) by \( a_{g,h}(x) \) with the function \( a_{g,h}(x) \) defined to be zero if \( x \in I \) and equal to \( a_{g,h}(x) \) otherwise, we have that for any \( \{ \xi_g \}, \{ \eta_h \} \in l^2(G_0) \),
\[ \left| \sum_{g,h \in G_0} a_{g,h}(x) \xi_g \eta_h \right| \leq \| A \| \left( \sum_{g,h \in G_0} \| \xi_g \| \right)^{1/2} \left( \sum_{g,h \in G_0} \| \eta_h \| \right)^{1/2} \]
for all \( x \). Now we shall define \( \langle A_{g,h} \rangle \) as follows: \( A_{g,h} = 0 \) except when \( x-y = j \cdot 2^{-k} \) for some \( k \geq 1, \ -2^k < j < 2^k, \ A_{g,X}(x) = a_{g,h}(x) \leq x < 1, \ g \in G_0 \), then \( x \rightarrow A_{2^{-k}(i+x), 2^{-k}(j+x)} \) is a bounded Baire function on \([0, 1)\). To see that \( \langle A_{g,h} \rangle \) determines a bounded linear operator \( B \) on \( \mathfrak{S}_0 \), we have only to show that \( \sum_{0 \leq x, Y < 1} A_{x,y} \xi_{x,y} \eta_{x,y} \) is a bounded Measurable function on \([0, 1)\). In fact,
\[ \sum_{0 \leq x < 1} \sum_{g,h \in G_0} A_{g,X}(x) \xi_{g,h} \eta_{g,h} \]  \( \text{(where } G_{h_0} = \{ \sigma_{X-k_0}; i = 0, 1, 2, \ldots, 2^{k_0} - 1 \} \) \)
\[ = \sum_{0 \leq x < 1} \sum_{g,h \in G_0} \sum_{k \in G_{h_0}} A_{g,h+k}(x) \xi_{g+h,k} \eta_{g+h,k} \]  \( \text{and hence there is a } B \in \mathfrak{N}_0 \) such that \( (Be_x, e_y) = A_{x,y} \) for all \( x, y \). By the construction, it is easy to check that \( \varphi(B + \mathfrak{S}_0) = A \). Thus \( \varphi \) is onto. Hence \( \mathfrak{N}_0/\mathfrak{S}_0 \cong M(\mathfrak{A}, G_0) \). This completes the proof of Theorem 4.

4. Remarks. (1) We shall remark first that every \( \sigma \)-finite type III
A $\text{AW}^*$-factor is simple. Certain standard arguments tell us that for any pair $e$ and $f$ of non-zero projections in each $\sigma$-finite type III $\text{AW}^*$-factor $M$, $e \sim f$ in $M$. In fact, since “comparability theorem” of projections and “additivity of equivalence” of projections hold in any $\text{AW}^*$-algebra ([6]), we can easily show that for any non-zero projection $e$ in $M$, there exists a mutually orthogonal sequence of projections $\{e_i\}_{i=1}^\infty$ in $M$ such that $e = \sum_{i=1}^\infty e_i$, $e \sim e_i$ for all $i$. Let $\{f_j\}_{j\in J}$ be a maximal family of orthogonal projections such that $f_j \prec e$ for all $j$. Then the $\sigma$-finiteness of $M$ implies that the cardinal of $J$ is at most countable. The maximality of $\{f_j\}_{j\in J}$ tells us that $1 - \sum_{j\in J} f_j = 0$. Thus $1 = \sum_{j\in J} f_j < \sum_{i=1}^\infty e_i = e$ and $e \sim 1$ in $M$.

Now let $I$ be any non-zero uniformly closed two-sided ideal of $M$, then by F. B. Wright’s theorem [14], $I$ contains a non-zero projection $e$. Thus, by the above argument, $e \sim 1$ and $1 \in I$, that is, $I = M$ and $M$ is simple.

(2) We note also that every type $\text{I}_\infty$ or type $\text{II}_\infty$ $\text{AW}^*$-factor is not simple because the uniformly closed two-sided ideal generated by all finite projections in it is non-trivial.

Using this, the regular $\sigma$-completion $\hat{A}$ of a simple, infinite dimensional, separable unital $C^*$-algebra $A$ is neither of type $\text{I}_\infty$ nor of type $\text{II}_\infty$ (because $\hat{A}$ is simple), that is, $\hat{A}$ is of type $\text{II}_1$ or of type $\text{III}$. Since $\hat{A}$ has a faithful state ([18, Theorem M]), [17, Theorem 6] tells us that $\hat{A}$ is of type $\text{III}$.

(3) Next we shall show that for any $\sigma$-finite, type III non-$\text{W}^*$, $\text{AW}^*$-factor $M$, $M$ does not have any non-trivial separable representations. Suppose, on the contrary, that $M$ has a non-trivial separable representation $(\pi, \mathcal{D}_\pi)$ ($\mathcal{D}_\pi$ is separable). Then we may assume without loss of generality that $\pi(1) = 1_{\mathcal{D}_\pi}$ (the identity operator on $\mathcal{D}_\pi$). Feldman and Fell [4] state that $\pi$ is completely additive on projections and by the argument in (1) ($M$ is simple), $\pi$ is faithful. This implies that $M$ has sufficiently many $\text{c.a.}$ states. Thus $M$ is a $W^*$-algebra by [9]. This is a contradiction and $M$ has no non-trivial separable representations.

Thus the examples $M(\mathcal{G}, G_\theta)$, $M(\mathcal{G}, G_\xi)$ and $\hat{A}$ are simple and do not have any non-trivial separable representations.

We note that the above statements also hold for any $\sigma$-finite, properly infinite $\text{AW}^*$-algebra without any $W^*$-direct summands, but we will omit the details.

(4) We shall also remark that there is a monotone closed $C^*$-factor which is not a $W^*$-algebra ($M(\mathcal{G}, G_\theta)$, $M(\mathcal{G}, G_\xi)$, $\hat{A}$) see [5, Corollary 3.10].
REFERENCES