HELSON-SZEGÖ-SARASON THEOREM FOR DIRICHLET ALGEBRAS

YOSHIKI OHNO

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Two linear subspaces in a Hilbert space are said to be at positive angle if
\[ \sup \, |(f, g)| < 1 \]
where \( f \) and \( g \) range over the elements of the respective linear subspaces with norm at most 1. Let \( \mu \) be a finite positive Borel measure on the unit circle \( T \) in the complex plane. Let \( \chi \) be the function on \( T \) defined by \( \chi(e^{i\theta}) = e^{i\theta} \). For each integer \( n \), we form in the Hilbert space \( L^2(d\mu) \) the linear subspace \( \mathcal{F}_n \) spanned by the functions \( \chi^n, \chi^{n+1}, \ldots \). For any set \( S \) of complex-valued functions, write \( S = \{ f \mid f \in S \} \).

Helson and Szegö [6] and Helson and Sarason [5] have proved the following result.

**Theorem 1.** Let \( n \) be a natural number. In order for \( \mathcal{F}_0 \) and \( \mathcal{F}_n \) to be at positive angle in \( L^2(d\mu) \) it is necessary and sufficient that \( d\mu \) is absolutely continuous with respect to the Lebesgue measure \( d\theta \) on \( T \), strictly so that \( d\mu = w(d\theta), w \in L^1(d\theta) \), and \( w \) has the form
\[
w = \left| P \right|^2 e^{i\alpha} c,\]
where \( P \) is a polynomial of degree less than \( n \), \( r \) is a real bounded function, and \( C_{\alpha} \) is the conjugate of a real function \( s \) with bound smaller than \( \pi/2 \).

For \( n = 1 \), Devinatz [1, 2] and Ohno [9] extended this result to a Dirichlet algebra setting. See also Merrill [7] as regards another result of this prediction type. The main object of this note is to show that the result is valid for a general natural number \( n \).

Let \( X \) be a compact Hausdorff space and let \( A \) be a Dirichlet algebra on \( X \), i.e., \( A \) is a uniform algebra on \( X \) such that the real parts of the functions in \( A \) are uniformly dense in the real continuous functions on \( X \). Let \( m \) be the unique representing measure on \( X \) for a complex homomorphism of \( A \). We assume that the Gleason part of \( m \) contains
more than one point. If \(0 < p < \infty\), \(H^p\) shall be the closure of \(A\) in \(L^p(dm)\) and \(H^\infty\) shall be the weak*-closure of \(A\) in \(L^\infty(dm)\). We put
\[
A_0 = \left\{ f \in A \left| \int f \, dm = 0 \right. \right\} \quad \text{and} \quad H^p_0 = \left\{ f \in H^p \left| \int f \, dm = 0 \right. \right\} \quad (1 \leq p \leq \infty).
\]
We denote by \((H^p_0)^*\) (resp. \(A^\circ\)) the ideal generated by products of \(n\) elements in \(H^\infty_0\) (resp. \(A_0\)). We shall make use of the theory of the abstract Hardy spaces. We refer to Gamelin [4] in this connection.

Let \(\nu\) be a positive finite measure on \(X\) and consider a condition for \(A\) and \(A^\circ\) to be at positive angle in \(L^2(d\nu)\). For the sake of simplicity we shall, at the outset, assume that \(d\nu\) is absolutely continuous with respect to \(dm\), i.e., \(d\nu = wd\nu\), where \(w\) is a non-negative function in \(L^{1}(dm)\).

For \(n = 1, 2, \ldots\), we define
\[
\rho_n = \sup \left| \int fg \, w dm \right|
\]
where \(f\) and \(g\) range over the elements of \(A\) and \(A^\circ\), respectively, subject to the restriction
\[
\int |f|^p w dm \leq 1 \quad \text{and} \quad \int |g|^p w dm \leq 1.
\]

**PROPOSITION 2.** If \(\rho_n < 1\), then \(\int \log w dm > -\infty\).

**PROOF.** We denote the norm in \(L^2(w dm)\) by \(\| \cdot \|_p\). Suppose that \(\int \log w dm = -\infty\). By Szegö’s theorem ([4], Theorem 8.2),
\[
\inf \left\{ \int |1-f|^p w dm \mid f \in A_0 \right\} = 0
\]
for \(0 < p < \infty\). We consider the case \(p = 4n\) and choose \(f_k \in A_0\) such that \(\|f_k - 1\|_{4n} \to 0\) \((k \to \infty)\). Then there exists a constant \(M\) such that \(\|f_k\|_{4n} \leq M\) \((k = 1, 2, \ldots)\). By Hölder’s inequality and Minkowski’s inequality,
\[
\int |f_k|^p w dm - 1^p w dm \\
\leq \left\{ \int |f_k - 1|^p w dm \right\}^{1/2} \left\{ \int |f_k|^{4n-1} + |f_k|^{4n-2} + \cdots + 1^{4n} w dm \right\}^{1/2} \\
\leq \|f_k - 1\|_4 \left\{ |f_k|^{4n-1} + |f_k|^{4n-2} + \cdots + |1|_{4n} \right\}^2 \\
\leq \|f_k - 1\|_4 \left\{ (K^{4n-1} + K^2 M^{4n-2} + \cdots + K^n) \right\}^2
\]
where \(K = \left\{ \int w dm \right\}^{1/4n}\). It follows that \(\|f_k - 1\|_{4n} \to 0\) \((k \to \infty)\). This shows that \(1\), hence any constant function lies in the \(L^p(w dm)\)-closure...
of $A_n^\circ$. It follows that $\rho_n = 1$. This contradicts the hypothesis and we have $\int \log wd\mu > -\infty$.

Thus, in order to characterize $w$ with $\rho_n < 1$, we may as well assume, from the outset, that $\int \log wd\mu > -\infty$. In this case, it is easy to see that

$$\rho_n = \sup \left\| \int fgwd\mu \right\|$$

where $f$ and $g$ range over the elements of $H^\infty$ and $(H_0^n)^n$, respectively, subject to the restriction

$$\int |f|^2wd\mu \leq 1 \quad \text{and} \quad \int |g|^2wd\mu \leq 1.$$

We begin by following an idea of Helson and Szegö [6]. Such a function $w$ is of the form $|h|^2$, $h$ being an outer function in $H^2$ ([3], Theorem 6). Define the function $e^{-i\phi}$ by equation $w = h^2e^{-i\phi}$. Then $\rho_n$ is given by

$$\rho_n = \sup \left\| (fh)(gh)e^{-i\phi}d\mu \right\|$$

where the supremum is taken over all $f \in H^\infty$ and $g \in (H_0^n)^n$ such that

$$\int |fh|^2d\mu \leq 1 \quad \text{and} \quad \int |gh|^2d\mu \leq 1.$$

By Wermer's embedding theorem ([4], Theorem 7.2), there exists an inner function $Z$ in $H^\infty$ such that $H_0^n = ZH^\infty$. Then $(H_0^n)^n = Z^nH^\infty$ and we have

$$\rho_n = \sup \left\| (fh)(gh)Z^n e^{-i\phi}d\mu \right\|$$

where $f$ and $g$ range over the elements of $H^\infty$ subject to the respective restriction

$$\int |fh|^2d\mu \leq 1 \quad \text{and} \quad \int |gh|^2d\mu \leq 1.$$

Since $h$ is outer in $H^2$, $\{fh \mid f \in H^\infty\}$ is dense in $H^2$ and more specifically $\{fh \mid f \in H^\infty, \int |fh|^2d\mu \leq 1\}$ is dense in the unit ball of $H^2$. Thus $\{fgZ^\phi \mid f, g \in H^\infty, \int |fh|^2d\mu \leq 1, \int |gh|^2d\mu \leq 1\}$ is dense in the unit ball of $H^2$ (cf. [2], Lemma 6). Therefore (2) can be written in the form

$$\rho_n = \sup \left\| \int fZ^n e^{-i\phi}d\mu \right\|$$
where \( f \) ranges over the functions in \( H^1 \) such that \( \int |f| \, dm \leq 1 \). Evidently (3) expresses \( \rho_n \) as the norm of the linear functional on \( H^1 \) defined by

(4) \[ \int f Z^* e^{-i\phi} \, dm \]

for \( f \in H^1 \). By the Hahn-Banach theorem, this linear functional has a norm-preserving extension to the whole \( L^1(dm) \). Since \( L^1(dm)^* = L^\infty(dm) \), we can identify the above extension with \( Z^* e^{-i\phi} - g_0 \in L^\infty(dm) \). In this case \( g_0 \in H_0^\infty \), because

\[
\int f Z^* e^{-i\phi} \, dm = \int f(Z^* e^{-i\phi} - g_0) \, dm \quad (f \in A)
\]

and so \( \int f g_0 \, dm = 0 \) (\( f \in A \)), implying \( g_0 \in H_0^\infty \). Furthermore, for every \( g \in H_0^\infty \), \( Z^* e^{-i\phi} - g \) gives an extension of the linear functional given by (4) to the whole \( L^1(dm) \) and so

\[
\rho_n = \| Z^* e^{-i\phi} - g_0 \| \leq \| Z^* e^{-i\phi} - g \| \quad (g \in H_0^\infty)
\]

where \( \| \cdot \| \) denotes the norm in \( L^\infty(dm) \). It follows that

(5) \[
\rho_n = \inf_{g \in H_0^\infty} \| Z^* e^{-i\phi} - g \| = \inf_{F \in H^\infty} \| 1 - Z^{-n} e^{i\phi} F \|.
\]

**Proposition 3.** \( \rho_n < 1 \) if and only if for some \( \varepsilon > 0 \) and \( F \in H^\infty \), we have

(6) \[
|F| > \varepsilon
\]

(7) \[
|\text{Arg} (F h^2 Z^{-n})| < \pi/2 - \varepsilon
\]

where \( -\pi \leq \text{Arg} \, z < \pi \).

**Proof.** If \( \rho_n < 1 \), take \( \varepsilon > 0 \) such that \( \rho_n \leq 1 - 2\varepsilon \). Then by (5), there exists an \( F \in H^\infty \) such that \( \| 1 - Z^{-n} e^{i\phi} F \| < 1 - \varepsilon \). This implies \( |F| > \varepsilon \), and then (7) is geometrically obvious, perhaps with a smaller value of \( \varepsilon \).

Conversely if \( F \) satisfies (6) and (7), then it is easy to see that

\[
\| Z^{-n} e^{-i\phi} - \lambda F \| < 1
\]

for some \( \lambda > 0 \), and so \( \rho_n < 1 \).

**Proposition 4.** If \( F \in H^1 \) and

(8) \[
|\text{Arg} (F Z^{-k})| < \pi/2 - \varepsilon
\]
then there exist an integer \(m(0 \leq m \leq k)\) and \(B \in H^1\) such that \(\int Bdm \neq 0\) and \(F = Z^mB\).

**Proof.** Since \(H_0^1 = ZH^1\), it suffices to show that if \(F = Z^kB\) and \(B \in H^1\), then \(\int Bdm \neq 0\). By (8),

\[
|\text{Arg } B| = |\text{Arg } (FZ^{-k})| < \frac{\pi}{2} - \varepsilon
\]

and we have \(\text{Re } B \geq 0\). Since \(B \in H^1\), it follows from Theorem 12 of Devinatz [2] that \(B\) is outer in \(H^1\). Hence we have \(\int Bdm \neq 0\).

We denote by \(\mathscr{H}^p\) the closure in \(L^p(dm)\) of the set of polynomials in \(Z\) and denote by \(\mathscr{L}^p\) the closure in \(L^p(dm)\) of the set of polynomials in \(Z\) and \(\bar{Z}\) (the norm closure for \(1 \leq p < \infty\); the weak*-closure for \(p = \infty\)). For \(1 \leq p \leq \infty\), we put

\[
I^p = \left\{ f \in H^p \mid \int f\bar{Z}^k dm = 0 \ (k = 0, 1, 2, \cdots) \right\}.
\]

**Lemma 5.** (Merrill and Lal [8], Lemma 5.) If \(1 \leq p \leq \infty\), then

\[
H^p = \mathscr{H}^p \oplus I^p,
\]

\[
L^p = \mathscr{L}^p \oplus N^p
\]

where \(\oplus\) denotes the algebraic direct sum and \(N^p\) denotes the closure of \(I^p + I^p\) in \(L^p(dm)\) (the norm closure for \(1 \leq p < \infty\); the weak*-closure for \(p = \infty\)).

**Theorem 6.** In order for \(A\) and \(\overline{A_0^s}\) to be at positive angle in \(L^2(dm)\), it is necessary and sufficient that \(w\) has the form

(9)

\[
w = |P|^r e^{iCs}
\]

where \(P\) is a function in \(H^\infty\) such that \(P \perp A_0^s\) in \(L^2(dm)\), \(r, s \in L^2_H(dm)\), \(||s|| < \pi/2\) and \(Cs\) is the conjugate of \(s\).

**Proof.** We assume \(\rho_n < 1\). By Proposition 3, there exist \(\varepsilon > 0\) and \(F \in H^\infty\) such that \(|F| > \varepsilon\) and

(10)

\[
|\text{Arg } (Fh^2Z^{1-n})| < \frac{\pi}{2} - \varepsilon.
\]

Let \(s\) be the function bounded by \(\pi/2 - \varepsilon\) such that

(11)

\[
s + \text{Arg } (Fh^2Z^{1-n}) = 0.
\]

We put

(12)

\[
S = Fh^2Z^{1-n} e^{-Cs + is},
\]
then, by (11), \( S \geq 0 \). From Theorem 10 of [2], we conclude that \( e^{-C_2 + is} \in H^1 \) is outer. By (10) and Proposition 4, we may write \( Fh^2 = Z^m B \), where \( B \in H^1 \), \( \int Bdm \neq 0 \) and \( 0 \leq m \leq n - 1 \). Therefore

\[
S = BZ^{-k}e^{-C_2 + is} \geq 0
\]

and so

\[
Z^k S = Be^{-C_2 + is} \in H^{1/2}
\]

where \( k = n - m - 1 \). Furthermore, by Jensen’s inequality,

\[
\int \log |Z^k S| dm = \int \log |B| dm + \int \log |e^{-C_2 + is}| dm
\]

\[
\geq \log \left| \int Bdm \right| + \log \left| \int e^{-C_2 + is} dm \right| > -\infty
\]

and so we have

\[
\exp \int \log |Z^k S| dm > 0 .
\]

Using Theorem 2 of [3], it follows from (14) and (15) that there exist an outer function \( P \) in \( H^1 \) and an inner function \( q \) in \( H^\infty \) such that

\[
Z^k S = qP^2 .
\]

Since \( S = |S| \) and \( |S| = |P|^2 \), we have from (16) that

\[
qP^2 = Z^k |P|^2 .
\]

Since \( P \) is outer, it follows that \( P \) is not zero. Thus we may divide (17) by \( P \) and we obtain

\[
qP = Z^k \bar{P} .
\]

By Lemma 5, we may write

\[
P = \sum_{j=0}^n a_j Z^j + \alpha_i \in H^{p_1} \oplus I^1
\]

where \( \alpha_i \) belongs to \( I^1 \). Now

\[
Z^k \bar{P} = \bar{a}_0 Z^k + \bar{a}_1 Z^{k-1} + \cdots + \bar{a}_{k-1} Z + \bar{a}_k
\]

\[
+ \bar{a}_{k+1} \bar{Z} + \bar{a}_{k+2} \bar{Z}^2 + \cdots + Z^k \bar{\alpha}_i .
\]

Because \( a_{k+1} Z + a_{k+2} Z^2 + \cdots \in H^0_1 \) and \( \bar{Z} \bar{\alpha}_i \in I^1 \cap H^1_1 \), we have

\[
g = a_{k+1} Z + a_{k+2} Z^2 + \cdots + \bar{Z} \bar{\alpha}_i \in H^1_1 .
\]

By (18), \( Z^k \bar{P} \in H^1 \) and we conclude \( \bar{g} \in H^1 \) by (19). Hence \( g \in H^1 \cap H^1 \). Since \( \bar{A} + A_0 \) is weak*-dense in \( L^m(dm) \), we have \( g = 0 \) and
\[ Z^k \bar{P} = \bar{a}_0 Z^k + \bar{a}_1 Z^{k-1} + \cdots + \bar{a}_{k-1} Z + \bar{a}_k. \]

Hence \( P \) has the form
\[ P = a_0 + a_1 Z + \cdots + a_k Z^k \]
where \( 0 \leq k \leq n - 1 \). Therefore \( P \in H^\infty \) and \( P \perp A^*_o \) in \( L^2(dm) \). Indeed, if \( G \in A^*_o \subset (H^\infty)^n \), then \( G = Z^k K \) for some \( K \in H^\infty \) and we have
\[
(P, G) = \left( \sum_{j=0}^{k} a_j Z^j \right) \bar{Z}^k K dm = \sum_{j=0}^{k} a_j \int \bar{Z}^{k-j} \bar{K} dm = 0,
\]
since \( m \) is multiplicative on \( H^\infty \) and \( n - 1 \geq k \). Now by (16) and (12) we have
\[
|P|^2 = S = |S| = |F| |h|^2 e^{-Cs}
\]
and since \( w = |h|^2 \),
\[
w = |P|^2 |F|^{-1} e^{Gs} = |P|^2 e^{r} e^{Gs}
\]
where \( r = -\log |F| \). In this case \( r, s \in L^\infty(dm) \) and \( |s| < \pi/2 \).

Conversely, suppose \( w \) has the form (9). We put \( S = |P|^2 \). Since \( Z^{n-1} Pf \in (H^\infty)^n \) for \( f \in I^\infty \), we have
\[
\int Z^{n-1} S f dm = \int Z^{n-1} Pf, P = 0 \quad (f \in I^\infty).
\]
If \( f \in I^\infty \), then it is easy to see that \( \bar{Z}^{2(n-1)} f \) is also in \( I^\infty \). Therefore, by (20),
\[
\int Z^{n-1} S f dm = \int Z^{n-1} S \bar{Z}^{2(n-1)} f dm = 0 \quad (f \in I^\infty).
\]
Since \( S = \bar{S} \),
\[
\int Z^{n-1} S \bar{f} dm = 0 \quad (f \in I^\infty).
\]
It follows from (20) and (21) that
\[
\int Z^{n-1} S f dm = 0 \quad (f \in I^\infty \oplus I^\infty).
\]
By Lemma 5, \( Z^{n-1} S \in \mathcal{L}^1 \). Furthermore, we have
\[
\int Z^{n-1} S \bar{Z}^k dm = \begin{cases} \int (Z^{n-1-k} P, P) = 0 & (n - 1 - k \geq n, \text{i.e., } k = -1, -2, \cdots) \\ \int (P, Z^{k+1-n} P) = 0 & (k + 1 - n \geq n, \text{i.e., } k = 2n - 1, 2n, \cdots). \end{cases}
\]
We conclude that \( Z^{n-1} S \) has the form
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\[ Z^{-1}S = a_0 + a_1Z + \cdots + a_{2n-2}Z^{2n-2}. \]

We put

\[ k = \max \{m | 0 \leq m \leq n - 1, a_{m+n-1} \neq 0 \} \]

Since \( S \neq 0 \) and \( \overline{S} = S \), such \( k \) exists. Then \( Z^kS \in H^\infty \) and \( \left\{ Z^kSdm \neq 0 \right\} \), therefore by Theorem 2 of [3], \( Z^kS \) has the factoring

\[ Z^kS = qG^2 \]

where \( q \) is inner and \( G \) is outer in \( H^\infty \). If we take an outer function \( F \) in \( H^\infty \) such that \( |F| = e^{-\tau} \), then

\[ Z^k\overline{q}Se^{\tau-i\tau} = \overline{F}h^2, \]

up to constant factors of modulus 1. Indeed, by Theorem 10 of [2], \( e^{\tau-i\tau} \) is outer in \( H^1 \), and \( Z^k\overline{q}S = G^2 \) is also outer in \( H^\infty \), so that the left hand side of (22) is outer in \( H^1 \). Furthermore, since \( F \) is outer in \( H^\infty \) and \( h \) is outer in \( H^2 \), the right hand side of (22) is also outer in \( H^1 \).

Now by the assumption on \( w \)

\[ Z^k\overline{q}Se^{\tau-i\tau} = \overline{F}h^2. \]

Since an outer function is determined up to a constant factor by its modulus, (22) follows. By (22), \( S = \overline{F}h^2Z^{-k}qe^{-\tau}e^{i\tau} \) and \( S \geq 0 \), it follows that

\[ \text{Arg} (\overline{F}h^2Z^{-k}qe^{-\tau}e^{i\tau}) = 0. \]

Hence for sufficiently small \( \varepsilon > 0 \),

\[ |\text{Arg} (\overline{F}h^2Z^{-k}q)| = |s| \leq ||s|| < \pi/2 - \varepsilon \]

and

\[ |F| = e^{-\tau} > \varepsilon. \]

If we put \( B = FqZ^{-1-k} \), then \( B \in H^\infty \) and

\[ |\text{Arg} (Bh^2Z^{1-k})| = |\text{Arg} (\overline{F}h^2Z^{-k}q)| < \pi/2 - \varepsilon \]

\[ |B| = |F| > \varepsilon. \]

The assertion follows from Proposition 3.

**Corollary.** In order for \( A \) and \( \overline{A}_n \) to be at positive angle in \( L^2(wdm) \), it is necessary and sufficient that \( w \) has the form

\[ w = |P|^2e^{r+C\tau} \]

where \( P \) is a polynomial in \( Z \) of degree less than \( n \), \( r, s \in L^\infty(dm) \), \( ||s|| < \pi/2 \) and \( Cs \) is the conjugate of \( s \).
EXAMPLE. Put \( S = \{(k, l) \in \mathbb{Z}^2 | k > 0\} \cup \{(0, l) \in \mathbb{Z}^2 | l \geq 0\} \). Let \( A = A(T^2) \) be the Dirichlet algebra of continuous functions on \( T^2 \) which are uniform limits of polynomials in \( e^{ikx}e^{il_2} \) where \((k, l) \in S\). Let \( m \) denote the normalized Haar measure on \( T^2 \). Then the Gleason part of \( m \) can be identified with \( \{(0, \alpha) \in \mathbb{C}^2 | |\alpha| < 1\} \) and is non-trivial. Wermer’s embedding function \( Z \) is given by \( Z(e^{ix}, e^{iy}) = e^{iy} \). In this case, \( A^\pi \) is the uniformly closed linear span of \( \{e^{ikx}e^{il_2} | (k, l) \in S\} \) and the function \( P \) in Theorem 6 is a polynomial in \( e^{iy} \) of degree less than \( n \).

REMARK. We used the setting such that \( A \) is a Dirichlet algebra and the Gleason part of the unique representing measure \( m \) is non-trivial since it is easier to work with. However, similar proofs will show that all results are valid for a setting such that \( A \) is a weak*-Dirichlet algebra on a given probability measure space \((X, m)\) and there exists a non-zero weak*-continuous multiplicative linear functional on \( A \) which is different from \( dm \) (for the relevant definition, see, Srinivasan and Wang [10]).

REFERENCES


COLLEGE OF GENERAL EDUCATION
Tôhoku University
Kawauchi, Sendai, 980 Japan.