REAL ANALYTIC SL(n, R) ACTIONS ON SPHERES

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0. Introduction. Let SL(n, R) denote the group of all n×n real matrices of determinant 1. In the previous paper [12], we classified real analytic SL(n, R) actions on the standard n-sphere for each n ≥ 3. In this paper we study real analytic SL(n, R) actions on the standard m-sphere for 5 ≤ n ≤ m ≤ 2n - 2. We shall show that such an action is characterized by a certain real analytic R× action on a homotopy (m - n + 1)-sphere. Here R× is the multiplicative group of all non-zero real numbers.

In Section 1 we construct a real analytic SL(n, R) action on the standard (n+k-1)-sphere from a real analytic R× action on a homotopy k-sphere satisfying a certain condition for each n+k ≥ 6. In Section 3 we state a structure theorem for a real analytic SL(n, R) action which satisfies a certain condition on the restricted SO(n) action, and in Section 5 we state a decomposition theorem and a classification theorem. In Section 6 we construct real analytic R× actions on the standard k-sphere. It can be seen that there are infinitely many (at least the cardinality of the real numbers) mutually distinct real analytic SL(n, R) actions on the standard m-sphere.

1. Construction. Let ψ: R× × Σ → Σ be a real analytic R× action on a real analytic closed manifold Σ which is homotopy equivalent to the k-sphere. Define a real analytic involution T of Σ by T(x) = ψ(-1, x) for x ∈ Σ. Put F = F(R×, Σ), the fixed point set. We say that the action ψ satisfies the condition (P) if
   (i) there exists a compact contractible k-dimensional submanifold X of Σ such that X ∪ TX = Σ and X ∩ TX = F,
   (ii) there exists a real analytic R× equivariant isomorphism j of R× × F onto an open set of Σ such that j(0, x) = x for x ∈ F. Here R× acts on R by the scalar multiplication.

Notice that F = F(T, Σ), the fixed point set of the involution T by the condition (i), and hence F is a real analytic (k-1)-dimensional closed submanifold of Σ. Define a map

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by \( f(u, x) = (u, j(1, x)) \) for \( u \in \mathbb{R}^n - 0, x \in F \). Then the map \( f \) is a real analytic \( SL(n, \mathbb{R}) \) equivariant isomorphism of \((\mathbb{R}^n - 0) \times F\) onto an open set of \((\mathbb{R}^n - 0) \times_\mathbb{R}(\Sigma - F)\), where \( SL(n, \mathbb{R}) \) acts naturally on \( \mathbb{R}^n, \mathbb{R}^{\times} \) acts on \( \mathbb{R}^n \) by the scalar multiplication and \( \mathbb{R}^{\times} \) acts on \( \Sigma \) by the given action \( \psi \). Here \((\mathbb{R}^n - 0) \times_\mathbb{R}(\Sigma - F)\) is the quotient of \((\mathbb{R}^n - 0) \times (\Sigma - F)\) obtained by identifying \((u, y)\) with \((t^{-1}u, \psi(t, y))\) for \( u \in \mathbb{R}^n - 0, y \in \Sigma - F, t \in \mathbb{R}^{\times}\). Put

\[
M(\psi, j) = \mathbb{R}^n \times F \sqcup \left((\mathbb{R}^n - 0) \times_\mathbb{R}(\Sigma - F)\right),
\]

which is the space formed from the disjoint union of \( \mathbb{R}^n \times F \) and \((\mathbb{R}^n - 0) \times_\mathbb{R}(\Sigma - F)\) by identifying \((u, x)\) with \( f(u, x)\) for \( u \in \mathbb{R}^n - 0, x \in F\). By the construction, it can be seen that the space \( M(\psi, j) \) is a compact Hausdorff space with \( SL(n, \mathbb{R}) \) action, and \( M(\psi, j) \) admits a real analytic structure so that the \( SL(n, \mathbb{R}) \) action is real analytic.

**PROPOSITION 1.1.** (a) Let \( j_1: \mathbb{R}\times F \to \mathbb{R}^{\times} \) be a real analytic \( \mathbb{R}^{\times} \) equivariant isomorphism of \( \mathbb{R}\times F \) onto an open set of \( \Sigma \) such that \( j_1(0, x) = x \) for \( x \in F \). Then \( M(\psi, j_1) \) is real analytically isomorphic to \( M(\psi, j) \) as \( SL(n, \mathbb{R}) \) manifolds.

(b) Suppose \( n \geq 1 \) and \( n + k \geq 6 \). Then \( M(\psi, j) \) is real analytically isomorphic to the standard \((n + k - 1)\)-sphere.

**PROOF.** It is easy to see that there is a real analytic function \( s: F \to \mathbb{R}^{\times} \) such that \( j_1(t, x) = j(s(x)t, x) \) for \( t \in \mathbb{R}, x \in F \). Let \( g \) be a real analytic automorphism of the disjoint union of \( \mathbb{R}^n \times F \) and \((\mathbb{R}^n - 0) \times_\mathbb{R}(\Sigma - F)\) defined by

\[
g(u, x) = (s(x)u, x) \quad \text{for} \quad u \in \mathbb{R}^n, \quad x \in F,
\]

\[
g(v, y) = (v, y) \quad \text{for} \quad v \in \mathbb{R}^n - 0, \quad y \in \Sigma - F.
\]

Then it is easy to see that \( g \) induces a real analytic \( SL(n, \mathbb{R}) \) equivariant isomorphism of \( M(\psi, j_1) \) onto \( M(\psi, j) \).

To show (b), we consider the restricted \( SO(n) \) action on \( M(\psi, j) \). We can assume \( j([0, \infty) \times F) \subset X \) by the condition (P). Put \( X_1 = X - j([0, 1) \times F) \). Let \( D^* \) denote the closed unit disk of \( \mathbb{R}^n \). Let \( \partial Y \) denote the boundary of a given manifold \( Y \). Then it can be seen that there exists an equivariant diffeomorphism

\[
M(\psi, j) = D^* \times F \sqcup \partial D^* \times X_1,
\]

as smooth \( SO(n) \) manifolds, where \( h: \partial D^* \times F \to \partial D^* \times \partial X_1 \) is a \( C^\infty \) diffeomorphism defined by \( h(u, x) = (u, j(1, x)) \) for \( u \in \partial D^*, x \in F \). Hence
$M(\psi, j)$ is $C^\infty$ diffeomorphic to $\partial(D^* \times X_i)$. Here $X_i$ is a compact contractible $k$-manifold; hence $\partial(D^* \times X_i)$ is simply connected for $n \geq 1$. Therefore $M(\psi, j)$ is $C^\infty$ diffeomorphic to the standard $(n+k-1)$-sphere for $n+k \geq 6$ by the $h$-cobordism theorem (cf. Milnor [8, Theorem 9.1]). It is known by Grauert [3] and Whitney [13, Part III] that two real analytic paracompact manifolds are real analytically isomorphic if they are $C^\infty$ diffeomorphic. Consequently, $M(\psi, j)$ is real analytically isomorphic to the standard $(n+k-1)$-sphere for $n+k \geq 6$. q.e.d.

REMARK. By the condition (P), it is shown that $\Sigma$ is real analytically isomorphic to the standard $k$-sphere for $k \geq 5$ by the $h$-cobordism theorem.

2. Certain subgroups of $SL(n, R)$. As usual we regard $M_n(R)$ with the bracket operation $[A, B] = AB - BA$ as the Lie algebra of $GL(n, R)$. Let $\mathfrak{sl}(n, R)$ and $\mathfrak{so}(n)$ denote the Lie subalgebras of $M_n(R)$ corresponding to the subgroups $SL(n, R)$ and $SO(n)$ respectively. Then

$$\mathfrak{sl}(n, R) = \{ X \in M_n(R) : \text{trace } X = 0 \} ,$$
$$\mathfrak{so}(n) = \{ X \in M_n(R) : X \text{ is skew symmetric} \} .$$

Define certain linear subspaces of $\mathfrak{sl}(n, R)$ as follows:

$$\mathfrak{sl}(n-r, R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} : A \text{ is } (n-r) \times (n-r) \text{ matrix of trace } 0 \right\} ,$$
$$\mathfrak{so}(n-r) = \mathfrak{so}(n) \cap \mathfrak{sl}(n-r, R) ,$$
$$\mathfrak{sym}(n-1) = \{ X \in \mathfrak{sl}(n-1, R) : X \text{ is symmetric} \} ,$$
$$a = \{ (a_{ij}) \in \mathfrak{sl}(n, R) : a_{ij} = 0 \text{ for } i \neq 1 \} ,$$
$$a^* = \{ (a_{ij}) \in \mathfrak{sl}(n, R) : a_{ij} = 0 \text{ for } j \neq 1 \} ,$$
$$b = \{ (a_{ij}) \in \mathfrak{sl}(n, R) : a_{ij} = 0 \text{ for } i \neq j, a_{22} = a_{33} = \cdots = a_{nn} \} .$$

Then

$$\mathfrak{sl}(n, R) = \mathfrak{sl}(n-1, R) \oplus a \oplus a^* \oplus b ,$$
$$\mathfrak{sl}(n-1, R) = \mathfrak{so}(n-1) \oplus \mathfrak{sym}(n-1)$$

as direct sums of vector spaces. Moreover we have

$$[a, a^*] = \mathfrak{sl}(n-1, R) \oplus b ,$$

(2.1) $[a, a] = [a^*, a^*] = [b, b] = [b, \mathfrak{sl}(n-1, R)] = 0 ,$
$$[a, b] = [a, \mathfrak{sl}(n-1, R)] = a , \quad [a^*, b] = [a^*, \mathfrak{sl}(n-1, R)] = a^* .$$

Let $SL(n-r, R)$ and $SO(n-r)$ denote the connected subgroups of $SL(n, R)$ corresponding to the Lie subalgebras $\mathfrak{sl}(n-r, R)$ and $\mathfrak{so}(n-r)$, respectively.
Let $\text{Ad}: \text{SL}(n,\mathbb{R}) \rightarrow \text{GL}(\mathfrak{sl}(n, \mathbb{R}))$ be the adjoint representation defined by $\text{Ad}(A)X = AXA^{-1}$ for $A \in \text{SL}(n,\mathbb{R})$, $X \in \mathfrak{sl}(n,\mathbb{R})$. Then the linear subspaces $\mathfrak{sl}(n-1,\mathbb{R})$, $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{b}$ are $\text{Ad}(\text{SL}(n-1,\mathbb{R}))$ invariant, and the linear subspaces $\mathfrak{so}(n-1)$ and $\mathfrak{sym}(n-1)$ are $\text{Ad}(\text{SO}(n-1))$ invariant. Moreover, the linear subspaces $\mathfrak{sym}(n-1)$, $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{b}$ are irreducible $\text{Ad}(\text{SO}(n-1))$ spaces respectively for each $n \geq 3$. Put

$$t(p, q) = \begin{cases} 
0 & qx_2 \cdots qx_n \\
p x_2 & 0 \cdots 0 \\
0 & 0 \cdots 0 \\
q x_2 & \cdots & \cdots & \cdots & \cdots & \cdots \\
p x_n & 0 & \cdots & 0 
\end{cases} : x_i \in \mathbb{R}$$

for $p, q \in \mathbb{R}$. Then $t(p, q)$ is an $\text{Ad}(\text{SO}(n-1))$ invariant linear subspace of $\mathfrak{a} \oplus \mathfrak{a}^*$, and we have

$$[t(p, q), \mathfrak{sym}(n-1)] = [t(p, q), \mathfrak{b}] = t(p, -q), \quad (2.2)$$

$$[t(p, q), t(p, q)] = \begin{cases} 0 & \text{for } pq = 0 \\
\mathfrak{so}(n-1) & \text{for } pq \neq 0. 
\end{cases}$$

**Lemma 2.3.** Suppose $n \geq 3$. Let $\mathfrak{g}$ be a proper Lie subalgebra of $\mathfrak{sl}(n, \mathbb{R})$ which contains $\mathfrak{so}(n-1)$. Then $\mathfrak{g}$ is one of the following: $\mathfrak{so}(n-1)$, $\mathfrak{so}(n-1) \oplus \mathfrak{b}$, $\mathfrak{so}(n-1) \oplus \mathfrak{a}$, $\mathfrak{so}(n-1) \oplus \mathfrak{a}^*$, $\mathfrak{so}(n-1) \oplus t(p, q)$ for $pq \neq 0$, $\mathfrak{so}(n-1) \oplus \mathfrak{a} \oplus \mathfrak{b}$, $\mathfrak{so}(n-1) \oplus \mathfrak{a}^* \oplus \mathfrak{b}$, $\mathfrak{sl}(n-1, \mathbb{R})$, $\mathfrak{sl}(n-1, \mathbb{R}) \oplus \mathfrak{b}$, $\mathfrak{sl}(n-1, \mathbb{R}) \oplus \mathfrak{a}$, $\mathfrak{sl}(n-1, \mathbb{R}) \oplus \mathfrak{a}^*$, $\mathfrak{sl}(n-1, \mathbb{R}) \oplus \mathfrak{a} \oplus \mathfrak{b}$, $\mathfrak{sl}(n-1, \mathbb{R}) \oplus \mathfrak{a}^* \oplus \mathfrak{b}$.

**Proof.** Since $\mathfrak{g}$ contains $\mathfrak{so}(n-1)$, $\mathfrak{g}$ is an $\text{Ad}(\text{SO}(n-1))$ invariant linear subspace of $\mathfrak{sl}(n, \mathbb{R})$. Hence we have $\mathfrak{g} = \mathfrak{so}(n-1) \oplus (\mathfrak{g} \cap \mathfrak{sym}(n-1)) \oplus (\mathfrak{g} \cap (\mathfrak{a} \oplus \mathfrak{a}^*)) \oplus (\mathfrak{g} \cap \mathfrak{b})$ as a direct sum of $\text{Ad}(\text{SO}(n-1))$ invariant linear subspaces. Since $\mathfrak{sym}(n-1)$ is irreducible, we have $\mathfrak{g} \cap \mathfrak{sym}(n-1) = 0$ or $\mathfrak{sym}(n-1)$. Since $\mathfrak{g}$ is a proper Lie subalgebra of $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{g}$ does not contain $\mathfrak{a} \oplus \mathfrak{a}^*$ by (2.1). Suppose $n \geq 4$. Then we derive that $\mathfrak{g} \cap (\mathfrak{a} \oplus \mathfrak{a}^*)$ coincides with certain $t(p, q)$. If $\mathfrak{g}$ contains $\mathfrak{sym}(n-1)$, then (2.2) implies that $\mathfrak{g} \cap (\mathfrak{a} \oplus \mathfrak{a}^*) = 0$, $\mathfrak{a}$ or $\mathfrak{a}^*$. Now we can prove the lemma for $n \geq 4$ by a routine work from (2.1) and (2.2). The proof for $n = 3$ is similar, so we omit the detail.

**Remark.** Let $G(p, q)$ denote the connected Lie subgroup of $\text{SL}(n, \mathbb{R})$ corresponding to the Lie subalgebra $\mathfrak{so}(n-1) \oplus t(p, q)$ for $pq \neq 0$. If $pq < 0$, then $G(p, q)$ is conjugate to $G(1, -1) = \text{SO}(n)$. If $pq > 0$, then $G(p, q)$ is conjugate to $G(1, 1)$, which is non-compact.
LEMMA 2.4. (i) Assume that \( g \) is one of the following:
\[
\mathfrak{so}(n-1), \mathfrak{so}(n-1) \oplus b, \mathfrak{so}(n-1) \oplus a, \mathfrak{so}(n-1) \oplus a \oplus b,
\]
\[
\mathfrak{so}(n-1) \oplus \mathfrak{fr}(p, q) \text{ for } pq \neq 0, \mathfrak{sl}(n-1, R), \mathfrak{sl}(n-1, R) \oplus b.
\]
Then \( \mathfrak{so}(n) \cap \text{Ad}(X_1)g = \mathfrak{so}(n-2) \).

(ii) Assume that \( g \) is one of the following:
\[
\mathfrak{so}(n-1)(+a^*), \mathfrak{so}(n-1)(+a^*+b).
\]
Then \( \mathfrak{so}(n) \cap \text{Ad}(X_{-1})g = \mathfrak{so}(n-2) \).

PROOF. Since \( \mathfrak{so}(n) \cap \text{Ad}(X_1)g = \{ A \in \mathfrak{so}(n) : X_1^{-1}AX_1 \in g \} \), we have the desired equations by a routine work from the following relation:

\[
\begin{pmatrix}
a_{11} + a_{12} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} + a_{22} - a_{11} - a_{12} & a_{22} - a_{12} & a_{23} - a_{13} & \cdots & a_{2n} - a_{1n} \\
a_{31} + a_{32} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} + a_{n2} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{pmatrix}
\]

Let \( L(n), L^*(n), N(n) \) and \( N^*(n) \) denote the connected Lie subgroups of \( SL(n, R) \) corresponding to the Lie subalgebras \( \mathfrak{sl}(n-1, R) \oplus a, \mathfrak{sl}(n-1, R) \oplus a^* \), \( \mathfrak{sl}(n-1, R) \oplus a \oplus b \) and \( \mathfrak{sl}(n-1, R) \oplus a^* \oplus b \), respectively. Then these are closed subgroups of \( SL(n, R) \).

PROPOSITION 2.5. Suppose \( n \geq 3 \). Let \( M \) be an \( SL(n, R) \) space. Assume that the restricted \( SO(n) \) action on \( M \) has at most two orbit types \( SO(n)/SO(n-1) \) and \( SO(n)/SO(n) \). Then the identity component of an isotropy group of the \( SL(n, R) \) action on \( M \) is conjugate to one of the following: \( L(n), L^*(n), N(n) N^*(n) \) and \( SL(n, R) \).

PROOF. Let \( g \) be the Lie algebra corresponding to an isotropy group. By the assumption on the restricted \( SO(n) \) action, we see that \( \text{Ad}(x)g \) contains \( \mathfrak{so}(n-1) \) for some \( x \in SL(n, R) \). Such a Lie subalgebra is determined by Lemma 2.3. Moreover, we can derive \( \mathfrak{so}(n) \cap \text{Ad}(y)g \neq \mathfrak{so}(n-2) \) for any \( y \in SL(n, R) \) by the assumption on the restricted \( SO(n) \) action. Hence we see that \( g \) is one of the following up to conjugation: \( \mathfrak{sl}(n-1, R) \oplus c, \mathfrak{sl}(n-1, R) \oplus a^*, \mathfrak{sl}(n-1, R) \oplus a \oplus b, \mathfrak{sl}(n-1, R) \oplus a^* \oplus b, \mathfrak{sl}(n, R) \) by Lemma 2.3 and Lemma 2.4. On the other hand, it is easy
to see that the restricted $SO(n)$ actions on the homogeneous spaces $SL(n, \mathbb{R})/L(n)$, $SL(n, \mathbb{R})/L^*(n)$, $SL(n, \mathbb{R})/N(n)$ and $SL(n, \mathbb{R})/N^*(n)$ have only one orbit type $SO(n)/SO(n-1)$ respectively. q.e.d.

3. Structure theorem. Let $\phi: G \times M \to M$ be a real analytic $G$ action. Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on $G$. Let $L(M)$ denote the Lie algebra of all real analytic vector fields on $M$. Then we can define a Lie algebra homomorphism $\phi^+: \mathfrak{g} \to L(M)$ as follows (cf. Palais [10, Chapter II, Theorem II]):

$$\phi^+(X)_q(f) = \lim_{t \to 0} \frac{f(\phi(\exp(-tX), q)) - f(q)}{t}$$

for $X \in \mathfrak{g}$, $q \in M$ and a real analytic function $f$ defined on a neighborhood of $q$. It is easy to see that $\phi^+(X)_q = 0$ iff $q$ is a fixed point of the one-parameter subgroup $\{\exp tX\}$. For each subgroup $H$ of $G$, let $F(H, M)$ denote the fixed point set of the restricted $H$ action of $\phi$. Then $F(H, M)$ is a closed subset of $M$.

**Lemma 3.1.** Let $\phi: SL(n, \mathbb{R}) \times M \to M$ be a real analytic action. Let $p \in F(SL(n, \mathbb{R}), M)$. Suppose that there exists an analytic system of coordinates $(U; u_1, \cdots, u_m)$ with origin at $p$, such that

$$(*) \quad \phi^+(x_{ij})_q = -\sum_{i,j=1}^n x_{ij}u_j(q)\partial/\partial u_i$$

for $(x_{ij}) \in \mathfrak{sl}(n, \mathbb{R})$, $q \in U$. Then, (i) there exists an open neighborhood $V$ of $p$ in $F(SL(n, \mathbb{R}), M)$ and there exists an analytic isomorphism $h$ of $\mathbb{R}^n \times V$ onto an open set of $M$ such that

(a) $h(0, v) = v$ for $v \in V$,
(b) $h(gu, v) = \phi(g, h(u, v))$ for $g \in SL(n, \mathbb{R})$, $u \in \mathbb{R}^n$, $v \in V$.

Moreover, (ii) if pairs $(V_1, h_1)$ and $(V_2, h_2)$ satisfy the conditions (a), (b), then

$$h_1(\mathbb{R}^n \times V_1) \cap h_2(\mathbb{R}^n \times V_2) = h_1(\mathbb{R}^n \times (V_1 \cap V_2)),$$

and there exists a unique real analytic real valued function $f$ on $V_1 \cap V_2$ such that $h_1(u, v) = h_2(f(v)u, v)$ for $u \in \mathbb{R}^n$, $v \in V_1 \cap V_2$.

**Proof.** The assumption $(*)$ implies $F(SL(n, \mathbb{R}), M) \cap U = \{q \in U: u_i(q) = \cdots = u_n(q) = 0\}$. Define a real analytic isomorphism $k$ of $U$ onto an open set of $\mathbb{R}^n$ by $k(q) = (u_1(q), \cdots, u_m(q))$. There is a positive real number $r$ such that $D^r_r \times D^{n-r}_r \subset k(U)$, namely $(u_1, \cdots, u_m) \in k(U)$ for $(u_1, \cdots, u_m) \in D^r_r \times (u_{n+1}, \cdots, u_m) \in D^{n-r}_r$. Here we denote $D^r_r = \{(v_1, \cdots, v_n) \in \mathbb{R}^n: v_1^2 + \cdots + v_n^2 < r^2\}$. Consider the following curves

$$a(t) = a(t; X, u, v) = k(\phi(\exp tX, k^{-1}(u, v))) \quad \text{and} \quad b(t) = b(t; X, u, v) = ((\exp tX)u, v)$$
for $X \in \mathfrak{so}(n, \mathbb{R})$, $u \in D^*_r$, $v \in D^m_{-n}$. The curve $b(t)$ is defined for each $t \in \mathbb{R}$, the curve $a(t)$ is defined on an interval $(-t_1, t_2)$ for some positive real numbers $t_1, t_2$. Put $X = (x_{ij})$, $a(t) = (a_1(t), \ldots, a_m(t))$ and $b(t) = (b_1(t), \ldots, b_m(t))$. Then it follows from the assumption (*) that

$$
\left(\frac{d}{dt}\right)a_i(t) = \sum_{j=1}^n x_{ij} a_j(t) \quad \text{for} \quad 1 \leq i \leq n,
$$

$$
\left(\frac{d}{dt}\right)a_i(t) = 0 \quad \text{for} \quad n < i \leq m.
$$

On the other hand,

$$
\left(\frac{d}{dt}\right)b_i(t) = \sum_{j=1}^n x_{ij} b_j(t) \quad \text{for} \quad 1 \leq i \leq n,
$$

$$
\left(\frac{d}{dt}\right)b_i(t) = 0 \quad \text{for} \quad n < i \leq m
$$

by the definition of $b(t)$. Since $a(0) = b(0)$, we can derive that

$$
(**) \quad a(t; X, u, v) = b(t; X, u, v)
$$

on the interval $(-t_1, t_2)$. Put $u_0 = (r/2, 0, \ldots, 0) \in D^r$. Then it follows from the equation (**) that the identity component of an isotropy group at $k^{-1}(u_0, v)$ coincides with $L(n)$ for each $v \in D^m_{-n}$. Hence we can define a map $h': R^n \times D^m_{-n} \rightarrow M$ by

$$
h'(u, v) = \begin{cases} k^{-1}(0, v) & \text{for } u = 0, \\ \phi(g, k^{-1}(u_0, v)) & \text{for } u = gu_0, \quad g \in SL(n, \mathbb{R}). \end{cases}
$$

First we shall show that $kh' = \text{identity on } D^*_r \times D^m_{-n}$. Let $u \in D^r$ and $u \neq 0$. Then $u$ can be expressed as follows: $u = (\exp X_1 \cdot \exp X_2)u_0$ for $X_1 \in \mathfrak{so}(n)$, and $X_2$ is a diagonal matrix with diagonal components $c, -c, 0, \ldots, 0$ for $c \in \mathbb{R}$. The equation (**) implies that $k(\phi(\exp tX_2, k^{-1}(u_0, v))) = ((\exp tX_2)u_0, v)$ for $|t| \leq 1$ and $k(\phi(\exp tX_1, k^{-1}((\exp X_2)u_0, v))) = ((\exp tX_1)(\exp X_2)u_0, v)$ for $t \in \mathbb{R}$. Then we have $kh' = \text{identity on } D^*_r \times D^m_{-n}$. Since $k: U \rightarrow k(U)$ is a real analytic isomorphism, it follows that the restriction of $h'$ to $D^*_r \times D^m_{-n}$ is a real analytic isomorphism of $D^*_r \times D^m_{-n}$ onto an open set of $M$. On the other hand, the restriction of $h'$ to $(R^n - 0) \times D^m_{-n}$ is real analytic by definition. Moreover, the map $h'$ is $SL(n, \mathbb{R})$ equivariant by definition. Hence the map $h'$ is a real analytic local isomorphism at each point of $R^*_r \times D^m_{-n}$.

Now we shall show that $h'$ is an injection. Assume that $h'(g, u_0, v_1) = h'(g, u_0, v_2)$ for some $g \in SL(n \in \mathbb{R})$, $v_1 \in D^m_{-n}$. Since $h'$ is equivariant, we have $k^{-1}(u_0, v_1) = \phi(g, k^{-1}(u_0, v_2))$. Put $g = g_1^{-1}g_2$. Let $L_i$ be the identity component of the isotropy group at $k^{-1}(u_0, v_i)$. Then $L_i = gL_0g^{-1}$ and $L_i = L(n)$ by the assumption (*). Hence $g \in NL(n)$, the normalizer of $L(n)$ in $SL(n, \mathbb{R})$. The equation (**) implies that $k(\phi((x_{ij}), k^{-1}(u_0, v_2))) = \phi(g_1^{-1}g_2, k^{-1}(u_0, v_2))$.
\[(x_{ij}, u_0, v) \quad \text{for} \quad v \in D_{n}^{-n}, \quad (x_{ij}) \in NL(n), \quad 0 < |x_{ii}| < 2. \] We can choose \( g \) or \( g^{-1} \) as \((x_{ij})\) such that \( 0 < |x_{ii}| < 2. \) It follows that \( v_1 = v_2 \) and \( g = g_1^{-1}g_2 \in L(n). \) Hence \( g_0u_0 = g_2u_0. \) Therefore \( h' \) is an injection. The map \( v \rightarrow h'(0, v) \) is a real analytic isomorphism of \( D_{n}^{-n} \) onto an open neighborhood \( V \) of \( p \) in \( F(SL(n, R), M). \)

Define a map \( h: R^n \times V \rightarrow M \) by \( h(u, v) = h'(u, k(v)) \) for \( u \in R^n, \) \( v \in V. \) Then it is easy to see that \( h \) is a real analytic isomorphism of \( R^n \times V \) onto an open set of \( M \) satisfying the conditions (a), (b).

Next, let \( h_i: R^n \times V_i \rightarrow M \) be a real analytic into isomorphism satisfying the conditions (a), (b) for \( i = 1, 2. \) Put \( e = (1, 0, \ldots, 0) \in R^n. \) Assume that \( \phi(g_i, h_i(e, v_i)) = \phi(g_2, h_2(e, v_2)) \) for some \( g_i \in SL(n, R), \) \( v_i \in V_i. \) Then \( h_i(e, v_i) = \phi(g_i^{-1}g_2, h_2(e, v_2)), \) and hence \( g_i^{-1}g_2 \in NL(n), \) because the isotropy group at \( h_i(e, v_i) \) coincides with \( L(n). \) Put \( x_i \) the diagonal matrix with diagonal components \( t, t^{-1}, 1, \ldots, 1. \) Then \( x_t \in NL(n). \) Since \( h_i(te, v_i) = \phi(x_t, h_i(e, v_i)) \) and \( NL(n)/L(n) \) is abelian, it follows that \( h_i(te, v_i) = \phi(g_i^{-1}g_2, h_2(te, v_2)) \) for \( t \neq 0. \) Let \( t \rightarrow 0. \) Then \( v_i = \phi(g_i^{-1}g_2, v_2) = v_2. \) It follows that \( h_i(R^n \times V_i) \cap h_2(R^n \times V_2) \) is contained in \( h_i(R^n \times V) \) for \( V = V_i \cap V_2. \) Since \( h_i(R^n \times V) \) is a smallest open \( SL(n, R) \) invariant neighborhood of \( V = h_i(0 \times V), \) we can derive that \( h_i(R^n \times V) = h_2(R^n \times V), \) and hence \( h_i(R^n \times V_i) \cap h_2(R^n \times V_2) = h_i(R^n \times V). \)

From the above argument, there exists a unique real analytic function \( f: V \rightarrow R \) such that \( h_i(e, v) = h_2(f(v)e, v) \) for \( v \in V. \) Then \( h_i(u, v) = h_2(f(v)u, v) \) for \( u \in R^n, \) \( v \in V, \) because \( h_1 \) and \( h_2 \) are \( SL(n, R) \) equivariant. q.e.d.

**Remark 3.2.** Let \( M \) be a real analytic paracompact manifold. Then \( M \) admits a real analytic Riemannian metric, because \( M \) is real analytically isomorphic to a real analytic closed submanifold of \( R^n \) (cf. Grauert [3, Theorem 3]). Suppose that \( M \) admits a real analytic action of a compact Lie group \( H. \) Then \( M \) admits a real analytic \( H \) invariant Riemannian metric, by averaging a given real analytic Riemannian metric. In particular, each connected component of \( F(H, M) \) is a real analytic closed submanifold of \( M. \)

**Lemma 3.3.** Suppose \( n \geq 3. \) Let \( \phi: SL(n, R) \times M \rightarrow M \) be a real analytic \( SL(n, R) \) action on a connected paracompact \( m \)-manifold. Suppose that the restricted \( SO(n) \) action of \( \phi \) has just two orbit types \( SO(n)/SO(n-1) \) and \( SO(n)/SO(n). \) Then

(a) each connected component of \( F(SO(n), M) \) is \((m-n)\)-dimensional,
(b) \( F(SO(n-1), M) \) is connected and \((m-n+1)\)-dimensional,
(c) \( F(SO(n-1), M) \) coincides with either \( F(L(n), M) \) or \( F(L^*(n), M). \)
Moreover, if $F(SO(n-1), M) = F(L(n), M)$, then there is an equivariant decomposition:

$$M - F = SL(n, R) \times F(L(n), M - F),$$

where $F = F(SL(n, R), M) = F(SO(n), M)$.

**Proof.** It follows from the assumption that the isotropy representation at a point of $F(SO(n), M)$ is equivalent to $\rho_n \oplus$ trivial. Here $\rho_n$ is the canonical representation of $SO(n)$. Hence (a) follows. Put $X = F(SO(n-1), M) - F(SO(n), M)$. There is an equivariant decomposition:

$$M - F = SO(n)/SO(n-1) \times X,$$

where $W = NSO(n-1)/SO(n-1) = \mathbb{Z}_2$. In particular, $\dim X = m - n + 1$. Let $\pi: M \to M^* = SO(n) \setminus M$ be the canonical projection to the orbit space $M^*$. Then $M^* = \pi(F(SO(n-1), M))$ by the assumption. Put $g_0$ the diagonal matrix with diagonal components $-1, -1, 1, \ldots, 1$. Define a map $T: F(SO(n-1), M) \to F(SO(n-1), M)$ by $T(x) = \phi(g_0, x)$. Then $T$ is an involution on $F(SO(n-1), M)$ and the fixed point set agrees with $F(SO(n), M)$. Then orbit space $T \setminus F(SO(n-1), M)$ is naturally homeomorphic to a connected space $M^*$. Let $Y$ be a connected component of $F(SO(n-1), M)$ such that $Y \cap F(SO(n), M)$ is non-empty. Then $TY = Y$ and the orbit space $T \setminus Y$ is a connected component of $T \setminus F(SO(n-1), M)$. Hence $Y = F(SO(n-1), M)$ is connected. Hence (b) follows. By the assumption, Lemma 2.3 and Proposition 2.5, we have the following:

$$F(SO(n-1), M) = F(L(n), M) \cup F(L^*(n), M),$$

$$F(SO(n), M) = F(L(n), M) \cap F(L^*(n), M) = F(SL(n, R), M).$$

It follows from the above argument that $X$ has at most two connected components. If $X$ is connected, then it is easy to see that $F(SO(n-1), M)$ coincides with either $F(L(n), M)$ or $F(L^*(n), M)$. Suppose that $X$ has two connected components $X_1$ and $X_2$. Then $TX_1 = X_2$. Since $g_0L(n)g_0^{-1} = L(n)$ and $g_0L^*(n)g_0^{-1} = L^*(n)$, we see that if $X_1$ is contained in $F(L(n), M)$ (resp. $F(L^*(n), M)$), then $X_2$ is also contained in $F(L(n), M)$ (resp. $F(L^*(n), M)$). Hence (c) follows.

Suppose now that $F(SO(n-1), M) = F(L(n), M)$. Consider the following commutative diagram:

$$\begin{array}{cccc}
SO(n) \times X & \xrightarrow{i \times 1} & SL(n, R) \times X \\
| \downarrow \pi & & \downarrow \pi' \\
SO(n) \times X & \xrightarrow{\phi_1} & M - F & \xleftarrow{\phi_2} \\
| \phi_1 \downarrow & & \phi_2 \downarrow & \phi_2 \downarrow \\
SO(n) \times X & \xleftarrow{\phi_1'} & SL(n, R) \times X \\
| \phi_2 \downarrow & & \phi_2 \downarrow \\
& & \end{array}$$
Here \( X = F(SO(n-1), M) - F(SO(n), M) = F(L(n), M) - F(SL(n, R), M) \); \( \pi, \pi' \) are the natural projections; \( \phi, \phi' \) are the restrictions of the map \( \phi; \phi', \phi' \) are the induced maps. Then \( \phi'_* \) is an \( SO(n) \) equivariant real analytic isomorphism. Since \( SL(n, R) = SO(n) \cdot N(n) \), it is easy to see that the map \( j \) is a surjection. Here the group \( N(n) \) is defined in Section 2. It follows that \( \phi'_* \) is an \( SL(n, R) \) equivariant real analytic isomorphism.

We require the following result due to Guillemin and Sternberg [4]:

**Lemma 3.4.** Let \( \mathfrak{g} \) be a real semi-simple Lie algebra and let \( \rho: \mathfrak{g} \to L(M) \) be a Lie algebra homomorphism of \( \mathfrak{g} \) into a Lie algebra of real analytic vector fields on a real analytic m-manifold \( M \). Let \( p \) be a point at which the vector fields in the image \( \rho(\mathfrak{g}) \) have common zero. Then there exists an analytic system of coordinates \((U; u_1, \cdots, u_m)\), with origin at \( p \), in which all of the vector fields in \( \rho(\mathfrak{g}) \) are linear. Namely, there exists \( a_{ij} \in g^* = \text{Hom}_R(\mathfrak{g}, R) \) such that

\[
\rho(X)_q = -\sum_{i,j} a_{ij}(X) u_j(q) \frac{\partial}{\partial u_i} \quad \text{for} \quad X \in \mathfrak{g}, \quad q \in U.
\]

**Remark 3.5.** The correspondence \( X \to (a_{ij}(X)) \) defines a Lie algebra homomorphism of \( \mathfrak{g} \) into \( \mathfrak{gl}(m, R) \). Let \( \mathfrak{P} = (p_{ij}) \in GL(m, R) \). Define an analytic system of coordinates \((U; v_1, \cdots, v_m)\) by \( v_i(q) = \sum_{j=1}^m p_{ij} u_j(q), \quad q \in U \). Then \( \rho(X)_q = -\sum_{i,j} b_{ij}(X) v_j(q) \frac{\partial}{\partial v_i} \) for \( X \in \mathfrak{g}, \quad q \in U \). Here \( b_{ij}(X) = \rho(a_{ij}(X)) \mathfrak{P}^{-1} \).

**Lemma 3.6.** Suppose \( n \geq 3 \). Let \( \phi: SL(n, R) \times M \to M \) be a real analytic action on m-manifold. Suppose that the restricted \( SO(n) \) action of \( \phi \) has just two orbit types \( SO(n)/SO(n-1) \) and \( SO(n)/SO(n) \). Suppose \( F(SO(n-1), M) = F(L(n), M) \). Then for each \( p \in F(SL(n, R), M) \) there exists an analytic system of coordinates \((U; u_1, \cdots, u_m)\), with origin at \( p \), such that

\[
\phi^*(x_{ij})(q) = -\sum_{i,j=1}^n x_{ij} u_j(q) \frac{\partial}{\partial u_i} \quad \text{for} \quad (x_{ij}) \in \mathfrak{sl}(n, R), \quad q \in U.
\]

**Proof.** By Lemma 3.4, there exists an analytic system of coordinates \((U; v_1, \cdots, v_m)\) with origin at \( p \) and there exists \( a_{ij} \in \mathfrak{sl}(n, R)^* \) such that \( \phi^*(X)_q = -\sum_{i,j=1}^m a_{ij}(X) v_j(q) \frac{\partial}{\partial v_i} \) for \( X \in \mathfrak{sl}(n, R), \quad q \in U \). Then \( F(SO(n), M) \cap U = \{ q \in U: \phi^*(X)_q = 0 \text{ for } X \in \mathfrak{s}o(n) \} = \{ q \in U: \sum_{i=1}^n a_{ij}(X) v_j(q) = 0 \text{ for } X \in \mathfrak{s}o(n), \quad 1 \leq i \leq m \} \). Since \( \dim F(SO(n), M) = m - n \) by Lemma 3.3 (a), we can assume \( F(SO(n), M) \cap U = \{ q \in U: v_i(q) = \cdots = v_m(q) = 0 \} \) by Remark 3.5. Then \( a_{ij}(X) = 0 \) for \( n + 1 \leq j \leq m, \quad 1 \leq i \leq m \) for each \( X \in \mathfrak{sl}(n, R) \), because \( F(SO(n), M) = F(SL(n, R), M) \) by Lemma 3.3. Therefore
fore the representation $X \to (a_{ij}(X))$ of $\mathfrak{sl}(n, \mathbb{R})$ has $(m - n)$-dimensional trivial subspace. It is well known that any real representation of $\mathfrak{sl}(n, \mathbb{R})$ is completely reducible (cf. Humphreys [6, Section 6]). Hence the representation $X \to (a_{ij}(X))$ is a direct sum of an $n$-dimensional representation and $(m - n)$-dimensional trivial representation. It is known that an $n$-dimensional real representation of $\mathfrak{sl}(n, \mathbb{R})$ is equivalent to the canonical representation $X \to X$ or the contragredient representation $X \to -\otimes X$. By Remark 3.5, there exists an analytic system of coordinates $(U; u_1, \cdots, u_m)$, with origin at $p$, such that

$$\phi^+((x_{ij}))_q = -\sum_{i,j=1}^n x_{ij} u_j(q) \frac{\partial}{\partial u_i}$$

or

$$\phi^+((x_{ij}))_q = \sum_{i,j=1}^n x_{ij} u_j(q) \frac{\partial}{\partial u_i}$$

for $(x_{ij}) \in \mathfrak{sl}(n, \mathbb{R})$, $q \in U$. The case (b) contradicts the assumption $F(SO(n-1), M) = F(L(n), M)$. q.e.d.

**Theorem 3.7.** Suppose $n \geq 3$. Let $\phi: SL(n, \mathbb{R}) \times M \to M$ be a real analytic action on a connected paracompact $m$-manifold. Suppose that the restricted $SO(n)$ action of $\phi$ has just two orbit types $SO(n)/SO(n-1)$ and $SO(n)/SO(n)$. Suppose $F(SO(n-1), M) = F(L(n), M)$. Put $F = F(SL(n, \mathbb{R}), M)$. Then (i) there exists a real analytic left principal $R^\times$ bundle $p: E \to F$, and there exists a real analytic isomorphism $h$ of $R^n \times_{R^\times} E$ onto an open set of $M$ such that

(a) $h(0, u) = p(u)$ for $u \in E$

(b) $h(gx, u) = \phi(g, h(x, u))$ for $g \in SL(n, \mathbb{R})$, $x \in R^n$, $u \in E$.

Moreover, (ii) if there exists a real analytic left principal $R^\times$ bundle $p': E' \to F$ and if there exists a real analytic isomorphism $h'$ of $R^n \times_{R^\times} E'$ onto an open set of $M$ such that

(a') $h'(0, u') = p'(u')$ for $u' \in E'$

(b') $h'(gx, u') = \phi(g, h'(x, u'))$ for $g \in SL(n, \mathbb{R})$, $x \in R^n$, $u' \in E'$,

then there exists a real analytic $R^\times$ bundle isomorphism $f: E \to E'$ such that $h(x, u) = h'(x, f(u))$ for $x \in R^n$, $u \in E$.

**Proof.** From Lemma 3.1 and Lemma 3.6, there exists an open covering $\{V_\alpha, \alpha \in A\}$ of $F$ and there exists a real analytic $SL(n, \mathbb{R})$ equivariant isomorphism $h_\alpha$ of $R^n \times V_\alpha$ onto an open set of $M$ for each $\alpha \in A$, such that $h_\alpha(0, v) = v$ for $v \in V_\alpha$. Put $U = \bigcup_{\alpha \in A} h_\alpha(R^n \times V_\alpha)$. Then $U$ is an $SL(n, \mathbb{R})$ invariant open neighborhood of $F$ in $M$. Put $E = F(L(n)$,
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$U - F$), and define $k_a : R^x \times V_a \to E$ by $k_a(t, v) = h_a(t e, v)$ for $t \in R^x, v \in V_a$. Here $e = (1, 0, \ldots, 0) \in R^n$. The group $NL(n)/L(n) = R^x$ acts naturally on $E$, and the map $k_a$ is $R^x$ equivariant. It follows from Lemma 3.1 that $E = \bigcup_{a \in A} k_a(R^x \times V_a)$ and $k_a(R^x \times V_a) \cap k_b(R^x \times V_b) = k_a(R^x \times (V_a \cap V_b))$ for $a, b \in A$, and there exists a unique real analytic function $g_{a b} : V_a \cap V_b \to R^x$ such that $k_b(t, v) = k_a(g_{a b}(v)t, v)$ for $t \in R^x, v \in V_a \cap V_b$.

Define $p : E \to F$ by $p k_a(t, v) = v$ for $t \in R^x, v \in V_a$. This is a desired real analytic left principal $R^x$ bundle. We can define a map $h : R^x \times R^x E \to M$ by $h(x, k_a(t, v)) = h_a(tx, v)$ for $x \in R^x, t \in R^x, v \in V_a$. The map $h$ is a real analytic $SL(n, R)$ equivariant isomorphism onto $U$. This is a desired map. Suppose finally that there exists a real analytic left principal $R^x$ bundle $p' : E' \to F$ and there exists a real analytic isomorphism $h' : R^x \times R^x E' \to R^x \times R^x E'$ onto an open set of $M$, satisfying the conditions (a'), (b'). It is easy to see from Lemma 3.1 (ii) that image $h = U = image h'$. It follows that there exists a unique $SL(n, R)$ equivariant real analytic isomorphism $f : R^x \times E \to R^x \times E'$ such that $h(x, u) = h'(f(x, u))$ for $x \in R^x, u \in E$. Considering the fixed point sets of the restricted $L(n)$ action, we have a real analytic $R^x$ equivariant isomorphism $f : E \to E'$ such that $f(te, u) = (te, f(u))$ for $t \in R$, $u \in E$. Then $f : E \to E'$ is a bundle isomorphism of principal $R^x$ bundles, because $p(u) = h(0, u) = h'(f(0, u)) = h'(0, f(u)) = p'(f(u))$ for $u \in E$.

4. Smooth SO(n) actions on homotopy spheres. First we state the following two lemmas of which proofs are given in Section 7.

**Lemma 4.1.** Suppose $n \geq 5$. Let $G$ be a closed connected proper subgroup of $O(n)$ such that $\dim O(n)/G \leq 2n - 2$. Then it is one of the following listed in Table 1 up to an inner automorphism of $O(n)$. Here

$\rho_k : SO(k) \to O(k), \mu_k : U(k) \to O(2k), \mu_k^* : SU(k) \to O(2k)$

are the canonical inclusions, $\theta^k$ is the trivial representation of degree $k$, and $\Delta, \omega, \beta$ are irreducible representations, respectively.

**Lemma 4.2.** Suppose $5 \leq n \leq k \leq 2n - 2$. Then an orthogonal non-trivial representation of $SO(n)$ of degree $k$ is equivalent to $\rho_n \oplus \theta^{k-n}$ by an inner automorphism of $O(k)$.

Now we shall prove the following result.

**Lemma 4.3.** Suppose $5 \leq n \leq k \leq 2n - 2$. Let $\Sigma^k$ be a homotopy $k$-sphere with a non-trivial smooth $SO(n)$ action. Then the principal
isotropy type is \((SO(n-1))\) and the fixed point set \(F(SO(n), \Sigma^k)\) is non-empty.

Let us start with some observations. In the following, let \(M\) be a closed connected \(k\)-dimensional manifold with a non-trivial smooth \(SO(n)\) action, let \((H)\) be the principal isotropy type, and suppose \(5 \leq n \leq k \leq 2n - 2\). Denote by \(H^0\) the identity component of \(H\).

**Observation 4.4.** If \(F(SO(n), M)\) is non-empty, then \((H) = (SO(n-1))\). This is a direct consequence of Lemma 4.2, by considering the isotropy representation at a fixed point.

**Observation 4.5.** Suppose that \(M\) is 2-connected and the \(SO(n)\) action is transitive. Then \(M = SO(n)/SO(n-2)\) or \(M = SO(5)/\beta SO(3)\). This is a direct consequence of Lemma 4.1.

**Observation 4.6.** Suppose that the principal isotropy type \((H)\) is one of the following listed in Table 2. Then \(M\) is not 3-connected.

**Proof.** Since \(F(SO(n), M)\) is empty by Observation 4.4 and \(H^0\) is a proper maximal connected subgroup of \(SO(n)\) by Lemma 4.1, there is an equivariant decomposition: \(M = SO(n)/H^0 \times_w F(H^0, M)\), where \(W = N(H^0)/H^0\) is a finite group. If \(M\) is simply connected, then \(M = SO(n)/\)
\( H^0 \times F \) and it is not 3-connected, where \( F \) is a connected component of \( F(H^0, M) \).

**Table 2**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( H^0 )</th>
<th>( \pi_1(\text{SO}(n)/H^0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>Spin(7)</td>
<td>( \pi_2 = \mathbb{Z} )</td>
</tr>
<tr>
<td>8</td>
<td>U(4)</td>
<td>( \pi_2 = \mathbb{Z} )</td>
</tr>
<tr>
<td>7</td>
<td>( G_2 )</td>
<td>( \pi_1 = \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>7</td>
<td>( \text{SO}(3) \times \text{SO}(4) )</td>
<td>( \pi_2 = \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>6</td>
<td>( \text{SO}(3) \times \text{SO}(3) )</td>
<td>( \pi_2 = \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>6</td>
<td>U(3)</td>
<td>( \pi_2 = \mathbb{Z} )</td>
</tr>
<tr>
<td>5</td>
<td>( \tilde{\text{SO}}(3) )</td>
<td>( \pi_2 \neq 0 )</td>
</tr>
</tbody>
</table>

**Observation 4.7.** Suppose that \((H)\) is one of the following:

\[ H^0 = \text{SO}(n-2) \times \text{SO}(2); \ U(4), \ n = 8; \ U(3), \ n = 6; \ U(2), \ n = 5. \]

Then \( M \) is not stably parallelizable.

**Proof.** If \( M \) is stably parallelizable, then the principal orbit \( \text{SO}(n)/H \) is stably parallelizable; hence \( \text{SO}(n)/H^0 \) is also stably parallelizable.

**Observation 4.8.** Suppose that \( \dim M = 2n - 2, \pi_1(M) = \{1\}, \chi(M) \neq 0, \) and \( H^0 \) is conjugate to \( \text{SO}(n-2) \). Then \( \chi(M) \geq 4 \). Here \( \chi(M) \) is the Euler characteristic of \( M \).

**Proof.** The principal orbit \( \text{SO}(n)/H \) is of codimension one. Since \( \pi_1(M) = \{1\} \), there are just two singular orbits (cf. Uchida [11, Lemma 1.2.1]). By Observation 4.4, \( F(\text{SO}(n), M) \) is empty. Hence the following are the only possibilities of the singular orbit types:

\[ \text{SO}(n)/\text{SO}(n-1) = S^{n-1}, \quad \text{SO}(n)/\text{SO}(n-2) \times \text{SO}(2) = Q_{n-2}, \quad \text{SO}(n)/\text{SO}(n-2) \times \text{SO}(2) = Q_{n-2}/\mathbb{Z}_2. \]

By the general position theorem and the assumption \( \pi_1(M) = \{1\} \), it is easy to see that the pair of singular orbits is none of the following: \( (S^{n-1}, \text{SO}(n-1)/\mathbb{R}), (S^{n-1}, Q_{n-2}/\mathbb{Z}_2), (P_{n-1}(R), P_{n-1}(R)), (P_{n-1}(R), Q_{n-2}/\mathbb{Z}_2). \) Since \( \chi(M) = \chi \) (singular orbits), we have the desired result.

**Observation 4.9.** Suppose that \( \dim M = 2n - 2 \) and \( (H) \) is one of the following:

\[ H^0 = \text{Spin}(7), \ n = 9; \ U(4), \ n = 8; \ U(2), \ n = 5. \]

Then \( \pi_1(M) \neq \{1\} \) or \( \chi(M) \geq 4 \).

This is similarly proved as Observation 4.8.
Observation 4.10. Suppose that $n = 6$ and $H^c$ is conjugate to $SU(3)$. Then $M$ is not 2-connected.

Proof. By Observation 4.4, $F(SO(6), M)$ is empty. Hence the identity component of an isotropy group is conjugate to $SU(3)$ or $U(3)$ for each point of $M$. It follows that there is an equivariant decomposition: $M = SO(6)/SU(3) \times W F(SU(3), M)$, where $W = NSU(3)/SU(3) = U(1)$. Then it is seen that $M$ is not 2-connected by the following homotopy exact sequence:

$$\pi_3(M) \to \pi_1(W) \to \pi_1(SO(6)/SU(3)) \to \pi_1(F(SU(3), M)) \to \pi_1(M).$$

Proof of Lemma 4.3. It is sufficient to prove that the set $F(SO(n), \Sigma^k)$ is non-empty by Observation 4.4. It is well known that every homotopy sphere is stably parallelizable (cf. Kervaire and Milnor [7, Theorem 3.1]). Let $(H)$ be the principal isotropy type of a non-trivial smooth $SO(n)$ action on a homotopy $k$-sphere $\Sigma^k$. Then it follows that $H^c$ is conjugate to $SO(n - 1)$ by Lemma 4.1 and the above Observations. Suppose that $F(SO(n), \Sigma^k)$ is empty. Then there is an equivariant decomposition: $\Sigma^k = SO(n)/SO(n - 1) \times W F(SO(n - 1), \Sigma^k)$, where $W = NSO(n - 1)/SO(n - 1) = Z_2$. But this is impossible for $k \leq n$. q.e.d.

Theorem 4.11. Suppose $5 \leq n \leq k \leq 2n - 2$. Let $\Sigma^k$ be a homotopy $k$-sphere with a non-trivial smooth $SO(n)$ action. Then there is an equivariant decomposition: $\Sigma^k = \delta(D^n \times Y)$ as a smooth $SO(n)$ manifold. Here $Y$ is a compact contractible $(k - n + 1)$-manifold with trivial $SO(n)$ action, and $D^n$ is the standard $n$-disk with the canonical $SO(n)$ action.

Proof. Put $F = F(SO(n), \Sigma^k)$. By Lemma 4.3, $F$ is non-empty. It follows from Lemma 4.2 that each connected component of $F$ is of $(k - n)$-dimension. Let $U$ be a closed $SO(n)$ invariant tubular neighborhood of $F$ in $\Sigma^k$. Then $U$ is regarded as an $n$-disk bundle over $F$ with a smooth $SO(n)$ action as bundle isomorphisms. It follows that there is an equivariant decomposition: $U = D^n \times W F(SO(n - 1), \partial U)$, where $W = NSO(n - 1)/SO(n - 1) = Z_2$. Put $E = \Sigma^k - \text{int}U$. Then there is an equivariant decomposition: $E = SO(n)/SO(n - 1) \times W F(SO(n - 1), E)$. Notice that $F(SO(n - 1), \partial U) = \partial F(SO(n - 1), E)$. It is easy to see that $\pi_1(E) = \{1\}$ by the general position theorem. Hence $F(SO(n - 1), E)$ has just two connected components. Let $Y$ be a connected component of $F(SO(n - 1), E)$. Then $Y$ is a compact simply connected $(k - n + 1)$-manifold with non-empty boundary, and there is an equivariant decomposition: $\Sigma^k = U \cup E = \partial(D^n \times Y)$. 
It remains to prove that $Y$ is contractible. By the Poincaré Lefschetz duality, $H_i(D^n \setminus Y, \Sigma^k; Z) = H^{k-i}(D^n \times Y; Z) = \{0\}$ for each $i < n$. Consider the homology exact sequence: $H_{i+1}(D^n \times Y, \Sigma^k; Z) \to H_i(\Sigma^k; Z) \to H_i(D^n \setminus Y; Z) \to H_i(D^n \times Y; \Sigma^k; Z)$. Then $H_i(Y; Z) = \{0\}$ for $0 < i \leq n - 2$. On the other hand, $Y$ is a compact simply connected manifold with non-empty boundary, and $\dim Y \leq n - 1$ by the assumption $h \leq 2n - 2$. It follows that $Y$ is contractible. q.e.d.

REMARK. Theorem 4.11 for $n \geq 9$ has been proved already by Hsiang [5, Theorem III].

5. Decomposition and classification. Suppose $5 \leq n \leq m \leq 2n - 2$. Let $\phi$ be a non-trivial real analytic $SL(n, R)$ action on $S^m$. Consider the restricted $SO(n)$ action of $\phi$. By Theorem 4.11, there exists an equivariant decomposition: $S^m = \partial(D^n \times Y)$ as a smooth $SO(n)$ manifold. In particular, the $SO(n)$ action has just two orbit types $SO(n)/SO(n-1)$ and $SO(n)/SO(n)$. Then, by Lemma 3.3, $F(SO(n-1), S^m)$ coincides with either $F(L(n), S^m)$ or $F(L^*(n), S^m)$. We shall show first the following decomposition theorem.

**THEOREM 5.1.** Suppose $5 \leq n \leq m \leq 2n - 2$. Let $\phi$ be a non-trivial real analytic $SL(n, R)$ action on $S^m$. Suppose

$$F(SO(n-1), S^m) = F(L(n), S^m).$$

Then, (i) $\Sigma = F(L(n), S^m)$ is a real analytic $(m - n + 1)$-dimensional closed submanifold of $S^m$ which is homotopy equivalent to a sphere, and $R^c = NL(n)/L(n)$ acts naturally on $\Sigma$, (ii) $F = F(SL(n, R), S^m)$ is a real analytic $(m - n)$-dimensional closed submanifold of $\Sigma$, and there exists a real analytic $R^c$ equivariant isomorphism $j$ of $R \times F$ onto an open set of $\Sigma$ such that $j(0, x) = x$ for $x \in F$, (iii) there exists an equivariant decomposition:

$$S^m = R^c \times F \cup \bigcup_{i=1}^n (R^c - 0) \times (\Sigma - F)$$

as a real analytic $SL(n, R)$ manifold, where $SL(n, R)$ acts naturally on $R^n$, $R^c$ acts on $R^n - 0$ by the scalar multiplication, and $f$ is an equivariant isomorphism of $(R^c - 0) \times F$ onto an open set of $(R^n - 0) \times (\Sigma - F)$ defined by $f(u, x) = (u, j(1, x))$ for $u \in R^n - 0$, $x \in F$.

**PROOF.** Consider the restricted $SO(n)$ action of $\phi$. By Theorem 4.11, there exists an equivariant decomposition: $S^m = \partial(D^n \times Y)$ as a smooth $SO(n)$ manifold. Here $Y$ is a compact contractible smooth $(m - n + 1)$-manifold. Then $\Sigma = F(SO(n-1), S^m)$ is a real analytic $(m - n + 1)$-dimensional closed submanifold of $S^m$ which is $C^\infty$ diffeomorphic to a
double of $Y$; hence $\Sigma$ is a homotopy sphere. By Lemma 3.3, $F = F(SO(n), S^n)$ is a real analytic $(m - n)$-dimensional closed submanifold of $S^n$ which is $C^\infty$ diffeomorphic to $\partial Y$; hence $F$ is homology equivalent to a sphere. Moreover, there exists an equivariant decomposition:

$$S^n - F = SL(n, R)/L(n) \times_{\mathbb{R}^+} (\Sigma - F) = (R^m - 0) \times (\Sigma - F)$$

as a real analytic $SL(n, R)$ manifold. By Theorem 3.7, there exists a real analytic left principal $R^+$ bundle $p: E \to F$ and there exists a real analytic $SL(n, R)$ equivariant isomorphism $h$ of $R^m \times R^+ E$ onto an open set of $S^n$ such that $h(0, u) = p(u)$ for $u \in E$. It is easy to see that the bundle $p: E \to F$ is trivial as a $C^\infty$ bundle by the decomposition $S^n = \partial(D^m \times Y)$.

To show that $E$ is trivial as a real analytic $R^+$ bundle, we need the following.

**Lemma 5.2.** Let $p: V \to X$ be a real analytic vector bundle over a paracompact real analytic manifold $X$. Then the bundle $V$ admits a real analytic Riemannian metric.

**Proof.** Let $i: X \to V$ be the zero section. Then it follows from a calculation of transition functions that there is an isomorphism $i^*\tau(V) = V \oplus \tau(X)$ as real analytic vector bundles. Here $\tau$ denotes the tangent bundle. Since $V$ is a paracompact real analytic manifold, there exists a real analytic embedding $f: V \to R^n$ such that $f(V)$ is a closed real analytic submanifold of $R^n$ (cf. Grauert [3]). It follows that there is an isomorphism $\tau(V) \oplus \nu = R^n \times V$ as real analytic vector bundles. Here $\nu$ denotes the normal bundle. Therefore there is an isomorphism $V \oplus \tau(X) \oplus i^*\nu = R^n \times X$ as real analytic vector bundles. The product bundle $R^n \times X$ admits canonically a real analytic Riemannian metric; hence its real analytic subbundle $V$ admits a real analytic Riemannian metric.

q.e.d.

We now return to the proof of Theorem 5.1. Let $R \times R^+ E \to F$ be the line bundle associated to the principal bundle $p: E \to F$. Then it has a real analytic Riemannian metric; hence the associated sphere bundle is a real analytic double covering over $F$. Since $p: E \to F$ is trivial as a $C^\infty$ bundle, the sphere bundle is trivial as a real analytic bundle, and hence the principal bundle $p: E \to F$ has a real analytic cross-section. Therefore $E$ is trivial as a real analytic $R^+$ bundle. It follows that there exists a real analytic $SL(n, R)$ equivariant isomorphism $h: R^m \times F \to S^n$ onto an open set of $S^n$ such that $h(0, x) = x$ for $x \in F$.

Consider the fixed point sets of restricted $L(n)$ actions. We have
a real analytic $R^\times$ equivariant isomorphism $j: R \times F \to \Sigma$ onto an open set of $\Sigma = F(L(n), S^n)$, defined by $j(t, x) = h(te, x)$ for $t \in R, x \in F$. Here $e = (1, 0, \ldots, 0) \in R^n$, and $R^\times$ acts canonically on $\Sigma$ through the identification $R^\times = NL(n)/L(n)$. It is easy to see that there exists an equivariant decomposition:

$$S^n = R^n \times F \cup \left( (R^n - 0) \times_{R^\times} (\Sigma - F) \right)$$

as a real analytic $SL(n, R)$ manifold. Here $f$ is an equivariant isomorphism of $(R^n - 0) \times_{R^\times} F$ onto an open set of $(R^n - 0) \times_{R^\times} (\Sigma - F)$ defined by $f(u, x) = (u, j(1, x))$ for $u \in R^n - 0, x \in F$. This completes the proof of Theorem 5.1.

**REMARK.** By this theorem, the action $\phi$ on $S^n$ is completely determined up to an equivariant isomorphism by $\Sigma = F(L(n), S^n)$ with $R^\times$ action and an equivariant map $j: R^\times \times F \to \Sigma$.

To state a classification theorem, we introduce the following notions. Let $G$ be a Lie group, and let $\phi_i: G \times M_i \to M_i$ be a real analytic $G$ action for $i = 1, 2$. We say that $\phi_1$ is weakly $C^r$ equivariant to $\phi_2$ if there exists an automorphism $h$ of $G$ and there exists a $C^r$ diffeomorphism $f: M_1 \to M_2$ such that the following diagram is commutative:

$$\begin{array}{ccc}
G \times M_1 & \xrightarrow{\phi_1} & M_1 \\
\downarrow{h \times f} & & \downarrow{f} \\
G \times M_2 & \xrightarrow{\phi_2} & M_2
\end{array}$$

(5-a)

In particular, $\phi_1$ is said to be $C^r$ equivariant to $\phi_2$ if the identity map of $G$ can be chosen as the automorphism $h$.

Let $h$ be an automorphism of $G$, and let $\phi: G \times M \to M$ be a real analytic $G$ action. Define a new real analytic $G$ action $h^*\phi$ on $M$ as follows: $(h^*\phi)(g, x) = \phi(h(g), x)$ for $g \in G, x \in M$. Then the action $h^*\phi$ is weakly $C^m$ equivariant to $\phi$, because the following diagram is commutative:

$$\begin{array}{ccc}
G \times M & \xrightarrow{h^*\phi} & M \\
\downarrow{h \times \text{id}} & & \downarrow{\text{id}} \\
G \times M & \xrightarrow{\phi} & M
\end{array}$$

(5-b)

Let $I_g$ denote the inner automorphism of $G$ defined by $I_g(g') = gg'g^{-1}$ for $g, g' \in G$. Then, for any real analytic $G$ action $\phi$ on $M$, $\phi$ is $C^m$ equivariant to $I_g^*\phi$, because the following diagram is commutative:
\begin{align*}
G \times M & \xrightarrow{\phi} M \\
\downarrow \text{id} \times f & \quad \downarrow f \\
G \times M & \xrightarrow{I_\phi} M,
\end{align*}

where \( f(x) = \phi(g, x) \) for \( x \in M \).

**Theorem 5.3.** Suppose \( 5 \leq n \leq m \leq 2n - 2 \). Then there is a natural one-to-one correspondence between the weak \( C^r \) equivariance classes of non-trivial real analytic \( SL(n, R) \) actions on the standard \( m \)-sphere and the \( C^r \) equivariance classes of real analytic \( R^\times \) actions on homotopy \( (m - n + 1) \)-spheres satisfying the condition \( (P) \), for each \( r = 0, 1, \ldots, \infty, \omega \). The correspondence is given by the construction in Section 1.

**Proof.** Let \( A_r(n, m) \) denote the weak \( C^r \) equivariance classes of non-trivial real analytic \( SL(n, R) \) actions on the standard \( m \)-sphere, let \( A'_r(n, m) \) denote the \( C^r \) equivariance classes of non-trivial real analytic \( SL(n, R) \) actions on the standard \( m \)-sphere such that \( F(SO(n - 1), S^m) = F(L(n), S^m) \), and let \( B_r(k) \) denote the \( C^r \) equivariance classes of real analytic \( R^\times \) actions on homotopy \( k \)-spheres satisfying the condition \( (P) \) in Section 1.

Let \( \phi: R^\times \times \Sigma \to \Sigma \) be a real analytic \( R^\times \) action on a homotopy \( k \)-sphere \( \Sigma \) satisfying the condition \( (P) \). We constructed, in Section 1, a compact real analytic \( SL(n, R) \) manifold \( M(\psi, j) \) such that the \( C^\infty \) equivariance class of \( M(\psi, j) \) does not depend on the choice of \( j \), \( F(SO(n - 1), M(\psi, j)) = F(L(n), M(\psi, j)) \), and \( M(\psi, j) \) is real analytically isomorphic to the standard \( (n + k - 1) \)-sphere for \( n + k \geq 6 \). The correspondence \( \psi \to M(\psi, j) \) defines a mapping \( c_r: B_r(k) \to A'_r(n, n + k - 1) \) for \( r = 0, 1, \ldots, \infty, \omega \) and each \( n + k \geq 6 \). It follows from Theorem 5.1 that \( c_r \) is a bijection \( (r = 0, 1, \ldots, \infty, \omega) \) if \( n \geq 5 \) and \( 1 \leq k \leq n - 1 \).

It remains to show that there is a natural one-to-one correspondence between \( A'_r(n, m) \) and \( A_r(n, m) \). Let \( \phi \) be a real analytic non-trivial \( SL(n, R) \) action on \( S^m \) such that \( F(SO(n - 1), S^m) = F(L(n), S^m) \). Then \( \phi \) represents a class of \( A'_r(n, m) \) and a class of \( A_r(n, m) \). Hence there is a natural mapping \( i_r: A'_r(n, m) \to A_r(n, m) \).

We shall show that \( i_r \) is a bijection \( (r = 0, 1, \ldots, \infty, \omega) \) if \( 5 \leq n \leq m \leq 2n - 2 \). Let \( \sigma \) be the automorphism of \( SL(n, R) \) defined by \( \sigma(X) = X^{-1} \) for \( X \in SL(n, R) \). Then it is seen that \( \sigma \) is an involution and \( \sigma(L(n)) = L^*(n) \). Let \( \phi \) be a real analytic non-trivial \( SL(n, R) \) action on \( S^m \). Then, by Lemma 3.3 (c) we have that \( F(SO(n - 1), S^m) \) coincides with \( F(L(n), S^m) \) or \( F(L^*(n), S^m) \). Since \( \sigma(L(n)) = L^*(n) \), we see that if
Let $\phi, \phi'$ be real analytic non-trivial $SL(n, R)$ actions on $S^n$. Suppose that $\phi'$ is weakly $C^r$ equivariant to $\phi$. Then by the diagrams (5-a), (5-b), (5-c) $\phi'$ is $C^r$ equivariant to one of the following: $\phi, \sigma^r \phi, \gamma^r \phi, \sigma^r \gamma^r \phi$. Notice that if $F(SO(n-1), S^n) = F(L(n), S^n)$ for $\phi$, then $F(SO(n-1), S^n) = F(L(n), S^n)$ for $\sigma^r \phi$, and $F(SO(n-1), S^n) = F(L^*(n), S^n)$ for $\sigma^r \gamma^r \phi$. Therefore, if $\phi$ and $\phi'$ represent classes of $A'_r(n, m)$, respectively, and if $\phi'$ is weakly $C^r$ equivariant to $\phi$, then $\phi'$ is $C^r$ equivariant to $\phi$ or $\gamma^r \phi$. To show that $i_r$ is injective, it suffices to prove $\sigma^r \phi$ is $C^r$ equivariant to $\phi$. Consider the real analytic $SL(n, R)$ manifold

$$M(\psi, j) = R^n \times F \cup (R^n - 0) \times (\Sigma - F)$$

constructed in Section 1. Define a real analytic isomorphism $g: M(\psi, j) \to M(\psi, j)$ by

$$g(u, x) = (Y \cdot u, x) \text{ for } (u, x) \in R^n \times F,$$
$$g(v, y) = (Y \cdot v, y) \text{ for } (v, y) \in (R^n - 0) \times (\Sigma - F).$$

Here the matrix $Y$ is as before. Then the following diagram is commutative:

$$\begin{array}{ccc}
SL(n, R) \times M(\psi, j) & \xrightarrow{\phi} & M(\psi, j) \\
\downarrow \gamma \times g & & \downarrow g \\
SL(n, R) \times M(\psi, j) & \xrightarrow{\phi} & M(\psi, j),
\end{array}$$

where $\phi$ is the natural $SL(n, R)$ action on $M(\psi, j)$. By the diagram (5-b), we have the following commutative diagram:
Since $\gamma^2 = 1$, it follows that $\gamma^* \phi$ is $C^\infty$ equivariant to $\phi$; hence the mapping $i$, is bijective. q.e.d.

6. $R^\infty$ actions on spheres. In the previous section, we showed that the classification of real analytic $SL(n, R)$ actions on the $m$-sphere can be reduced to that of real analytic $R^\infty$ actions on homotopy $(m-n+1)$-spheres satisfying the condition (P). So we study now $R^\infty$ actions on spheres.

Let $S^k$ be the standard $k$-sphere in $R^{k+1}$, $k \geq 1$. Let $T$ be an involution of $S^k$ defined by $T(x_0, x_1, \ldots, x_k) = (-x_0, x_1, \ldots, x_k)$. Put

$$\xi^a = x_0(1 - x_0^2)a(x_0^2)(\sqrt{x_0^2} - x_0^2a(x_0^2)\sum_{i=1}^k x_i(\partial/\partial x_i),$$

where $a(t)$ is a real analytic function defined on an open neighborhood of the closed interval $[0, 1]$. It is easy to see that $\xi^a$ is a real analytic tangent vector field on $S^k$ such that $T_\ast \xi^a = \xi^a$. Let $\{\theta_t; t \in R\}$ be the one-parameter group of real analytic transformations of $S^k$ associated with the vector field $\xi^a$. It follows from $T_\ast \xi^a = \xi^a$ that $T_\ast \theta_t = \theta_t \cdot T$ for $t \in R$. Now we can define a real analytic $R^\infty$ action $\psi^a$ on $S^k$ by

$$\psi^a((-1)^n e^t, x) = T^n(\theta_t(x)) \quad \text{for} \quad x \in S^k, \quad t \in R, \quad n \in Z.$$

It is easy to see that the $R^\infty$ action $\psi^a$ satisfies the condition (P)-(i). We shall give a sufficient condition for $\psi^a$ to satisfy the condition (P)-(ii).

**Proposition 6.1.** If $a(0) = 1$, then the $R^\infty$ action $\psi^a$ satisfies the condition (P).

**Proof.** It is sufficient to construct a real analytic into isomorphism $j: R \times F \to S^k$ satisfying the following conditions:

(1) \quad $j(0, x) = x$ ,

(2) \quad $T(j(t, x)) = j(-t, x)$ ,

(3) \quad $j(e^t, x) = \psi^a(e^t, j(t, x))$

for $x \in F; t \in R$. Here $F$ is the fixed point set of $T$. It is easy to see that the condition (3) is equivalent to the following condition:

(3') \quad $j_\ast(t(\partial/\partial t)) = \xi^a$. 

\[ \begin{align*} 
SL(n, R) \times M(\psi, j) & \xrightarrow{\gamma^* \phi} M(\psi, j) \\
\downarrow \phi \times \text{id} & \quad \downarrow \text{id} \\
SL(n, R) \times M(\psi, j) & \xrightarrow{\phi} M(\psi, j). 
\end{align*} \]
By the assumption \( a(0) = 1 \), there is a real analytic function \( b(t) \) such that \( a(t) = 1 + t \cdot b(t) \). Put \( F(t, u) = -tu^3 + tu^3 b(t^2u^2) - t^2u^b(t^2u^2) \). Then there is a unique real analytic function \( c(t) \) defined on an interval \( (-\varepsilon, \varepsilon) \) for a positive real \( \varepsilon \) such that \( \frac{d}{dt} c(t) = F(t, c(t)) \), \( c(0) = 1 \), \( -1 < t \cdot c(t) < 1 \).

Define a real analytic mapping \( j_1: (-\varepsilon, \varepsilon) \times F \to S^k \) by \( j_1(t, x) = (t \cdot c(t), (1 - t^2c(t)^2)^1/2, x) \). Then it is easy to see that \( j_1(t, x) \) is an integral curve of the vector field \( \xi^a \). By the definition of the action \( \psi^a \), the curve \( s \to \psi^a(e^s, j_1(t, x)) \) is an integral curve of the vector field \( \xi^a \). By the condition \( \psi^a \), the curve \( s \to j_1(e^s, x) \) is also an integral curve of \( \xi^a \). It follows that

\[
(\ast) \quad \psi^a(e^s, j_1(t, x)) = j_1(e^s, x)
\]

for \( x \in F, -\varepsilon < t < \varepsilon, -\varepsilon < e^t \varepsilon < \varepsilon \). Define a mapping \( j: R \times F \to S^k \) by

\[
j(t, x) = \begin{cases} (\psi^a(2t/\varepsilon, j_1(\varepsilon/2, x)) & \text{for } t \neq 0 \\ (0, x) & \text{for } t = 0 \end{cases}
\]

Then \( j \) is an extension of \( j_1 \) by (\( \ast \)); hence \( j \) is real analytic. By definition, the map \( j \) satisfies the conditions (1), (2) and (3).

Finally, we shall show that \( j \) is an into isomorphism. Let \( O(k) \) be the orthogonal transformation group of the Euclidean space \( R^{k+1} \) leaving fixed the \( x_0 \)-coordinate. Then the vector field \( \xi^a \) and the map \( j_1 \) are \( O(k) \) invariant by definition. Hence we have

\[
(\ast\ast) \quad A(j(t, x)) = j(t, Ax) \text{ for } A \in O(k), \quad (t, x) \in R \times F.
\]

Since \( c(0) = 1 \), the map \( j \) is non-singular at each point of \( 0 \times F \). It remains to show that \( j \) is injective. Assume \( j(t_1, x_1) = j(t_2, x_2) \) for some \( (t_1, x_1) \in R \times F \). Then \( j(st_1, x_1) = j(st_2, x_2) \) for any \( s \neq 0 \) by the definition of \( j \). Let \( s \to 0 \). Then \( j(0, x_1) = j(0, x_2) \). Hence we have \( x_1 = x_2 \) and \( j(t_1, x_1) = j(t_2, x_2) \). It follows from \( (\ast\ast) \) that \( j(t_1, x) = j(t_2, x) \) for any \( x \in F \). Assume \( t_1 \neq t_2 \). Then \( j \) induces a real analytic isomorphism of \( S^1 \times F \) onto an open set of \( S^k \). This is a contradiction. Therefore the map \( j \) is injective.

By Proposition 6.1, we can construct many examples of real analytic \( R^k \) actions on the standard \( k \)-sphere satisfying the condition (P). Let

\[
a = (a_1, a_2, \ldots, a_N) \in R^N \quad \text{for } \ N = 1, 2, \ldots,
\]

and define a real analytic tangent vector field \( \xi^a \) on \( S^k \) as follows:
\[ \xi_a = \left( \prod_{i=1}^{k} (1 - a_i x_0^i) \right) \cdot (x_0(1 - x_0^2)(\partial/\partial x_0) - x_0^2 \sum_{i=1}^{k} x_i(\partial/\partial x_i)) \]  

Let \( \psi^a \) be the real analytic \( R^a \) action on \( S^k \) determined by the vector field \( \xi^a \) and the involution \( T \). Then the action \( \psi^a \) satisfies the condition (P).

**Proposition 6.2.** Let \( a = (a_1, \ldots, a_N) \) and \( a' = (a_1', \ldots, a_N') \).

(i) If \( \psi^a \) is \( C^0 \) equivariant to \( \psi^{a'} \), then the cardinality of the set \( \{a_j: a_j > 1\} \) is equal to that of the set \( \{a'_j: a'_j > 1\} \).

(ii) If \( \psi^a \) is \( C^2 \) equivariant to \( \psi^{a'} \), then \( \prod_{j=1}^{N} (1 - a_j) = \prod_{j=1}^{N} (1 - a'_j) \).

**Proof.** The points \( x_0 = \pm 1 \) are isolated zeros of the vector field \( \xi^a \), and the other zeros of \( \xi^a \) are the hypersurfaces

\[ x_0 = 0 \quad \text{and} \quad x_0 = \pm 1/a_j^{1/2} \quad \text{for} \quad a_j > 1. \]

If there is an equivariant homeomorphism of \( S^k \) with the \( R^a \) action \( \psi^a \) to \( S^k \) with the \( R^{a'} \) action \( \psi^{a'} \), then the zeros of the vector field \( \xi^a \) is homeomorphic to the zeros of the vector field \( \xi^{a'} \). Hence the cardinality of the set \( \{a_j: a_j > 1\} \) is equal to that of the set \( \{a'_j: a'_j > 1\} \).

Suppose next that there is an equivariant \( C^2 \) diffeomorphism \( f \) of \( S^k \) with the \( R^a \) action \( \psi^a \) to \( S^k \) with the \( R^{a'} \) action \( \psi^{a'} \). We shall show that there is an equivariant \( C^2 \) diffeomorphism \( g \) of \( S^1 \) with the \( R^a \) action \( \psi^a \) to \( S^1 \) with the \( R^{a'} \) action \( \psi^{a'} \). Put

\[ A(x) = \{(t, (1 - t^2)^{1/2} x) \in S^k: -1 < t < 1 \}, \]
\[ C(x) = \{(\sin \theta, \cos \theta \cdot x) \in S^k: \theta \in R \}, \]

for \( x \in F \). Then \( C(x) \) is the closure of the union \( A(x) \cup A(-x) \). Since the map \( f \) is equivariant, we have \( f(A(x)) = A(f(x)) \) for \( x \in F \). Then we have \( f(-x) = -f(x) \) for \( x \in F \), by the differentiability of \( f \) at \( x_0 = 1 \). Hence \( f(C(x)) = C(f(x)) \) for \( x \in F \). Since the \( R^a \) action \( \psi^a \) is compatible with the \( O(k) \) action (see the proof of Proposition 6.1), we can assume \( f(y) = y \) for some \( y \in F \). Then the restriction \( f: C(y) \to C(y) \) can be regarded as an equivariant \( C^2 \) diffeomorphism \( g \) of \( S^1 \) with the \( R^a \) action \( \psi^a \) to \( S^1 \) with the \( R^{a'} \) action \( \psi^{a'} \).

Finally we shall show that the existence of \( g \) implies \( \prod_{j=1}^{N} (1 - a_j) = \prod_{j=1}^{N} (1 - a'_j) \). Since \( g \) is equivariant, we have \( g_*(\xi^a) = \xi^{a'} \). Let \( \pi: S^1 \to R \) be a map defined by \( \pi(x_0, x_1) = x_1 \). Then \( \pi \) is a local diffeomorphism at \( x_0 = \pm 1 \), and

\[ \pi_*(\xi^a) = -x_1(1 - x_1^2) \prod_{j=1}^{N} (1 - a_j(1 - x_1^2))(d/dx_1) \].

There is a local \( C^2 \) diffeomorphism \( h \) of \( R \) such that \( h(0) = 0, \pi \cdot g = \)
h \cdot \pi$. Then it follows from \( h_{a}(\pi_{a}(\xi^*)\pi_{a}(\xi^*)) = \pi_{a}(\xi^*) \) that
\[-x_{i}(1-x_{i}^{2}) \prod_{j=1}^{N}(1-a_{j}(1-x_{j}^{2})) \frac{dh}{dx_{i}}(x_{i}) = -y_{i}(1-y_{i}^{2}) \prod_{j=1}^{N}(1-a_{j}^{*}(1-y_{j}^{2})) \] for \( y_{i} = h(x_{i}) \). Differentiate by \( x_{i} \), and put \( x_{i} = 0 \). Then we have the desired equation, because \( \frac{dh}{dx_{i}}(0) \neq 0 \). q.e.d.

7. Closed subgroups of \( O(n) \). In this section, we shall prove Lemmas 4.1 and 4.2. The method used here is essentially due to Dynkin[2].

PROOF OF LEMMA 4.1. Let \( G \) be a connected closed subgroup of \( O(n) \). Suppose that
\[(*) n \geq 5, \quad 0 < \dim O(n)/G \leq 2n - 2.
The inclusion map \( i: G \rightarrow O(n) \) gives an orthogonal faithful representation of \( G \).

(A) Suppose first that the representation \( i \) is irreducible.

(A-1) Suppose that \( G \) is not semi-simple. Let \( T \) be a one-dimensional closed central subgroup of \( G \). Since \( i \) is irreducible, the centralizer of \( T \) in \( O(n) \) agrees with \( U(n/2) \) by an inner automorphism of \( O(n) \) (cf. Uchida [12, Lemma 5.1]). Put \( n = 2k \). Then it can be assumed that \( G \) is a subgroup of \( U(k) \) and the inclusion \( G \rightarrow U(k) \) is irreducible. It follows that the center of \( G \) is one-dimensional by Schur’s lemma. Moreover the condition \((*) \) implies \( k(k-1) = \dim O(2k)/U(k) \leq 4k - 2 \). Hence \( k = 3, 4 \). It is easy to see that \( SU(3) \) has no semi-simple proper subgroup of codimension \( \leq 4 \), and \( SU(4) \) has no semi-simple proper subgroup of codimension \( \leq 2 \). Therefore the case (A-1) occurs only when \( n = 6, 8; G \) agrees with \( U(n/2) \) up to an inner automorphism of \( O(n) \).

(A-2) Suppose that \( G \) is semi-simple and the complexification \( i^c \) of the representation \( i \) is reducible. Then \( n = 2k \), \( G \) is isomorphic to a subgroup \( G' \) of \( U(k) \), and the inclusion \( G' \rightarrow U(k) \) is irreducible. Hence \( k = 3, 4 \) and \( G' = SU(k) \). Calculating the centralizer of the center of \( G \) in \( O(n) \), we can show that \( G \) agrees with \( SU(n/2) \) up to an inner automorphism of \( O(n) \).

(A-3) Suppose that \( G \) is semi-simple, non-simple, and \( i^c \) is irreducible. Let \( G^* \) be the universal covering group of \( G \), and let \( p: G^* \rightarrow G \) be the projection. Since \( G \) is not simple, there are closed semi-simple normal subgroups \( H_1, H_2 \) of \( G^* \) such that \( G^* = H_1 \times H_2 \). Consider the representation \( i^{c}p: G^* \rightarrow U(n) \). Then there are irreducible complex representations \( r_t: H_t \rightarrow U(n_t) \) for \( t = 1, 2 \) such that the tensor product \( r_1 \otimes r_2 \) is equivalent to \( i^{c}p \). Since \( i^{c}p \) has a real form, the representations \( r_1, r_2 \) are self-conjugate; hence \( r_1 \) (resp. \( r_2 \)) has a real form or a quaternionic structure, but not both (cf. Adams [1, Proposition 3.56]).
Moreover, if \( r_1 \) has a real form (resp. quaternionic structure), then \( r_2 \) has also a real form (resp. quaternionic structure).

Suppose first that \( r_1, r_2 \) have quaternionic structures. Then it follows that \( n_1, n_2 \) are even, and \( \dim H_t \leq \dim Sp(n_t/2) = n_t(n_t + 1)/2 \) for \( t = 1, 2 \). The condition (*) implies \( \dim O(n) - \dim Sp(n_t/2) - \dim Sp(n_t/2) \leq 2n - 2, n = n_1n_2 \). Therefore \( n^2 - 3n + 4 \leq (n_1 + n_2)(n_1 + n_2 + 1) \leq (2 + n/2)^2 \). Hence \( n \leq 7 \). But \( n \) is a multiple of 4 and \( n \geq 5 \). Therefore \( r_1, r_2 \) cannot have quaternionic structures simultaneously.

Suppose next that \( r_1, r_2 \) have real forms. Then, since \( H_t \) is semisimple, it follows that \( n_t \geq 3 \) for \( t = 1, 2 \). Moreover, \( \dim H_t \leq \dim O(n_t) = n_t(n_t - 1)/2 \) for \( t = 1, 2 \). The condition (*) implies \( \dim O(n) - \dim O(n_t) - \dim O(n) \leq 2n - 2, n = n_1n_2 \). Therefore \( n^2 - 3n + 4 \leq (n_1 + n_2)(n_1 + n_2 - 1) \leq (3 + n/3)(2 + n/3) \). Hence \( n \leq 5 \). But \( n = n_1n_2 \geq 9 \). Therefore \( r_1, r_2 \) cannot have real forms simultaneously. Therefore the case (A-3) does not happen.

(A-4) Suppose finally that \( G \) is simple and \( i^c \) is irreducible. Put \( r = \text{rank } G \), and denote by \( G^* \) the universal covering group of \( G \). Denote by \( L_1, L_2, \ldots, L_r \) the fundamental weights of \( G^* \). Then there is a one-to-one correspondence between complex irreducible representations of \( G^* \) and sequences \( (a_1, \ldots, a_r) \) of non-negative integers such that \( a_1L_1 + \cdots + a_rL_r \) is the highest weight of a corresponding representation (cf. Dynkin [2, Theorems 0.8 and 0.9]; Humphreys [6, Section 21.2]). Denote by \( d(a_1L_1 + \cdots + a_rL_r) \) the degree of the complex irreducible representation of \( G^* \) with the highest weight \( a_1L_1 + \cdots + a_rL_r \). The degree can be computed by Weyl's dimension formula (cf. Dynkin [2, Theorem 0.24, (0.148)-(0.155)]; Humphreys [6, Section 24.3]). Notice that if \( a_i \geq a'_i \) for \( i = 1, 2, \ldots, r \), then \( d(a_1L_1 + \cdots + a_rL_r) \geq d(a'_1L_1 + \cdots + a'_rL_r) \) and the equality holds only if \( a_i = a'_i \) for \( i = 1, 2, \ldots, r \).

(A-4-1) Suppose that \( G \) is an exceptional Lie group. Then we have Table 3. Here \( m(G) \) is the least degree of non-trivial complex irreducible representations of \( G^* \) (cf. Dynkin [2, p. 378, Table 30]). The condition (*) implies that \( \dim G \geq \dim O(n) - (2n - 2) = (n - 1)(n - 4)/2 \). Hence \( (m - 1)(m - 4) \leq 2k \). The possibility remains only when \( G^* = G_2 \) and

<table>
<thead>
<tr>
<th>( G^* )</th>
<th>( k = \dim G )</th>
<th>( m = m(G) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_2 )</td>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>52</td>
<td>26</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>78</td>
<td>27</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>133</td>
<td>56</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>248</td>
<td>248</td>
</tr>
</tbody>
</table>
Since \( d(L_1) = 7, d(L_2) = 14, d(2L_1) = 27 \) for \( G^* = G_2 \), there is no complex irreducible representation of \( G_2 \) of degree 8. The complex irreducible representation of \( G_2 \) of degree 7 has a real form. Therefore the case (A-4-1) occurs only when \( n = 7 \) and \( G = G_2 \), where the inclusion \( G_2 \to O(7) \) is uniquely determined up to an inner automorphism of \( O(7) \).

(A-4-2) Suppose that \( G^* \) is isomorphic to \( SU(r + 1) \) for \( r \geq 1 \). Since \( \text{rank} \ G \leq \text{rank} SO(n) \), it follows that

\[
2r \leq n.
\]

The condition (*) implies that

\[
(n - 1)(n - 4)/2 \leq r(r + 2) \leq n(n - 1)/2, \quad n \geq 5.
\]

It is easy to see from (a), (b) that \( n \leq 13 \). If the pair \((n, r)\) satisfies the conditions (a), (b), then it is one of the following: (12, 6), (11, 5), (10, 5), (9, 4), (8, 4), (8, 3), (7, 3), (6, 2), (5, 2), (5, 1). Notice that \( d(L_1) = r + C_r, d(L_1 + L_r) = r(r + 2), d(2L_1) = d(2L_r) = (r + 1)(r + 2)/2 \). Hence there is no complex irreducible representation of \( SU(r + 1) \) of degrees \( 2r \) and \( 2r + 1 \) for \( r \geq 4 \). If \( r = 3 \), then \( d(L_1) = d(L_2) = 4, d(L_3) = 6, d(2L_1) = d(2L_2) = 10, d(2L_3) = d(L_1 + L_3) = 20, d(L_1 + L_3) = 15 \). Hence there is no complex irreducible representation of \( SU(4) \) of degrees 7 and 8. If \( r = 2 \), then \( d(L_1) = d(L_2) = 3, d(2L_1) = d(2L_2) = 6, d(L_1 + L_2) = 8 \). Hence there is no complex irreducible representation of \( SU(3) \) of degree 5. There are just two complex irreducible representations of \( SU(3) \) of degree 6 which are not self-conjugate. Therefore there is no possibility for \( r \geq 2 \). Finally there is only one complex irreducible representation of \( SU(2) \) of degree 5 which has a real form. Therefore the case (A-4-2) occurs only when \( n = 5 \) and \( G = SO(3) \), where the inclusion \( SO(3) \to O(5) \) is an irreducible representation uniquely determined up to an inner automorphism of \( O(5) \).

(A-4-3) Suppose that \( G^* \) is isomorphic to \( Sp(r) \) for \( r \geq 2 \). The condition (*) implies that \((n - 1)(n - 4)/2 \leq r(2r + 1) < n(n - 1)/2\). Hence \( n = 2r + 2 \) or \( n = 2r + 3 \). Notice that \( d(L_1) = 2r + 1, d(2L_1) = r(2r + 1) \). If \( r \geq 3 \), then \( d(L_1) < d(L_2) < \cdots < d(L_s) \geq d(L_{s+1}) > \cdots > d(L_r) \) for some \( s \). It is easy to see that there is no complex irreducible representation of \( Sp(r) \) of degrees \( 2r + 2 \) and \( 2r + 3 \) for \( r \geq 3 \). If \( r = 2 \), then \( d(L_1) = 4, d(L_2) = 5, d(2L_1) = 10, d(2L_2) = 14, d(L_1 + L_2) = 16 \). Hence there is no complex irreducible representation of \( Sp(r) \) of degrees \( 2r + 2 \) and \( 2r + 3 \) for \( r \geq 2 \). Therefore the case (A-4-3) does not happen.

(A-4-4) Suppose that \( G^* \) is isomorphic to \( Spin(k) \) for \( k \geq 5 \). The condition (*) implies that \((n - 1)(n - 4) \leq k(k - 1) < n(n - 1)\). Hence
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If $k = 2r$, then $d(L_i) = 2^{r-1}i$, for $1 \leq i \leq r - 2$, $d(L_{r-1}) = d(L_r) = 2^{r-1}i$, $d(2L_i) = (r + 1)(2r - 1)$, $d(2L_{r-1}) = d(L_r) = 2^{r-1}(2r - 1)$, $d(L_{r-1} + L_r) = 2^{r-1}C_{r-1}$. Hence there is no complex irreducible representation of $\text{Spin}(2r)$ of degrees $2r + 1$ and $2r + 2$. If $k = 2r + 1$, then $d(L_i) = 2^{r+1}i$, for $1 \leq i \leq r - 1$, $d(L_r) = 2^{r+1}i$, $d(2L_i) = r(2r + 3)$, $d(L_i + L_r) = 2^{r+1}i$, $d(2L_r) = 2^{2r}$. Hence there is no complex irreducible representation of $\text{Spin}(2r + 1)$ of degrees $2r + 2$ and $2r + 3$ for $r \geq 3$, there is no complex irreducible representation of $\text{Spin}(7)$ of degree 9, but there is only one complex irreducible representation of $\text{Spin}(7)$ of degree 8 which has a real form. Therefore the case (A-4-4) occurs only when $n = 8$ and $G = \text{Spin}(7)$, the inclusion $\text{Spin}(7) \hookrightarrow \text{O}(8)$ is a real spin representation uniquely determined up to an inner automorphism of $\text{O}(8)$.

Consequently, the case (A) occurs only when $G$ is one of the following listed in Table 4 up to an inner automorphism of $\text{O}(n)$. Here

<table>
<thead>
<tr>
<th>$n$</th>
<th>$G$</th>
<th>$i: G \to \text{O}(n)$</th>
<th>$	ext{dim} \text{O}(n)/G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$\text{Spin}(7)$</td>
<td>$A_7$</td>
<td>$7 = n - 1$</td>
</tr>
<tr>
<td>8</td>
<td>$U(4)$</td>
<td>$\mu_4$</td>
<td>$12 = 2n - 4$</td>
</tr>
<tr>
<td>8</td>
<td>$SU(4)$</td>
<td>$\mu_6$</td>
<td>$13 = 2n - 3$</td>
</tr>
<tr>
<td>7</td>
<td>$G_2$</td>
<td>$\omega$</td>
<td>$7 = n$</td>
</tr>
<tr>
<td>6</td>
<td>$U(3)$</td>
<td>$\mu_3$</td>
<td>$6 = n$</td>
</tr>
<tr>
<td>6</td>
<td>$SU(3)$</td>
<td>$\mu_6$</td>
<td>$7 = 2n - 5$</td>
</tr>
<tr>
<td>5</td>
<td>$SO(3)$</td>
<td>$\beta$</td>
<td>$7 = 2n - 3$</td>
</tr>
</tbody>
</table>

$\mu_4: U(k) \to \text{O}(2k)$, $\mu_6: SU(k) \to \text{O}(2k)$ are the canonical inclusions, and $A_7$, $\omega$, $\beta$ are irreducible representations uniquely determined up to an inner automorphism of $\text{O}(n)$, respectively.

(B) Suppose next that the representation $i: G \to \text{O}(n)$ is reducible. Then, by an inner automorphism of $\text{O}(n)$, $G$ is isomorphic to a subgroup $G'$ of $\text{O}(k) \times \text{O}(n - k)$ for some $k$ such that $0 < k \leq n/2$. The condition $(\ast)$ implies that

\[(c) \quad k(n - k) = \dim \text{O}(n)/\text{O}(k) \times \text{O}(n - k) \leq 2n - 2.\]

Hence $k = 1, 2$ or $k = 3$ and $n = 6, 7$. If $k = 3$ and $n = 6, 7$, then it is easy to see that $G' = SO(3) \times SO(3)$, $G' = SO(3) \times SO(4)$, respectively. Suppose $k = 2$. Then the inequality $(c)$ implies $2 + \dim G' \leq \dim \text{O}(2) \times \text{O}(n - 2)$. Since $SO(n - 2)$ is semi-simple for $n \geq 5$, $SO(n - 2)$ has no closed subgroup of codimension one. Therefore $G' = SO(n - 2)$, $SO(2) \times SO(n - 2)$ or $G' = SO(2) \times G''$, where $G''$ is a closed subgroup of $\text{O}(n - 2)$ of codimension 2. If the inclusion $G'' \to \text{O}(n - 2)$ is irreducible, then $n = 5, 6$.
by the case (A). Hence \( n = 6 \) and \( G'' = U(2) \). If the inclusion \( G'' \to O(n-2) \) is reducible, then \( n = 5 \) and \( G'' \) is a maximal torus of \( SO(3) \). Suppose \( k = 1 \). Then \( G' \) is a closed subgroup of \( O(n-1) \), and the inequality (c) implies \( \dim O(n-1)/G' \leq n-1 \). It can be assumed that the inclusion \( G' \to O(n-1) \) is irreducible. By the case (A), \( G' \) is one of the following listed in Table 5. Consequently, the case (B) occurs only when \( G \) is one of the following listed in Table 6 up to an inner automorphism of \( O(n) \). Here \( \rho_k: SO(k) \to O(k) \) is the canonical inclusion, and \( \theta^k \) is the trivial representation of degree \( k \). This completes the proof of Lemma 4.1.

**Table 5**

<table>
<thead>
<tr>
<th>( n-1 )</th>
<th>( G' )</th>
<th>( G' \to O(n-1) )</th>
<th>( \dim O(n-1)/G' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n-1 )</td>
<td>( SO(n-1) )</td>
<td>( \rho_{n-1} )</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>( Spin(7) )</td>
<td>( I_1 )</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>( G_2 )</td>
<td>( \omega )</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>( U(3) )</td>
<td>( \mu_3 )</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>( U(2) )</td>
<td>( \mu_2 )</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>( SU(2) )</td>
<td>( \mu_2^0 )</td>
<td>3</td>
</tr>
</tbody>
</table>

**Table 6**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( G )</th>
<th>( i: G \to O(n) )</th>
<th>( \dim O(n)/G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>( SO(n-1) )</td>
<td>( \rho_{n-1} \oplus \theta^1 )</td>
<td>( n-1 )</td>
</tr>
<tr>
<td>( n )</td>
<td>( SO(n-2) )</td>
<td>( \rho_{n-2} \oplus \theta^2 )</td>
<td>( 2n-3 )</td>
</tr>
<tr>
<td>( n )</td>
<td>( SO(n-2) \times SO(2) )</td>
<td>( \rho_{n-2} \oplus \rho_2 )</td>
<td>( 2n-4 )</td>
</tr>
<tr>
<td>9</td>
<td>( Spin(7) )</td>
<td>( I_1 \oplus \theta^1 )</td>
<td>( 15-2n-3 )</td>
</tr>
<tr>
<td>8</td>
<td>( G_2 )</td>
<td>( \omega \oplus \theta^1 )</td>
<td>( 14-2n-2 )</td>
</tr>
<tr>
<td>7</td>
<td>( U(3) )</td>
<td>( \mu_3 \oplus \theta^1 )</td>
<td>( 12-2n-2 )</td>
</tr>
<tr>
<td>7</td>
<td>( SO(3) \times SO(4) )</td>
<td>( \rho_3 \oplus \rho_4 )</td>
<td>( 12-2n-2 )</td>
</tr>
<tr>
<td>6</td>
<td>( SO(3) \times SO(3) )</td>
<td>( \rho_3 \oplus \rho_3 )</td>
<td>( 9-2n-3 )</td>
</tr>
<tr>
<td>6</td>
<td>( U(2) \times U(1) )</td>
<td>( \mu_2 \oplus \mu_1 )</td>
<td>( 10-2n-2 )</td>
</tr>
<tr>
<td>5</td>
<td>( U(2) )</td>
<td>( \mu_2 \oplus \theta^1 )</td>
<td>( 6-2n-4 )</td>
</tr>
<tr>
<td>5</td>
<td>( SU(2) )</td>
<td>( \mu_2^0 \oplus \theta^1 )</td>
<td>( 7-2n-3 )</td>
</tr>
<tr>
<td>5</td>
<td>( U(1) \times U(1) )</td>
<td>( \mu_1 \oplus \mu_1 \oplus \theta^1 )</td>
<td>( 8-2n-2 )</td>
</tr>
</tbody>
</table>

**Proof of Lemma 4.2.** It is sufficient to prove that there is no irreducible real representation of \( SO(n) \) of degree \( m \) for \( 5 \leq n < m \leq 2n-2 \), and a non-trivial orthogonal representation of \( SO(n) \) of degree \( n \) is equivalent to the canonical representation \( \rho_n \) up to an inner automorphism of \( O(n) \). The second half is well known and a proof is given in our previous paper [12, Section 5]. To prove the first half, suppose that there is an irreducible real representation \( \sigma \) of \( SO(n) \) of degree \( m \) for \( 5 \leq n < m \leq 2n-2 \). Then it is easy to see that the complexification \( \sigma^c \) of \( \sigma \) is irreducible. Let \( p: Spin(n) \to SO(n) \) be the covering pro-
jection. Then the composition $\sigma^p$ is an irreducible complex representation of $Spin(n)$, which has a real form. Suppose $n = 2r$. Then $d(L_i) = z_iC_i$ for $1 \leq i \leq r - 2$, $d(L_{r-1}) = d(L_r) = 2^{r-1}$, $d(2L_i) = (r + 1)(2r - 1)$, $d(2L_{r-1}) = d(2L_r) = z_{r-1}C_{r-1}$, $d(L_i + L_{r-1}) = d(L_i + L_r) = 2^{r-1}(2r - 1)$, $d(L_{r-1} + L_r) = z_rC_{r-1}$. Therefore the following are the only possibilities for the irreducible complex representation of $Spin(2r)$ of degree $m$ ($2r < m \leq 4r - 2$):

$$
\tau_r^+, \tau_r^-: Spin(2r) \rightarrow U(2^{r-1}) \quad \text{for} \quad r = 5
$$

$$
\tau, \tau^*: Spin(6) = SU(4) \rightarrow U(10).
$$

Here the representation space of $\tau$ is the second symmetric product of the canonical representation space $C^4$ of $SU(4)$, and $\tau^*$ is the dual representation. Hence $\tau, \tau^*$ have no real form. It is known that the half spin representations $\tau_{2r}^+, \tau_{2r}^-$ are not induced from a representation of $SO(2r)$. Suppose $n = 2r + 1$. Then $d(L_i) = z_{r+1}C_i$ for $1 \leq i \leq r - 1$, $d(L_r) = 2^r$, $d(2L_i) = r(2r + 3)$, $d(L_i + L_r) = 2^{r+1}r$, $d(2L_r) = 2^{2r}$. Therefore the following is the only possibility for the irreducible complex representation of $Spin(2r + 1)$ of degree $m$ ($2r + 1 < m \leq 4r$):

$$
\tau_{2r+1}: Spin(2r + 1) \rightarrow U(2^r) \quad \text{for} \quad r = 3, 4.
$$

It is known that the spin representation $\tau_{2r+1}$ is not induced from a representation of $SO(2r + 1)$. Consequently, we have the desired result.

q.e.d.

8. Concluding remark. If $5 \leq n \leq m \leq 2n - 2$, then there exists only one linear $SO(n)$ action $\rho_n \oplus \theta^{n-m+1}$ on the standard $m$-sphere (see Theorem 4.11). This action is the restriction of a linear $SL(n, R)$ action. We shall show a counterexample for $n = 4$.

Recall that there is a surjective homomorphism $\pi: SO(4) \rightarrow SO(3)$. Through this homomorphism, $SO(4)$ acts on $R^4$ and the action is transitive on the unit sphere $S^3$ with the isotropy group $U(2)$. Also $SO(4)$ acts naturally on $R^4$ and the action is transitive on the unit sphere $S^3$ with the isotropy group $SO(3)$. Thus we have the diagonal action of $SO(4)$ on the unit sphere $S^3$ of $R^4 \oplus R^4$. This action is a linear $SO(4)$ action on $S^3$, the principal orbit type is $SO(4)/SO(2)$ and there are just two singular orbit types $SO(4)/SO(3)$ and $SO(4)/U(2)$.

**Proposition 8.1.** The above $SO(4)$ action on $S^3$ is not extendable to any continuous $SL(4, R)$ action on $S^3$.

**Proof.** Suppose that there exists a continuous $SL(4, R)$ action on $S^3$ which is an extension of the $SO(4)$ action. Let $x \in S^3$ be a point such
that $SO(4)_x = U(2)$. Then

1. $U(2) \subset SL(4, \mathbb{R}) \neq SL(4, \mathbb{R})$,
2. $\dim SL(4, \mathbb{R})/SL(4, \mathbb{R})_x \leq 6$.

Here we shall show first the following result.

**Lemma 8.2.** Let $u(2)$ be the Lie algebra of $U(2)$. Let $g$ be a proper Lie subalgebra of $\mathfrak{sl}(4, \mathbb{R})$ which contains $u(2)$. Then $\dim g = 4, 6, 7$ or 10.

**Proof.** Recall

$$U(2) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in M_4(\mathbb{R}) : A^tA + B^tB = I_2, A^tB = B^tA \right\}.$$

Put

$$u(2) = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M_4(\mathbb{R}) : X + \imath X = 0, Y = \imath Y \right\},$$

$$\mathfrak{b}(2) = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M_4(\mathbb{R}) : X = \imath X, Y + \imath Y = 0, \text{trace } X = 0 \right\},$$

$$\mathfrak{a} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in M_4(\mathbb{R}) : X = \imath X, Y = \imath Y \right\},$$

$$\mathfrak{b} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in M_4(\mathbb{R}) : X + \imath X = Y + \imath Y = 0 \right\}.$$

Then $\mathfrak{sl}(4, \mathbb{R}) = u(2) \oplus \mathfrak{b}(2) \oplus \mathfrak{a} \oplus \mathfrak{b}$ as a direct sum of $Ad(U(2))$ invariant linear subspaces. Here $\mathfrak{b}(2)$, $\mathfrak{a}$ and $\mathfrak{b}$ are irreducible, respectively, and $\dim \mathfrak{b}(2) = 3$, $\dim \mathfrak{a} = 6$, $\dim \mathfrak{b} = 2$. Moreover, we have $[\mathfrak{b}(2), \mathfrak{a}] = \mathfrak{b}$, $[\mathfrak{b}(2), \mathfrak{b}] = \mathfrak{a}$, $[\mathfrak{a}, \mathfrak{b}] = \mathfrak{b}(2)$, $[\mathfrak{a}, \mathfrak{a}] \subset u(2)$, $[\mathfrak{b}, \mathfrak{b}] \subset u(2)$. Therefore $g$ is one of the following: $u(2)$, $u(2) \oplus \mathfrak{a}$, $u(2) \oplus \mathfrak{b}$, $u(2) \oplus \mathfrak{b}(2)$. Then $\dim g = 4, 10, 6$ or 7, respectively.

We now return to the proof of Proposition 8.1. By the condition (1), (2), it follows from Lemma 8.2 that $\dim SL(4, \mathbb{R})_x = 10$. Therefore the orbit $SL(4, \mathbb{R}) \cdot x$ contains the orbit $SO(4) \cdot x$ as a proper subset. Since the orbit $SO(4) \cdot x$ is isolated, the orbit $SL(4, \mathbb{R}) \cdot x$ must intersect a principal orbit of the $SO(4)$ action. Hence there is an element $g \in SL(4, \mathbb{R})$ such that $SO(4)_{x} = SO(2)$. Put $y = gx$. Then there is an embedding $SO(4) \cdot y \subset SL(4, \mathbb{R}) \cdot y = SL(4, \mathbb{R}) \cdot x$. But $\dim SO(4) \cdot y = \dim SL(4, \mathbb{R}) \cdot x = 5$. Hence $SO(4) \cdot y = SL(4, \mathbb{R}) \cdot x$. Since $SO(4) \cdot y$ is a principal orbit, we have $x \in SO(4) \cdot y$. This is a contradiction. Therefore there is no continuous $SL(4, \mathbb{R})$ action on $S^6$ which is an extension of the $SO(4)$ action.
References


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