NOTES ON THE CANCELLATION OF RIEMANNIAN MANIFOLDS

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Introduction. Let $M$, $N$ and $B$ be Riemannian manifolds. Then, we have a question "is $M$ isometric to $N$, if $M \times B$ is isometric to $N \times B$?" Uesu [1] proved that the answer is affirmative if $M$ and $N$ are complete and if $B$ is compact locally symmetric. In the present note, we shall show that the answer is affirmative also if the last condition on $B$ is replaced by one of the following (1) and (2):

1. $B$ is simply connected and complete.
2. $B$ is complete and $B$ has the irreducible restricted holonomy group.

The last assertion is stated as Theorems A and B in the next section.

We assume, in the present note, that all Riemannian manifolds are connected and $C^\infty$.

1. Proof of the theorems. First, we give some lemmas. Let $\Omega' = \{1, \ldots, r\}$ and $\Omega'' = \{r+1, \ldots, n\}$. For a subset $\Omega \subset \Omega' \cup \Omega''$, we denote by $S(\Omega)$ the symmetric group of $\Omega$. And, by $S_s$ and $S_r$, we denote $S(\Omega' \cup \Omega'')$ and $S(\Omega')$, respectively. Let $G$ be a subgroup of $S_r$ and $G' = \{i \in \Omega' \mid \tau(i) = i \text{ for all } \tau \in G\}$, $G'' = \{i \in \Omega'' \mid \tau(i) \neq i \text{ for some } \tau \in G\}$.

**Lemma 1.** Let $\sigma$ be an element of $S_r$.

(i) If $\omega \in S(\sigma(\Omega' \cup \Omega''))$, then $(\omega \sigma) \tau(\omega \sigma)^{-1} = \sigma \tau \sigma^{-1}$ for all $\tau \in G$.

(ii) If $\sigma G \sigma^{-1} \subset S_r$, then $\sigma(\Omega') \subset \Omega'$.

**Proof.** (i) We note $\tau(\Omega'_0) = \Omega'_0$ for all $\tau \in G$. If $i \in \Omega'_0$, then $\omega \sigma(i) = \sigma(i)$, and hence $\omega \sigma \tau \sigma^{-1}(\omega \sigma(i)) = \omega \sigma(\tau(i)) = \sigma \tau(i)$, $\sigma \tau \sigma^{-1}(\omega \sigma(i)) = \sigma \tau \sigma^{-1}(\sigma(i)) = \sigma \tau(i)$. If $i \in \Omega'_0 \cup \Omega''$, then $\omega \sigma \tau \sigma^{-1}(\omega \sigma(i)) = \omega \sigma \tau(i) = \sigma \tau(i)$ and $(\sigma \tau \sigma^{-1}) \omega \sigma(i) = \omega \sigma(i)$, as $\omega \sigma(i) = \sigma(j)$ for some $j \in \Omega'_0 \cup \Omega''$. (ii) Assume $i \in \Omega'_0$ and $\sigma(i) \in \Omega''$. Since $\sigma G \sigma^{-1} \subset S_r$, we have $\sigma \tau \sigma^{-1}(\sigma(i)) = \sigma(i)$ and hence $\tau(i) = i$ for all $\tau \in G$, a contradiction. q.e.d.

Throughout the present note, $I(M)$ denotes the group of all isometries of a simply connected and complete Riemannian manifold $M$.

Let $M_1 = M_2 = \cdots = M_n$ be a simply connected and complete Riemannian manifold whose homogeneous holonomy group is irreducible. Let $M$ be the direct product Riemannian manifold $M_1 \times M_2 \times \cdots \times M_n$. For
each $\sigma \in S_n$, $\lambda(\sigma): M \to M$ is defined by

$$\lambda(\sigma)(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$$

$\sigma \in S_n$. Then, $\lambda(\sigma) \in I(M)$ and $\lambda: S_n \to I(M)$ is an isomorphism. Briefly we denote $\lambda(\sigma)$ by $\sigma$. Then $S_n$ is a subgroup of $I(M)$.

**Lemma 2.** (i) $I(M)$ is generated by $S_n$ and $I(M_1) \times \cdots \times I(M_n)$. (ii) If $\sigma \in S_n$ and $(f_1, \ldots, f_n) \in I(M_1) \times \cdots \times I(M_n)$, then

$$\sigma(f_1, \ldots, f_n)\sigma^{-1} = (f_{\sigma^{-1}(1)}, \ldots, f_{\sigma^{-1}(n)})$$

where $(f_{\sigma^{-1}(1)}, \ldots, f_{\sigma^{-1}(n)}) \in I(M_1) \times \cdots \times I(M_n)$. In particular, $I(M_1) \times \cdots \times I(M_n)$ is a normal subgroup of $I(M)$.

**Proof.** (i) is easily seen by the uniqueness of de Rham's decomposition (cf. Uesu [1] and Wolf [2]). (ii) $\sigma(f_1, \ldots, f_n)\sigma^{-1}(x_1, \ldots, x_n) = \sigma(f_1(x_{\sigma(1)}), \ldots, f_n(x_{\sigma(n)})) = (f_{\sigma^{-1}(1)}(x_1), \ldots, f_{\sigma^{-1}(n)}(x_n)) = (f_{\sigma^{-1}(1)}, \ldots, f_{\sigma^{-1}(n)})(x_1, \ldots, x_n)$. q.e.d.

By the above lemma, $S_n$ is isomorphic to the quotient group $I(M)/I(M_1) \times \cdots \times I(M_n)$. Let $\mu$ be the natural projection of $I(M)$ onto $I(M)/I(M_1) \times \cdots \times I(M_n)$. Then, the image $\mu(\Gamma)$ of a subgroup $\Gamma$ of $I(M)$ is considered as a subgroup of $S_n$.

Let us decompose $M$ in Lemma 2 into $M'$ and $M''$, where $M' = M_1 \times \cdots \times M_r$ and $M'' = M_{r+1} \times \cdots \times M_s$. Then $M = M' \times M''$.

**Lemma 3.** Let $\Gamma$ be a subgroup of $I(M')$ and $f \in I(M)$. If $f \Gamma f^{-1} \subset I(M')$, then there exists $f' \in I(M')$ satisfying $f'hf^{-1} = fhf^{-1}$ for all $h \in \Gamma$.

**Proof.** Let $G$ be the subgroup $\mu(\Gamma)$ which is a subgroup of $S_r$. Then, we may apply Lemma 1 with the other notations used. $f$ is written as $f = \sigma(f_1, \ldots, f_n)$ by Lemma 2, where $\sigma \in S_n$ and $(f_1, \ldots, f_n) \in I(M_1) \times \cdots \times I(M_n)$. Let $h$ be an element of $\Gamma$. Then $h$ is written as $h = \tau(h_1, \ldots, h_r, h_{r+1}, \ldots, h_s)$, where $\tau \in S_r, \quad (h_1, \ldots, h_r, h_{r+1}, \ldots, h_s) \in I(M_1) \times \cdots \times I(M_r) \times I(M_{r+1}) \times \cdots \times I(M_s)$ and $h_{r+1} = \cdots = h_s = 1$. Then we have

$$(\ast) \quad fhf^{-1} = \sigma \tau \sigma^{-1}(f_{\sigma^{-1}(1)}h_{\sigma^{-1}(1)}f_{\sigma^{-1}(1)}^{-1}, \ldots, f_{\sigma^{-1}(n)}h_{\sigma^{-1}(n)}f_{\sigma^{-1}(n)}^{-1}) \in I(M_1) \times \cdots \times I(M_n).$$

Since $fhf^{-1} \in I(M')$, we have $\sigma \tau \sigma^{-1} \in S_r$ and $f_{\sigma^{-1}(i)}h_{\sigma^{-1}(i)}f_{\sigma^{-1}(i)}^{-1} = 1$ if $i \in \Omega'$. On the other hand, by (ii) of Lemma 1, if $i \in \Omega''$, then $\sigma^{-1}(i) \in \Omega'_s \cup \Omega''$. Hence $\sigma^{-1}(i) = \sigma'(i)$ and $h_{\sigma^{-1}(i)} = 1$. Thus, if $j \in \sigma^{-1}(\Omega'' \cup \Omega'')$, then $\tau(j) = j$ and $h_j = 1$. We may assume for brevity that $\sigma^{-1}(\Omega'') \cap \Omega' = \{1, \ldots, s\}, s \leq r$. Then $\Gamma' \subset I(M_{s+1} \times \cdots \times M_r)$. 


Now, we define \( w \in S(\{a(j_1), \ldots, a(j_s), a(1), \ldots, a(s)\}) \) by \( w(a(j_k)) = a(k) \) and \( \omega(a(k)) = a(j_k) \), \( k = 1, \ldots, s, \) where \( a^{-1}(\Omega') \cap \Omega'' = \{j_1, \ldots, j_s\} \), \( r + 1 \leq j_i < \cdots < j_s \leq n \). And we define \( f' \in I(M') \) by \( f' = \omega w(1, \ldots, 1, f_{s+1}, \ldots, f_r, 1, \ldots, 1) \). Then, \( f' \) is the desired one. Indeed, as \( f' \in I(M_{s+1} \times \cdots \times M_r) \), \( \sigma(1, \ldots, 1, f_{s+1}, \ldots, f_r, 1, \ldots, 1) = \sigma(1, \ldots, 1, f_{s+1}, \ldots, f_r, 1, \ldots, 1) \). And we define \( f'' = \omega w(1, \ldots, 1, f_{s+1}, \ldots, f_r, 1, \ldots, 1) \). Then, \( f'' \) is the desired one. Indeed, as \( f'' \in I(M_{s+1} \times \cdots \times M_r) \), \( \sigma(1, \ldots, 1, f_{s+1}, \ldots, f_r, 1, \ldots, 1) = \sigma(1, \ldots, 1, f_{s+1}, \ldots, f_r, 1, \ldots, 1) \).

Let \( E^n \) be an \( n \)-dimensional Euclidean space. Let \( E(n) = I(E^n) \). Then \( E(n) \) is the semi-direct product group \( O(n) + R^n \), where \( O(n) \) is the orthogonal group of the \( n \)-dimensional Euclidean vector space \( R^n \) and, if \( (A, a), (B, b) \in E(n) \), \( (A, a)(B, b) = (AB, Ab + a) \).

**Lemma 4.** Let \( G \) be a subgroup of \( O(n) \), \( A \in O(n) \) and \( G = AGA^{-1} \). Let \( V = \{v \in R^n \mid \mu(h)v = v \text{ for all } h \in G\} \) and \( W = \{w \in R^n \mid Yw = w \text{ for all } Y \in G\} \). Then \( A(V) = W \) and hence \( A(V^\perp) = W^\perp \), where \( V^\perp \) and \( W^\perp \) are orthogonal complements in \( R^n \) of \( V \) and \( W \), respectively.

Let us consider \( E^n \) as the direct product Riemannian manifold \( E^r \times E^{n-r} \) of the Euclidean spaces \( E^r \) and \( E^{n-r} \).

**Lemma 5.** Let \( I' \) be a subgroup of \( E(r) = I(E^r) \) and \( f \in E(n) \). If \( f I' f^{-1} \subset E(r) \), then there exists \( f' \in E(r) \) satisfying \( f' h f'^{-1} = f h f^{-1} \) for all \( h \in I' \).

**Proof.** Let \( I' = f I' f^{-1} \), \( V_0 = \{v \in R^n \mid \mu(h)v = v \text{ for all } h \in I'\} \) and \( W_0 = \{w \in R^n \mid \mu(h)w = w \text{ for all } h \in I'\} \), where \( \mu \) is the projection \( E(n) \to O(n) \). Let \( V = V_0 \oplus R^{n-r} \) and \( W = W_0 \oplus R^{n-r} \). Then \( R^r = V^\perp \oplus W_0 = W^\perp \oplus V_0 \) and \( f \) is considered as a mapping \( f: V^\perp \oplus V \to W^\perp \oplus W \), where \( V^\perp \) and \( W^\perp \) are orthogonal complements in \( R^n \) of \( V \) and \( W \), respectively. Then, by Lemma 4, \( \mu(f)(V^\perp) = W^\perp \) and \( \mu(f)(V) = W \). Let \( h = (X, x) \in I' \) and \( f = (A, a) \). Then \( X|_v = 1 \) and \( x \in R^r \). On the other hand, we have

\[
(*) \quad fh f^{-1} = (A, a)(X, x)(A^{-1}, -A^{-1}a) = (AXA^{-1}, -AXA^{-1}a + Ax + a).
\]

Since \( fh f^{-1} \in E(r) \), we have \( AXA^{-1} \in O(r) \), \( -AXA^{-1}a + Ax + a \in R^r \) and \( AXA^{-1}|_v = 1 \). Here, \( a \) is written as \( a = a' + a'' \), where \( a' \in W^\perp \) and \( a'' \in W \). Then \( -AXA^{-1}a + a = -AXA^{-1}a' + a' \in W^\perp \subset R^r \). Thus \( Ax \in R^r \), as \( -AXA^{-1}a + Ax + a \in R^r \).

Now, let \( U = \{v \in V_0 \mid Av \in R^r \} \). Then \( V_0 = U \oplus U^\perp \), where \( U^\perp \) is the orthogonal complement of \( U \) in \( V_0 \). Let \( A' \) be an element of \( O(r) \).
satisfying $A'_U v = A_U v$, $A'_{U^*} = 1$ and $A'(U^*) = A(U)^*$, where $A(U)^*$ is the orthogonal complement of $A(U)$ in $W_0$. Then $f' = (A', A') \in E(v)$ is the desired one. In fact, let $h = (X, x) \in \Gamma$. If $v \in V^\perp$, then $AXA^{-1}(Av) = AX(v) = A'XA'^{-1}(Av)$ as $Xv \in V^\perp$. If $v \in U$, then $AXA^{-1}(Av) = Av = A'XA'^{-1}(Av)$ as $Xv = v$. If $v \in R^e$, then $AXA^{-1}(Av) = Av = A'XA'^{-1}(Av)$ as $Xv = v$. If $v \in U_1$, then $AXA^{-1}(Av) = Av = A'XA'^{-1}(Av)$ as $Xv = v$ and $Av \in R^e$. Then we have $AXA^{-1} = A'XA'^{-1}$. Moreover, $Ax = A'x$ as $x \in V^\perp \oplus U$. Then $AXA^{-1} = f'hf'^{-1}$. q.e.d.

**Lemma 6.** Let $M$, $N$ and $B$ be complete Riemannian manifolds. If $M \times B$ is isometric to $N \times B$, then $\bar{M}$ is isometric to $\bar{N}$, where $\bar{M}$ and $\bar{N}$ are universal Riemannian covering manifolds of $M$ and $N$, respectively.

**Proof.** Let $p: \bar{M} \to M$, $p': \bar{N} \to N$ and $q: \bar{B} \to B$ be the universal Riemannian coverings. And let $\phi: M \times B \to N \times B$ be an isometry. Then, the covering $\phi^*(p, q): M \times B \to N \times B$ has a lift $\tilde{\phi}: M \times B \to \bar{N} \times \bar{B}$, since $M \times B$ and $\bar{N} \times \bar{B}$ are simply connected. Then $\tilde{\phi}$ is a covering and a local isometry as $(\phi^*, q) \circ \tilde{\phi} = \phi^*(p, q)$. Hence, $\tilde{\phi}$ is an isometry. Thus $\bar{M}$ is isometric to $\bar{N}$ by de Rham's decomposition theorem. q.e.d.

**Lemma 7.** Let $\bar{M}$ be a simply connected and complete Riemannian manifold. Let $\Gamma$ and $\bar{\Gamma}$ be subgroups of $I(\bar{M})$ acting freely and properly discontinuously on $\bar{M}$. Then the quotient $\bar{M}/\Gamma$ is isometric to the quotient $\bar{M}/\bar{\Gamma}$ if and only if there exists an element $f \in I(\bar{M})$ satisfying $f\Gamma f^{-1} = \bar{\Gamma}$.

**Proof.** See Wolf [2].

**Remark.** Let $\bar{M}$ and $\bar{N}$ be simply connected and complete Riemannian manifolds. Let $\tilde{\phi}: \bar{M} \to \bar{N}$ be an isometry. Let $\Gamma$ and $\bar{\Gamma}$ be subgroups of $I(\bar{M})$ acting freely and properly discontinuously on $\bar{M}$. Then, $\Delta = \tilde{\phi}\Gamma\tilde{\phi}^{-1}$ and $\bar{\Delta} = \tilde{\phi}\bar{\Gamma}\tilde{\phi}^{-1}$ are subgroups of $I(\bar{N})$ acting freely and properly discontinuously on $\bar{N}$. And $\tilde{\phi}$ induces natural isometries $\phi: \bar{M}/\Gamma \to \bar{N}/\Delta$ and $\tilde{\phi}: \bar{M}/\Gamma \to \bar{N}/\bar{\Delta}$. If there exists $f \in I(\bar{M})$ satisfying $\bar{\Gamma} = f\bar{\Gamma}f^{-1}$, then $\Delta = (\tilde{\phi}f\tilde{\phi}^{-1})\Delta(\tilde{\phi}f\tilde{\phi}^{-1})^{-1}$. Conversely, if there exists $g \in I(\bar{N})$ satisfying $\bar{\Delta} = g\Delta g^{-1}$, then $\bar{\Gamma} = (\tilde{\phi}^{-1}g\tilde{\phi})^*(\tilde{\phi}^{-1}g\tilde{\phi})^{-1}$. By Lemma 7, $\bar{M}/\Gamma$ is isometric to $\bar{M}/\Gamma$ if and only if $\bar{N}/\bar{\Delta}$ is isometric to $\bar{N}/\bar{\Delta}$.

**Theorem A.** Let $M$ and $N$ be complete Riemannian manifolds. Let $\bar{B}$ be a simply connected and complete Riemannian manifolds. If $M \times \bar{B}$ is isometric to $N \times \bar{B}$, then $M$ is isometric to $N$.

**Proof.** By Lemma 6, we may assume that $M$ and $N$ are isometric
to \(M/\Gamma\) and \(\tilde{M}/\tilde{\Gamma}\), respectively, where \(\Gamma\) and \(\tilde{\Gamma}\) are subgroups of \(I(M)\) acting freely and properly discontinuously on \(M\). Since \(M/\Gamma \times \tilde{B}\) is isometric to \(\tilde{M}/\tilde{\Gamma} \times \tilde{B}\), there exists \(f \in I(\tilde{M} \times \tilde{B})\) satisfying \(f \Gamma f^{-1} = \tilde{\Gamma}\) by Lemma 7. It is sufficient to prove that there exists \(f' \in I(\tilde{M})\) satisfying \(f'h f'^{-1} = fh f^{-1}\) for all \(h \in \Gamma\).

Now, by de Rham’s decomposition theorem, we may assume that \(\tilde{M}\) and \(\tilde{B}\) are isometric to the direct product Riemannian manifolds \(N_0 \times N_1 \times \cdots \times N_m \times N^*\) and \(B_0 \times B_1 \times \cdots \times B_m \times B^*\), respectively, which have the following properties (1)-(4):

1. \(N_0, \ldots, N_m, N^*, B_0, \ldots, B_m\) and \(B^*\) are all simply connected and complete.
2. \(N_0 \times B_0\) is a Euclidean space.
3. For each \(i \in \{1, \ldots, m\}\), \(N_i \times B_i\) is a product of some Riemannian manifolds which are all isometric to one simply connected and complete Riemannian manifold \(M_i\) whose homogeneous holonomy group is irreducible. And if \(i \neq j\), then \(M_i\) is not isometric to \(M_j\).
4. Any component of de Rham’s decomposition of \(N^* \times B^*\) has the irreducible homogeneous holonomy group. And any component of \(N^*\) is not isometric to any of \(B^*\).

By the above remark, we may suppose \(M = N_0 \times N_1 \times \cdots \times N_m \times N^*\) and \(B = B_0 \times B_1 \times \cdots \times B_m \times B^*\). Moreover, we have a natural isometry \(\phi: M \times B \to P = (N_0 \times B_0) \times \cdots \times (N_m \times B_m) \times N^* \times B^*\). By the uniqueness of de Rham’s decomposition, we have \(I(P) = I(N_0 \times B_0) \times \cdots \times I(N_m \times B_m) \times I(N^*) \times I(B^*)\), (cf. Uesu [1]). Since \(\Gamma\) and \(\tilde{\Gamma}\) are contained in \(I(M)\), \(\phi \Gamma \phi^{-1}\) and \(\phi \tilde{\Gamma} \phi^{-1}\) are contained in \(I(N_0) \times \cdots \times I(N_m) \times I(N^*) \times \{1\}\), where \(I(N_i)\) is interpreted as \(I(N_i) \subset I(N_i \times B_i)\) for each \(i \in \{0, 1, \ldots, m\}\). Again, by the remark, we may consider \(\phi \Gamma \phi^{-1}\) and \(\phi \tilde{\Gamma} \phi^{-1}\) as \(\Gamma\) and \(\tilde{\Gamma}\), respectively. Then, it is sufficient to prove the following: Let \(\Gamma\) and \(\tilde{\Gamma}\) be subgroups of \(I(N_0) \times \cdots \times I(N_m) \times I(N^*) \times \{1\}\). If there exists \(f \in I(P)\) satisfying \(f \Gamma f^{-1} = \Gamma\), then there exists \(f' \in I(N_0) \times \cdots \times I(N_m) \times I(N^*) \times \{1\}\) satisfying \(f'h f'^{-1} = fh f^{-1}\) for all \(h \in \Gamma\).

Indeed, \(f\) is written as \(f = (g_0, g_1, \cdots, g_m, g^*, g^{**})\), where \(g_i \in I(N_i \times B_i)\), \(g^* \in I(N^*)\) and \(g^{**} \in I(B^*)\). \(h\) is written as \(h = (k_0, k_1, \cdots, k_m, k^*, 1)\), where \(k_0 \in I(N_i)\) and \(k^* \in I(N^*)\). Then \(f'h f^{-1} = (g_0 k_0 g_0^{-1}, g_1 k_1 g_1^{-1}, \cdots, g_m k_m g_m^{-1}, g^* k^* g^{*-1}, 1)\). Now, the assertion is clear by Lemmas 3 and 5.

**Theorem B.** Let \(B\) be a complete Riemannian manifold whose restricted homogeneous holonomy group is irreducible. And let \(M\) and \(N\) be complete Riemannian manifolds. If \(M \times B\) is isometric to \(N \times B\), then \(M\) is isometric to \(N\).
Proof. Let $\tilde{B}$ and $\tilde{M}$ be universal Riemannian covering manifolds of $B$ and $M$, respectively. Then, by Lemma 6, $M$ and $B$ are isometric to the quotients $\tilde{M}/\Gamma$, $\tilde{M}/\tilde{\Delta}$ and $\tilde{B}/\Lambda$, respectively. Then, by Lemma 7, it is sufficient to prove: If there exists $f \in I(\tilde{M} \times \tilde{B})$ satisfying $f(\Gamma \times \Lambda)f^{-1} = \Gamma \times \Delta$, then there exists $f' \in I(\tilde{M})$ satisfying $f'\tilde{\Gamma}f'^{-1} = \tilde{\Gamma}$.

Let $M_1 \times \cdots \times M_n = \tilde{M}$ be de Rham's decomposition of $\tilde{M} \times \tilde{B}$, where $\tilde{M} = M_1 \times \cdots \times M_n$ and $\tilde{B} = B_n$. Then, by the uniqueness of de Rham's decomposition, $I(\tilde{M} \times \tilde{B})$ is generated by $I(M_1), \cdots, I(M_n)$ and by all permutations of $M_i$'s which are isometric to each other, where we identify $M_i$ with $M_j$ by an isometry if $M_i$ is isometric to $M_j$ (cf. Uesu [1]). Moreover, we have a statement similar to (ii) of Lemma 2. Then $f$ is written as $f = \sigma(f_1, \cdots, f_{n-1}, f_n)$, where $\sigma \in S_n$ and $(f_1, \cdots, f_{n-1}, f_n) \in I(M_1) \times \cdots \times I(M_{n-1}) \times I(M_n)$. Let $r = \sigma(n)$ and $s = \sigma^{-1}(n)$. Suppose $r \leq n - 1$ and hence $s \leq n - 1$. Then $M_r, M_s$ and $M_n$ are isometric to each other. We shall prove

$$\tilde{\Gamma} = g_r \sigma_r g_r^{-1} \times (f \tilde{\Gamma} f^{-1} \cap \tilde{\Gamma}) \quad \text{(the direct product group)},$$

where $g_r = (1, \cdots, f_r, 1 \cdots, 1, 1) \in I(M_r)$, $f \tilde{\Gamma} f^{-1} \cap \tilde{\Gamma} \subset I(M_1 \times \cdots \times M_{r-1} \times M_{r+1} \times \cdots \times M_n)$ and $\sigma_r$ is the group $\Delta$ considered as a subgroup of $I(M_r)$. Let $h = \tau(h_r, \cdots, h_{n-1}, h_n) \in \Gamma \times \Delta$, where $\tau \in S_{n-1}$ and $(h_1, \cdots, h_{n-1}, h_n) \in I(M_1) \times \cdots \times I(M_{n-1}) \times I(M_n)$. Then

$$f h f^{-1} = \sigma \tau \sigma^{-1}(f_{r-1}(1), h_{r-1}(1), f_{r+1}(1), \cdots, f_{n-1}(1), h_{n-1}(1), f_n(1)), \cdots, f_{r-1}(1), h_{r-1}(1), f_{r+1}(1), \cdots, f_{n-1}(1), h_{n-1}(1), f_n(1)) \in I(M_r),$$

where $(f_{r-1}(1), h_{r-1}(1), f_{r+1}(1), \cdots, f_{n-1}(1), h_{n-1}(1), f_n(1)), \cdots, f_{r-1}(1), h_{r-1}(1), f_{r+1}(1), \cdots, f_{n-1}(1), h_{n-1}(1), f_n(1)) \in I(M_r)$. Since $\sigma \tau \sigma^{-1}(r) = r$, we have $\tilde{\Gamma} \subset I(M_r) \times I(M_1 \times \cdots \times M_{r-1} \times M_{r+1} \times \cdots \times M_n)$. Moreover $g_r \sigma_r g_r^{-1} = f \tilde{\Gamma} f^{-1} \subset \tilde{\Gamma} \cap I(M_r)$. Next, let $\tilde{h} \in \tilde{\Gamma}$. Then $\tilde{h}$ is written as $\tilde{h} = f h f^{-1}$, where $h \in \Gamma \times \Delta$. But $h$ is written as $h = h' h''$, where $h' \in \Gamma$, $h'' \in \Delta$. Then $f h f^{-1} = f h' f^{-1} f h'' f^{-1}$. Since $f h' f^{-1} \in I(M_r)$, we have $f h' f^{-1} \in \tilde{\Gamma}$. Hence $f h' f^{-1} \in f \tilde{\Gamma} f^{-1} \cap \tilde{\Gamma}$. By the above argument, it is evident that $f h' f^{-1} \in I(M_1 \times \cdots \times M_{r-1} \times M_{r+1} \times \cdots \times M_n)$.

Now, let $\omega$ be the transposition $(r, n) \in I(M_r \times M_n)$ and $f' = g \omega \sigma(f_1, \cdots, f_{n-1}, 1) \in I(\tilde{M})$. Then $f'$ is the desired one. In fact, let $h = \tau(h_r, \cdots, h_{n-1}, 1) \in \Gamma$. As $\tau \in S_{n-1}$, we have $f h f^{-1} = \sigma(f_1, \cdots, f_{n-1}, 1) h(f_1, \cdots, f_{n-1}, 1)^{-1} \sigma^{-1}$. On the other hand, as $f h f^{-1} \in \tilde{\Gamma} \times \Delta$, we have $\sigma \tau \sigma^{-1}(n) = n$, that is, $\tau(s) = s$ and hence $f h f^{-1}$ is written as $f h f^{-1} = (h', f h_s f_s^{-1})$. where $(h', 1) \in f \tilde{\Gamma} f^{-1} \cap \tilde{\Gamma} \subset \tilde{\Gamma} \cap I(M_r \times \cdots \times M_{r-1} \times M_{r+1} \times \cdots \times M_n)$ and $(1, \cdots, 1, f h_s f_s^{-1}) \in \Delta \subset I(M_r)$. Then we have $\omega \sigma(f_1, \cdots, f_{n-1}, 1) h(f_1, \cdots, f_{n-1}, 1)^{-1} \sigma^{-1} \omega^{-1} = (1, \cdots, 1, f h_s f_s^{-1}, 1, \cdots, 1, 1, 1)(h', 1) \in I(M_r) \times (f \tilde{\Gamma} f^{-1} \cap \tilde{\Gamma})$. Hence $f' h f'^{-1} = g_r(1, f h_s f_s^{-1}, 1, \cdots, 1, 1, 1) \sigma(g', 1) \in (g \Delta, g_r^{-1} \times (f \tilde{\Gamma} f^{-1} \cap \tilde{\Gamma}) = \tilde{\Gamma}$. Thus, $f' \tilde{\Gamma} f'^{-1} \subset \tilde{\Gamma}$. By the above argument, it is evident that $f' \tilde{\Gamma} f'^{-1} = \tilde{\Gamma}$.

q.e.d.
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