EXISTENCE PROBLEM OF TRANSVERSE FOLIATIONS
FOR SOME FOLIATED 3-MANIFOLDS

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Existence problem of transverse foliations

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1. Introduction. Recently several foliators began the study of an ordered set $\mathcal{W} = (\mathcal{F}_1, \ldots, \mathcal{F}_k)$ of codimension one foliations of a manifold
\[ M^\ast \text{ in general position}, \text{ that is}, \]
\[ \dim T_xF_{\{i(1)\}} \cap \cdots \cap T_xF_{\{i(p)\}} = n - p \]
for all \( x \in M \) and \( \{i(1), \cdots, i(p)\} \subseteq \{1, \cdots, k\} \) with \( p \leq n \).

When \( k = n + 1 \), we call \( \mathcal{W} \) an octahedral web if for all \( x \in M \) there is a chart \( \phi: U \to \mathbb{R}^n \) such that \( x \in U \) and \( F_i|U = \phi^*\mathcal{G}_i \) for all \( i \), where \( \mathcal{G}_i = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_i = c\}_{c \in \mathbb{R}} \) for \( i = 1, \cdots, n \) and \( \mathcal{G}_{n+1} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_1 + \cdots + x_n = c\}_{c \in \mathbb{R}} \). In Nishimori \[10\] and \[11\], the author classified almost all the octahedral webs on closed manifolds.

When \( k = n \), we call \( \mathcal{W} \) a multifoliation (or a total foliation). Tischler \[16\] constructed multifoliations on the total spaces of \( S^1 \)-bundles over closed surfaces, and Silberstein \[14\] constructed multifoliations on \( M \sim S^1 \) where \( M \) is a stably parallelizable manifold. Furthermore Hardorp \[5\] showed that all the closed orientable manifolds of dimension three admit multifoliations.

When \( k = 2 \), we see that \( \mathcal{W} \) is a pair of transverse foliations. For such \( \mathcal{W} \), there exists the study by Tamura and Sato \[15\]. They regarded a foliated manifold \((M, \mathcal{F})\) as an underlying manifold, and a foliation \( \mathcal{G} \) of \( M \) transverse to \( \mathcal{F} \) as a structure on \((M, \mathcal{F})\). From this point of view, Tamura and Sato characterized codimension-one \( C^\omega \) foliations transverse to the Reeb component of \( S^1 \times D^2 \) or to the Reeb foliation of \( S^3 \) and classified them topologically by introducing TS diagrams. From these results and the theorem of Novikov \[12\], they derived that the foliation of \( S^3 \) obtained from a fibered knot with a fiber of non-zero genus has no transversely orientable transverse codimension-one \( C^\omega \) foliation. In contrast to this, they remarked that any codimension one \( C^r \) foliation of \( S^3 \) admits a transverse 2-plane field. Furthermore they raised several problems on transverse foliations. One of them is the following.

**Problem A** \[15, \text{Problem 10}\]. *Find conditions for \( C^\omega \) foliated manifolds to admit transverse foliations.*

From now on, a manifold is always of class \( C^\omega \) and a foliation is a codimension one \( C^\omega \) foliation, unless stated otherwise.

In Part I of this paper, we generalize the results of Tamura and Sato on the Reeb component to the foliated manifolds \((E(h), \mathcal{F}(h; \sigma))\) introduced as follows. Take a positive integer \( h \) and let \( \hat{E}(h) \) be a compact manifold obtained from \( S^2 \) by deleting \( h \) small open 2-disks. Let \( E(h) = S^1 \times \hat{E}(h) \). We treat \( S^1 \) and \( \hat{E}(h) \) as oriented manifolds. Denote by \( \hat{\Gamma}(h) \) the set of the connected components of \( \partial \hat{E}(h) \) and let \( \Gamma(h) = \{S^1 \times \hat{C} | \hat{C} \in \hat{\Gamma}(h)\} \). Note that each \( C \in \Gamma(h) \) is diffeomorphic to \( T^2 \). Take a continuous map \( \sigma: \partial \hat{E}(h) \to \{1, -1\} \). Frequently we regard \( \sigma \) as a map
from $\partial E(h)$, $\Gamma(h)$ and $\hat{\Gamma}(h)$ to $\{1, -1\}$ in a canonical way without caution. We turbulize the product foliation 
\[ \mathcal{F}(h, pr) = \{\{x\times\hat{E}(h)|x\in S'\} \]

of $E(h)$ so that for each $y \in \partial \hat{E}(h)$ the oriented closed path $\sigma(y)(S' \times \{y\})$ has an expanding holonomy with respect to the modified foliation $\mathcal{F}(h; \sigma)$. The turbulization will be stated precisely in §2.

Tamura and Sato decomposed foliations transverse to the Reeb component into three kinds of simple components, namely half Reeb components, foliated $I$-bundles over $S' \times I$ and $TS$ components. In our case we decompose foliations transverse to $\mathcal{F}(h; \sigma)$ into nine kinds of components (see Theorem 3 in §9). For a foliated manifold $(M, \mathcal{F})$, we denote by $t^l(M, \mathcal{F})$ (or simply $t^l(\mathcal{F})$) the set of transversely orientable foliations of $M$ transverse to $\mathcal{F}$. We can classify the foliations in $t^l(E(h), \mathcal{F}(h; \sigma))$ with respect to a certain equivalence relation by using generalized $TS$ diagrams (see Theorem 4 in §14).

In Part II, as an application of the results of Part I we consider Problem A for a certain class of foliated manifolds of dimension three introduced as follows. Roughly speaking, our foliated 3-manifolds are unions of foliated manifolds of the form $(E(h), \mathcal{F}(h; \sigma))$.

First take a connected finite graph $\Phi$ and fix an orientation for each side of $\Phi$. Denote by $V(\Phi)$ (or $S(\Phi)$) the set of vertices (or sides) of $\Phi$. For $v \in V(\Phi)$, let $S(\Phi; v) = \{s \in S(\Phi)|v$ is an end of $s\}$, where we take two copies $s^+, s^-$ of $s$ if the ends of $s$ coincide and are $v$. Let $h(v) = \#S(\Phi; v)$ and $E[v] = E(h(v))$. We fix a bijection $C[v]: S(\Phi; v) \to \Gamma[v] = \Gamma(h(v))$.

Take a map $\Psi: S(\Phi) \to \left\{ (k, l, m, n) | kn - lm = -1, k, l, m, n \in \mathbb{Z} \right\}$. For each side $s \in S(\Phi)$ with $\partial(s) = (v_i) - (v_z), v_i, v_z \in V(\Phi)$, we define a diffeomorphism $\Psi^*[s]: C[v_i](s) \to C[v_z](s)$ by 
\[ \psi^*[s](x, y) = (kx + ly, mx + ny), x, y \in R/Z, \]

where $\Psi(s) = (\frac{k}{l}, \frac{m}{n})$, $C[v_i](s) = S' \times \hat{C}_i$, and $S', \hat{C}_i$ and $\hat{C}_z$ are identified with $R/Z$. When the ends of $s$ coincide and are $v$, we use the convention that $C[v_i](s) = C[v_i](s^+)$ and $C[v_i](s) = C[v_i](s^-)$. Now we obtain a closed connected manifold $M(\Phi, \Psi)$ from the disjoint union $\bigcup \{E[v]|v \in V(\Phi)\}$ by identifying $C[v_i](s)$ with $C[v_z](s)$ by $\Psi^*[s]$ for all $s \in S(\Phi)$.

Take a continuous map $\sigma: \Gamma[v] = \bigcup \{\Gamma[v]|v \in V(\Phi)\} \to \{1, -1\}$. Then we have a foliation $\mathcal{F}(\Phi, \Psi; \sigma)$ of $M(\Phi, \Psi)$ such that $\mathcal{F}(\Phi, \Psi; \sigma)|E[v] = \mathcal{F}(\Gamma[v]; \sigma \partial E[v])$ for all $v \in V(\Phi)$, where $\mathcal{F}(\Phi, \Psi; \sigma)|E[v]$ is the foliation induced from $\mathcal{F}(\Phi, \Psi; \sigma)$ by the canonical immersion $\zeta: E[v] \to M(\Phi, \Psi)$. 
We denote by \( t_1^*(M(\mathcal{F}, \mathcal{F}; \sigma)) \) (or simply \( t_1^*(\mathcal{F}(\Phi, \Psi; \sigma)) \)) the set of foliations \( \mathcal{F} \) of \( M(\Phi, \Psi) \) transverse to \( \mathcal{F}(\Phi, \Psi; \sigma) \) such that \( \mathcal{F}|E[v] \) is transversely orientable. Clearly \( t_1^*(\mathcal{F}(\Phi, \Psi; \sigma)) \supseteq t_0^*(\mathcal{F}(\Phi, \Psi; \sigma)) \). Our main purpose is to investigate whether \( t_1^*(\mathcal{F}(\Phi, \Psi; \sigma)) \) is empty or not. Note that if \( \Phi \) is a tree (that is, a connected contractible graph) then \( \mathcal{F}(\Phi, \Psi; \sigma) \) is transversely orientable for all \( \Psi \) and \( \sigma \) and it follows that \( t_1^*(\mathcal{F}(\Phi, \Psi; \sigma)) = t_0^*(\mathcal{F}(\Phi, \Psi; \sigma)) \).

Our criterion for the existence of transverse foliations splits into two stages—an arithmetic criterion and a geometric one. Although the former is stronger than the homotopy theoretic one asking for the existence of transverse 2-plane fields (see Theorem 2 below), we do not know whether it is complete as a criterion or not. The latter is complete and takes the form of a jigsaw puzzle or a tangram (see Theorems 8, 8* and 8** in §21).

Now we formulate the arithmetic criterion precisely.

**Definition 1.1.** Let \( (N \times \mathbb{Z})_{\text{coprime}} = \{(a, b) \in N \times \mathbb{Z} | ap + bq = 1 \text{ for some } p, q \in \mathbb{Z}\} \), where \( N \) is the set of positive integers. Let \( (N \times \mathbb{Z})^* = (N \times \mathbb{Z})_{\text{coprime}} \cup \{(0, 1), (\infty, \infty)\} \).

**Definition 1.2.** An arithmetic model transverse to \( \mathcal{F}(\Phi, \Psi; \sigma) \) is a map \( (a, b; r): \Gamma[\Phi] \to (N \times \mathbb{Z})^* \times \mathbb{Z} \), where \( \mathbb{Z} \) is the set of even integers, satisfying the following conditions (A1)-(A5).

(A1) Consider \( v \in V(\Phi) \) with \( h(v) = 1 \), and let \( \Gamma[v] = \{C\} \). Then \( a(C) = 1 \).

(A2) Consider \( v \in V(\Phi) \) with \( h(v) = 2 \), and let \( \Gamma[v] = \{C_1, C_2\} \). Then

(i) \( r(C_2) = -r(C_1) \).

(ii) If \( r(C_i) \neq 0 \) and \( a(C_i) > 0 \), then \( a(C_2) = a(C_1)^{-1} \) and \( b(C_2) = -b(C_1) \).

(iii) If \( r(C_1) \neq 0 \) and \( a(C_1) = 0 \), then \( a(C_2) = 0 \) and \( \sigma(C_2) = -\sigma(C_1) \).

(A3) Consider \( v \in V(\Phi) \) with \( h(v) > 2 \), and let \( C \in \Gamma[v] \). If \( (a(C), b(C)) \neq (1, 0) \), then \( r(C) = 0 \).

(A4) (The TS formula). For each \( v \in V(\Phi) \),

\[
\sum_{c \in \Gamma[v]} a(C)r(C) = 4 - 2h(v),
\]

where we use the convention \( \infty \cdot 0 = 0 \).

(A5) (The compatibility condition). Let \( s \in S(\Phi) \) with \( \partial(s) = (v_i) - (v_i) \) and \( C_i = C[v_i](s) \) for \( i = 1, 2 \).

(i) If \( a(C_i) = \infty \), then \( a(C_2) = \infty \) and \( r(C_i) = r(C_2) = 0 \).

(ii) If \( a(C_i) \neq \infty \), then
\[
\begin{pmatrix}
a(C_2) \\
b(C_2)
\end{pmatrix} = \gamma_1 \begin{pmatrix} k & l \\ m & n \end{pmatrix} \begin{pmatrix} a(C_1) \\
b(C_1)
\end{pmatrix} \quad \text{and} \quad r(C_2) = \gamma_2 r(C_1).
\]

In the above, we put \( \Psi(s) = \begin{pmatrix} k & l \\ m & n \end{pmatrix} \) and

\[
\gamma_1 = \begin{cases} 
\text{sgn}(ka(C_1) + lb(C_1)) & \text{if } a(C_2) > 0, \\
\text{sgn}(ma(C_1) + nb(C_1)) & \text{if } a(C_2) = 0.
\end{cases}
\]

Furthermore, we put \( \delta(0) = 0 \) and \( \delta(a) = 1 \) for \( a > 0 \), and

\[
\gamma_2 = \gamma_1 \cdot \sigma(C_1)^{[(a(C_1))] - \sigma(C_2)^{[(a(C_2))]} - 1}.
\]

We denote by \( \text{am}(\Phi, \Psi; \sigma) \) the set of arithmetic models transverse to \( \mathcal{F}(\Phi, \Psi; \sigma) \).

Now we can state the arithmetic criterion.

**Theorem 1.** There exists a canonical map

\[
\alpha: t^*_* (F(\Phi, \Psi; \sigma)) \rightarrow \text{am}(\Phi, \Psi; \sigma).
\]

Roughly speaking, if \( \alpha(\mathcal{S}) = (a, b; r) \) for \( \mathcal{S} \in t^*_* (F(\Phi, \Psi; \sigma)) \), then \( (a(C), b(C)) \) represents the homology class of a compact leaf of \( \mathcal{F}|_C \) and \( r(C) \) is the difference of the numbers of the positive Reeb components (cf. Definition 4.1) and negative Reeb components of \( \mathcal{F}|_C \). The condition (A1) was already known in Davis and Wilson [1] and does not depend on the integrability of \( \mathcal{F}|E[v] \). The condition (A3) reflects the integrability of \( \mathcal{F}|E[v] \) (see Remark 19.3). The conditions (A4) and (A5) do not depend on the integrability of \( \mathcal{F} \), but it is not clear whether (A2) does or not.

The following is a direct consequence of Theorem 1.

**Theorem 1*. (The arithmetic criterion). If \( \text{am}(\Phi, \Psi; \sigma) = \emptyset \), then \( t^*_* (\mathcal{F}(\Phi, \Psi; \sigma)) = \emptyset \).

It is comparatively easy to see whether \( \text{am}(\Phi, \Psi; \sigma) \) is empty or not. We will give some examples in §19. The following will be proved in §24.

**Theorem 2.** If \( \text{am}(\Phi, \Psi; \sigma) \neq \emptyset \), then there is a 2-plane field of \( M(\Phi, \Psi) \) transverse to \( \mathcal{F}(\Phi, \Psi; \sigma) \).

Our criterion is practical. The algorithm is as follows. First determine whether \( \text{am}(\Phi, \Psi; \sigma) \) is empty or not. When \( \text{am}(\Phi, \Psi; \sigma) = \emptyset \), we are done. When \( \text{am}(\Phi, \Psi; \sigma) \neq \emptyset \), try to construct a TS model (cf. Definition 20.7) transverse to \( \mathcal{F}(\Phi, \Psi; \sigma) \). In many cases we find a TS model. So far we did not find any \( \mathcal{F}(\Phi, \Psi; \sigma) \) such that \( \text{am}(\Phi, \Psi; \sigma) \neq \emptyset \).
and $t^*\left(\mathcal{F}(\phi, \mathcal{T}; \sigma)\right) = \emptyset$. We give some examples in § 22. We hope that our criterion will give a hint to constructing a theoretical or general criterion. If a new criterion is found, then we can test it by the examples investigated by our criterion.

We wish to thank Professors I. Tamura and A. Sato for critical and valuable discussions.

PART I

A generalization of the results of Tamura and Sato

2. Turbulization I and Reeb components. We describe the turbulence precisely. Let $W$ be a compact manifold with boundary and $M$ a codimension-zero compact submanifold of $\partial W$. Let $\mathcal{F}$ be the set of connected components of $M$. Choose a small collar $\hat{k}: M \times [0, 1] \rightarrow W$ such that

$$
\hat{k}(y, 0) = y \quad \text{for } y \in M,
\hat{k}(y, t) \in \partial W \quad \text{for } y \in \partial M \text{ and } t \in [0, 1].
$$

Let $k: S^1 \times M \times [0, 1] \rightarrow S^1 \times W$ be the collar of $S^1 \times M$ defined by $k(x, y, t) = (x, \hat{k}(y, t))$ for $x \in S^1$, $y \in M$ and $t \in [0, 1]$. Let $W^\circ = \text{Cl}(W - \hat{k}(M \times [0, 1]))$. Take a $C^\infty$ function $f: [0, 1] \rightarrow [-\infty, 0]$ such that

- $(f_1)$ $f(t) = 0$ for all $t \in [1/2, 1]$,
- $(f_2)$ $\lim_{t \rightarrow 0} f(t) = -\infty$,
- $(f_3)$ $\frac{df}{dt} > 0$ in $[0, 1/2]$,
- $(f_4)$ the submanifolds $R \times \{0\}$ and $F_{\varepsilon}(f) = \{(f(t) + c, t) | t \in [0, 1]\}$, $c \in R$, of $R \times [0, 1]$ are leaves of a foliation of $R \times [0, 1]$.

Take a continuous map $\sigma: M \rightarrow \{-1, 1\}$. Let $\mathcal{F}$ be a foliation of $S^1 \times W$ such that $\mathcal{F}|k(S^1 \times M \times [0, 1]) = \{(x) \times \hat{k}(M \times [0, 1]) | x \in S^1\}$. Then the foliation $T[\mathcal{F}, M, \sigma]$ obtained by turbulizing $\mathcal{F}$ around $M$ in the direction of $\sigma$ is defined so that $T[\mathcal{F}, M, \sigma]|S^1 \times W^\circ = \mathcal{F}|S^1 \times W^\circ$ and that $T[\mathcal{F}, M, \sigma]|k(S^1 \times M \times [0, 1])$ consists of compact leaves $S^1 \times \hat{C}$ for $\hat{C} \in \hat{F}$ and non-compact leaves

$$
\{k(\sigma(y)f(t)) + x, y, t | y \in \hat{C}, t \in [0, 1]\}
$$

for $x \in S^1 = R/Z$ and $\hat{C} \in \hat{F}$, where $[z]$ means $z \text{ mod } 1$.

Now consider $E(h) = S^1 \times \hat{E}(h)$, $\Gamma(h)$ and $\sigma: \partial E(h) \rightarrow \{-1, -1\}$ as in § 1. Let $\hat{E}(h)$ (or $\partial \hat{E}(h)$) play the role of $W$ (or $M$ respectively). (Below we omit the word “respectively” in the similar description.) Then we have the turbulized foliation $\mathcal{F}(h; \sigma)$ in § 1:

$$
\mathcal{F}(h; \sigma) = T[\mathcal{F}(h, pr), \partial \hat{E}(h), \sigma].
$$
Therefore $\mathcal{F}(h; \sigma)$ consists of compact leaves $C \in \Gamma(h)$ and non-compact leaves

$$F^* = S^1 \times \hat{E}(h)^\circ \cup k([(\sigma(y)f(t)] + x, y, t) | y \in \partial \hat{E}(h), t \in [0, 1])$$

for $x \in S^1$, where $\hat{E}(h)^\circ$ and $k: \partial E(h) \times [0, 1] \to E(h)$ are constructed as above.

We recall some definitions.

**Definition 2.1.** Let $\mathcal{F}_{pr}^{n+1}$ be the product foliation $\{(x) \times D^n | x \in S^1\}$ of $S^1 \times D^n$, and $\sigma: \partial D^n \to \{1, -1\}$ be a constant map. Then the turbulized foliation $\mathcal{F}_{R}^{n+1}(\sigma) = T[\mathcal{F}_{pr}^{n+1}, \partial D^n, \sigma]$ is called a standard Reeb component of $S^1 \times D^n$. We called $\mathcal{F}_{R}^{n+1}(1)$ plus and $\mathcal{F}_{R}^{n+1}(-1)$ minus in Tamura-Sato [15].

**Definition 2.2.** Let $\sigma: \partial D^1 \to \{1, -1\}$ be a bijection. Then $T[\mathcal{F}_{pr}^{1}, \partial D^1, \sigma]$ is called a standard slope component of $S^1 \times D^1$.

![Reeb components and slope components](image-url)
 DEFINITION 2.3. Let $D^n_+ = \{(x_1, \cdots, x_n) \in D^n | x_n \geq 0\}$. Then $T[I_r^{n+1}] S^r \times D^n_+, D^{r-1} \times \{0\}, \pm 1]$ is called a standard half Reeb component of $S^r \times D^n_+$.

The following is a new component appearing in our decomposition theorem but not being contained in foliations transverse to the Reeb component $\mathcal{F}_n^r(1)$ of $S^r \times D^n_+$.

DEFINITION 2.4. Let $D^{n+[1/2, 1]} = \{(x_1, \cdots, x_n) \in R^n | 1/4 \leq x_1^2 + \cdots + x_n^2 \leq 1\}$. Then $T[I_r^{n+1}] S^r \times D^{n+[1/2, 1]} \delta D^n_+, \pm 1]$ is called a standard tunneled Reeb component of $S^r \times D^n_+$.

DEFINITION 2.5. Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be $C^r$ foliated manifolds. We say that $\mathcal{F}_1$ is $C^r$ isomorphic to $\mathcal{F}_2$ if there is a $C^r$ diffeomorphism $\phi: M_1 \rightarrow M_2$ with $\mathcal{F}_1 = \phi^* \mathcal{F}_2$.

DEFINITION 2.6. A foliation $\mathcal{F}$ is called a Reeb (or slope, half Reeb, tunneled Reeb, etc.) component if $\mathcal{F}$ is $C^0$ isomorphic to a standard Reeb (or slope, half Reeb, tunneled Reeb, etc.) component.

For better understanding, we give some figures in Figure 2.1.

3. Turbulization II and several components I. For some foliations of $S^r \times S^r \times [0, 1]$, we can introduce a somewhat sophisticated type of turbulizations, as follows. The foliations thus obtained will appear in the decomposition theorem for $\mathcal{E} \in \mathcal{E}(\mathcal{F}(h; \sigma))$.

As the data, we take a transversely orientable foliation $\mathcal{G}_0$ of $S^r \times S^r$ without Reeb components, a positive integer $\mu_0$, a map $\sigma: \{1, \cdots, \mu_0\} \rightarrow \{1, -1\}$ and an element $(a, b) \in (N \times Z)^{\text{prime}}$ such that there is a closed transversal $L$ intersecting all the leaves of $\mathcal{G}_0$ with $[L] = a[S^r \times \{\ast\}] + b[\{\ast\} \times S^r]$ in $H_1(S^r \times S^r; Z)$. What we will turbulize is the product foliation $\mathcal{G}_0 \times I$, where $I$ means the interval $[0, 1]$.

Put $\mu = a \cdot \mu_0$. Let $M_i = \{[y] \in S^r = R/Z | (i-1)/\mu \leq y \leq (2i-1)/2\mu\}$ for $i = 1, \cdots, \mu$, and $M = M_1 \cup \cdots \cup M_\mu$. Let $M^* = \{([at], [bt] + \gamma) \in S^r \times S^r | t \in R, \gamma \in M\}$. Then $M^* \cap \{[0]\} \times S^r = M$, and $M^*$ has $\mu_0$ connected components. Let $M_\nu^*$ be the connected component of $M^*$ containing $\{[0]\} \times M_i$ for $i = 1, \cdots, \mu_0$.

We can construct a diffeomorphism $\alpha: S^r \times S^r \times I \rightarrow S^r \times S^r \times I$ satisfying the following conditions (1)-(3).

1. $\alpha(S^r \times S^r \times \{t\}) = S^r \times S^r \times \{t\}$ for all $t \in I$.
2. $\alpha|S^r \times S^r \times \{2/3, 1\} = \text{id}$.
3. There is a neighborhood $U$ of $M^*$ in $S^r \times S^r$ such that the leaves of $\alpha^*(\mathcal{G}_0 \times I)|U \times [0, 1/3)$ are connected components of $\{([-bt], [at] + \gamma) \wedge R \cap U \times [0, 1/3]$ for some $\gamma \in S^r$. (See Figure 3.1.)
Bend $S^1 \times S^1 \times I$ along $\partial M^* \times \{0\}$ so that $\partial M^* \times \{0\}$ is a corner. Choose a small collar $k: M^* \times I \to U \times [0, 1/3]$ such that $k(\partial M^* \times I) \subset U \times [0]$ and that the leaves of $k^* \alpha^*(\mathcal{G}_0 \times I)$ are connected components of $\{(\lfloor -bt \rfloor, [at] + y) \mid t \in \mathbb{R}\} \cap M^* \times I$ for $y \in S^1$. Then the turbulized foliation $T[g_0; \sigma_0, \sigma; a, b]$ is defined so that $T[g_0; \sigma_0, \sigma; a, b] = \alpha^*(\mathcal{G}_0 \times I)$ on $S^1 \times S^1 \times I - k(M^* \times I)$ and that $T[g_0; \sigma_0, \sigma; a, b]|k(M^* \times I)$ consists of compact leaves $M_i^* \times \{0\}, i = 1, \ldots, \mu$, and non-compact leaves $[k([a \sigma(i)f(t/3)] + x, [b \sigma(i)f(t/3)] + y, t) \mid t \in [0, 1], (x, y, 0) \in F]$ for leaves $F$ of $k^* \alpha^*(\mathcal{G}_0 \times I)|M_i^* \times \{0\}, i = 1, \ldots, \mu$, where we use $f: [0, 1] \to ]-\infty, 0]$ in §2. (See Figure 3.2.)

**Definition 3.1.** We call $T[\mathcal{G}_0; \mu_0, \sigma; a, b]$ a standard gear component if $\sigma$ is constant.

The definitions below are not used until §9, and it is possible to omit them till then.
When two standard gear components $F_1 = T[G_1; \sigma, a, b]$ and $F_2 = T[G_2; \sigma, a, b]$ with $G_1 = G_2$ are given, we can glue $F_1$ and $F_2$ by identifying $G_1 \sim 1$ and $G_2 \sim 1$, and obtain a foliation of a manifold homeomorphic to $S^1 \times S^1 \times I$.

**Definition 3.2.** The foliation obtained above is called a standard double gear component if the values of $\sigma$ and $\alpha$ are different.

When $G_0$ is the foliation $F_{pr} = \{x \times S^1 | x \in S^1\}$, we can glue the foliation $\{x \times D^1 | x \in S^1\}$ to $T[F_{pr}; \sigma; a, b]$, and obtain a foliation $T*[F_{pr}; \sigma; a, b]$.

**Definition 3.3.** We call $T*[F_{pr}; \sigma; a, b]$ a standard arcade component if $\sigma > 1$ and $\sigma$ is constant.

**Definition 3.4.** Let $q_1$ and $q_2$ be non-negative integers with $q_1 + q_2 > 0$. We call $T*[F_{pr}; \sigma, a, b]$ a standard TS component of type $(q_1, q_2)$ if $\sigma_0 = q_1 + q_2 + 2$ and

$$\sigma(j) = \begin{cases} 1 & \text{for } j = 1, \ldots, q_1 + 1, \\ -1 & \text{for } j = q_1 + 2, \ldots, \sigma_0. \end{cases}$$

**Remark 3.5.** A TS component of type $(0, q)$ is a TS component of type $q$ defined in Tamura-Sato [15].

When the leaves of $G_0$ are all compact, we can turbulize $T[G_0; \sigma, a, b]$ around $S^1 \times S^1 \times \{1\}$ in the directions orthogonal to $G_0$.

**Definition 3.6.** The foliation obtained above by turbulization is called a standard turbulized gear component if $\sigma$ is constant and the turbulization around $S^1 \times S^1 \times \{1\}$ is performed in the direction of $-\sigma(1)(a'[S^1 \times \{*\}] + b'[\{\} \times S^1])$, where $(a', b') \in (N \times Z)^{\text{coprime}} \cup \{(0, 1)\}$ with $aa' + bb' = 0$.

Let $G$ be a standard turbulized gear component obtained from $T[G_0; \sigma, a, b]$. We may suppose that $K = \{(at), [bt + (1/4\sigma)] | t \in R\} \times I$ is transverse to $G$. Then $G|K$ is a slope component. Therefore $G|K$ admits a smooth $S^1$ action (see Imanishi-Yagi [6], Fukui-Ushiki [3] and Fukui [2]) if the turbulization is carefully performed. Let $\beta: K \to K$ be a diffeomorphism such that $\beta$ maps each non-compact leaf of $G|K$ to a different leaf of $G|K$. Cutting $S^1 \times S^1 \times I$ along $K$ and pasting by $\beta$, we have a foliation $G'$ of manifold homeomorphic to $S^1 \times S^1 \times I$.

**Definition 3.7.** The foliation $G'$ obtained above is called a standard perturbed gear component.

We give some figures.
4. Preliminaries, a lemma of Tamura and Sato, and the TS formula. Let $\mathcal{F} \in \mathcal{F}(\mathcal{F}(h; \sigma))$. We give some remarks on $\mathcal{F}|C$ for $C \in \Gamma(h)$. Note that $C$ is diffeomorphic to $T^2$. When $\mathcal{F}|C$ has a compact leaf $L$, the homology class $[L] \in H_1(C; \mathbb{Z})$ depends only on $\mathcal{F}|C$. We call $C$ vertical if $\mathcal{F}|C$ has no compact leaf homologous to $\{x\} \times \hat{C}$, and otherwise horizontal. If there is an immersion $g: S^1 \rightarrow D^1 \rightarrow C$ such that $g|\text{Int}(S^1 \times D^1)$ is an imbedding and $g^*\mathcal{F}$ is a Reeb component, then $g$ is an imbedding, and $\mathcal{F}|C$ contains an even number of Reeb components, since $\mathcal{F}|C$ is transversely orientable. As in Tamura-Sato [15], we can construct a $C^\infty$ isotopy $\{\phi_t\}_{t \in \mathbb{R}} \subset \text{Diff}(E(h))$ satisfying the following conditions (E1)-(E5).

(E1) $\phi_t = \text{id}$ for $t \leq 0$, and $\phi_t = \phi_1$ for $t \geq 1$.

(E2) $\phi_t^*\mathcal{F} \in \mathcal{F}(\mathcal{F}(h; \sigma))$ for all $t \in \mathbb{R}$.

(E3) When $\mathcal{F}|C$ has no compact leaf for $C \in \Gamma(h)$, each leaf of $\phi_t^*\mathcal{F}|C$ is transverse to $\{x\} \times \hat{C}$ and $S^1 \times \{y\}$ for all $x \in S^1$ and $y \in \hat{C}$. 
When $C$ is horizontal, each compact leaf $L$ of $\phi^*_{\mathcal{C}}|C$ has the form $L = \{x\} \times \hat{C}$ for some $x \in S^1$.

When $C$ is vertical and $\mathcal{C}|C$ has a compact leaf, each compact leaf $L$ has the form $L = \{(at) + x, [bt]\}|t \in \mathbb{R}\}$ for some $a \in \mathbb{N}$ and $b \in \mathbb{Z}$, and for each Reeb component $\mathcal{R}$ contained in $\phi^*_{\mathcal{C}}$ there is a circle $\Sigma(\mathcal{R}) \subset \text{Int } |\mathcal{R}|$ such that $\phi^*_{\mathcal{C}}$ is tangent to the curves $\{x\} \times \hat{C}$, $x \in S^1$, at and only at $\Sigma(\mathcal{R})$.

Since it is sufficient for our purpose to consider $\phi^*_{\mathcal{C}}$ instead of $\mathcal{C}$, hereafter we treat $\phi^*_{\mathcal{C}}$ and denote it by $\mathcal{C}$ for simplicity.

Since $\mathcal{C}$ is transverse to $\partial E(h)$, there is $\varepsilon > 0$ such that $\mathcal{C}$ is transverse to $k(\partial E(h) \times \{t\})$ for all $t \in [0, \varepsilon]$, where $k : \partial E(h) \times [0, 1] \rightarrow E(h)$ is the collar used in the definition of $\mathcal{F}(h; \phi)$. Let $A = E(h) - k(\partial E(h) \times [0, \varepsilon])$ and $A^* = F^* \cap A$ for $x \in S^1$, where $F^*$ is the non-compact leaf defined in §2. Let $\partial_c A = k(C \times \{x\})$ and $\partial_c A^* = F^* \cap \partial_c A$ for $C \in \Gamma(h)$. Then $\partial A = \cup \{\partial_c A | C \in \Gamma(h)\}$. We have a diffeomorphism $\gamma: \partial E(h) \rightarrow \partial A$ such that $x \in \partial E(h)$ and $\gamma(z)$ belong to the same leaf of $\mathcal{C}$. We may assume the following conditions.

(E4)' If $C \in \Gamma(h)$ is horizontal, then $\partial_c A^{[0]}$ is a compact leaf of $\mathcal{C}|\partial A$, where $[0] \in S^1 = \mathbb{R}/\mathbb{Z}$ means $0$ mod $1$.

(E5)' If $C \in \Gamma(h)$ is vertical, then for each Reeb component $\mathcal{R}$ contained in $\mathcal{C}|\partial_c A$ there is a circle $\Sigma(\mathcal{R}) \subset \text{Int } |\mathcal{R}|$ such that $\mathcal{C}|\partial_c A$ is tangent to the curves $\partial_c A^*$, $x \in S^1$, at and only at $\Sigma(C) = \cup \{\Sigma(\mathcal{R}) | \mathcal{R}$ is a Reeb component contained in $\mathcal{C}|\partial_c A\}$, where $|\mathcal{R}|$ means the underlying manifold of $\mathcal{R}$. (See Figure 4.1.)

In order to recall a lemma in Tamura-Sato [15], we make preperations. Let $C \in \Gamma(h)$ be vertical. When $\mathcal{C}|C$ has a compact leaf $L$, we appoint the orientation of $L$ so that $[L] = a[S^1 \times \{x\}] + b[\{x\} \times \hat{C}]$ in $H_1(C; \mathbb{Z})$ for some $a \in \mathbb{N}$ and $b \in \mathbb{Z}$.

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**Figure 4.1**

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DEFINITION 4.1. In the above situation, a Reeb component $R$ contained in $\mathcal{C}|C$ is called positive (or negative) if a compact leaf $L$ of $R$ has an expanding (or contracting) holonomy in the direction of $\sigma(C)\cdot L$. A Reeb component $R$ contained in $\mathcal{C}|\partial_t A$ is called positive (or negative) if $r^* R$ is positive (or negative).

The lemma which we need is the following.

LEMMA 4.2 [15, Lemma 1]. Let $R$ be a Reeb component contained in $\mathcal{C}|\partial_t A$ and $\Sigma(\mathcal{R}) \cap A^\circ = \{z\}$. If $R$ is positive, then $\mathcal{C}|A^\circ$ forms a family of concentric half circles with center $z$ in a neighborhood of $z$. If $R$ is negative, then $\mathcal{C}|A^\circ$ forms a family of confocal parabolas in a neighborhood of $z$. (See Figure 4.2.)

![Figure 4.2](image)

Now we introduce the TS formula for $\mathcal{C}$. When $C \in \Gamma(h)$ is vertical and $\mathcal{C}|C$ has no compact leaf, let $a(C) = b(C) = -\infty$ and $r(C) = 0$. When $C \in \Gamma(h)$ is vertical and $\mathcal{C}|C$ has a compact leaf $L$, define $(a(C), b(C)) \in (N \times Z)^{\text{coprime}}$ by

$$[L] = a(C) [S^1 \times \{\ast\}] + b(C) [\{\ast\} \times \check{C}] \text{ in } H_1(C; Z),$$

and let $r(C) = p(C) - q(C)$, where $p(C)$ (or $q(C)$) is the number of positive (or negative) Reeb components of $\mathcal{C}|C$. When $C \in \Gamma(h)$ is horizontal, let $(a(C), b(C)) = (0, 1)$ and define $r(C)$ in the same way as above but by replacing $\mathcal{C}|C$ by $c^*(\mathcal{C}|C)$, where $c: S^1 \times \hat{C} \to S^1 \times \hat{C}$ is defined by $c(x, y) = (y, x)$ for $x \in S^1, y \in \hat{C} = S^1$.

PROPOSITION 4.3 (The TS formula). In the above situation,

$$\sum_{C \in \Gamma(h)} a(C) r(C) = 4 - 2h,$$

where we use the convention that $-\infty \cdot 0 = 0$.

PROOF. Regard $\mathcal{C}|A^0$ as the set of orbits of a vector field $Y$ by giving an orientation. By patching two copies of $A^0$ along $\partial A^0$, we have a closed manifold $W$. We obtain a vector field $\check{Y}$ on $W$ from $Y \cup (-Y)$. By Lemma 4.2, the vector field $\check{Y}$ has $p$ (or $q$) singular points of index
1 (or $-1$), where
\[ p = \sum_{C \in \Gamma(h)} a(C)p(C) \quad \text{and} \quad q = \sum_{C \in \Gamma(h)} a(C)q(C). \]
Since the Euler number of $W$ equals $4 - 2h$, we have the formula.

5. The characteristic diffeomorphism of $\mathcal{F} | C$ to $F^s$. Let $\mathcal{F} \in \mathfrak{t}_i(\mathcal{F}(h; \sigma))$. Take a vector field $X$ of $E(h)$ tangent to $\mathcal{F}$ and transverse to $\mathcal{F}(h; \sigma)$ such that $X$ is inward (or outward) at $y \in \partial E(h)$ with $\sigma(y) = 1$ (or $-1$). We may suppose that $y \in \partial E(h)$ and $\gamma(y)$ is on the same orbit of $X$. For $x \in S^1$, let $F^s$ be the non-compact leaf of $\mathcal{F}(h; \sigma)$ defined in §2. Since $F^s$ is proper, for each $y \in F^s$ there is the first point $\varphi_x(y)$, of the orbits of $X$ starting from $y$, intersecting $F^s$. Then $\varphi_x(y)$'s give rise to a diffeomorphism $\varphi_x : F^s \rightarrow F^s$.

**DEFINITION 5.1.** We call $\varphi_x$ above the characteristic diffeomorphism of $\mathcal{F}$ with respect to $X$ for $F^s$.

**DEFINITION 5.2.** For a subset $B$ of $\partial E(h)$, the real projection $R_P^s(B)$ of $B$ to $F^s$ along $X$ is the set of $z \in F^s$ such that the orbit of $X$ passing through $z$ intersects $B$. For a leaf $L$ of $\mathcal{F} | \partial E(h)$, the projection $P^s_x(L)$ of $L$ to $F^s$ along $X$ is the saturation of $R_P^s(L)$ with respect to $\mathcal{F} | F^s$. We denote by $P^s_x(L)$ the set of leaves of $\mathcal{F} | F^s$ contained in $P^s_x(L)$.

Clearly $\varphi^s_x(\mathcal{F} | F^s) = \mathcal{F} | F^s$ and $\varphi_x(P^s_x(L)) = P^s_x(L)$. The set $R_P^s(L)$ is open in $P^s_x(L)$. For disjoint subsets $B$ and $B'$ of $\partial E(h)$, it follows that $R_P^s(B) \cap R_P^s(B') = \emptyset$ if $\sigma|B \cup B'$ is constant. Furthermore $\varphi_x$ and $P^s_x(L)$ have the following useful properties.

**PROPOSITION 5.3.** Let $L$ be a leaf of $\mathcal{F} \mid \partial E(h)$.

1. The group $\{\varphi_x^n | n \in \mathbb{Z}\}$ acts transitively on the set $P^s_x(L)$.
2. Let $L'$ be another leaf of $\mathcal{F} \mid \partial E(h)$. If $P^s_x(L) \cap P^s_x(L') \neq \emptyset$, then $P^s_x(L) = P^s_x(L')$.
3. If $L$ is a compact leaf and $C \in \Gamma(h)$ with $L \subset C$ is vertical, then $\# P^s_x(L) = a(C)$, where $a(C)$ was defined in §4.
4. If $\# P^s_x(L) < \infty$, then $C \in \Gamma(h)$ with $L \subset C$ is vertical and $L$ is a compact leaf or a non-compact leaf of a negative Reeb component contained in $\mathcal{F} | C$.

**PROOF.** (1) Let $K_1$ and $K_2$ be leaves of $\mathcal{F} | F^s$ contained in $P^s_x(L)$. By definition of $P^s_x(L)$, there are points $y_i$ and $y_i \in L$ such that the orbit of $X$ passing through $y_i$ intersects $K_i$ at some point $z_i$, $i = 1, 2$. Since $L$ is connected, there is a path $\omega : I \rightarrow L$ with $\omega(0) = y_i$ and $\omega(1) = y_i$. Transporting $\omega$ along $X$, we have a path $\tilde{\omega} : I \rightarrow K$ such that $\tilde{\omega}(0)$ equals
$z_1$ and $\omega(t)$ and $\tilde{\omega}(t)$ are on the same orbit of $X$. Then $\tilde{\omega}(1) \in K_1$, $y_2 = \omega(1)$ and $z_2 \in K_2$ are on the same orbit of $X$. This implies that $\psi^*_n(\omega(1)) = z_2$ for some $n \in \mathbb{Z}$. Then $\psi^*_n(K_1) = K_2$.

(2) Suppose that $P_x(L) \cap P_x(L') \neq \emptyset$. Then $P_x^*(L) \cap P_x^*(L') \neq \emptyset$. Let $K_0 \in P_x^*(L) \cap P_x^*(L')$. Then for each $K \in P_x^*(L)$ there is $n \in \mathbb{Z}$ with $\psi^*_n(K_0) = K$ by (1). Since $\psi^*_n(P_x^*(L')) = P_x^*(L')$ by (1), it follows that that $K \in P_x^*(L')$. This implies that $P_x(L) \subset P_x(L')$. In the same way we have $P_x(L') \subset P_x(L)$.

In order to prove (3) and (4), we make preparations. Let $H = \{(0) \times C$. Then $RP_x(H)$ consists of an infinite number of circles and we can number them so that $RP_x(H) = \{H_i | i \in \mathbb{Z}\}$ and that if $i < j$ then $H_i$ is between $H$ and $H_j$ in $E(h)$. With respect to the topology of $F^r$, the set $\bigcap_{n \in \mathbb{Z}} Cl(U_{i<n} H_i)$ is empty. For each $i$, the connected component $F_i$ of $F^r - H_i$ containing $H_{i-1}$ is diffeomorphic to $S^1 \times R$ and the closure of $F_i$ in $F^r$ is not compact. Furthermore the decreasing sequence $F_0, F_{-1}, \ldots$ determines an end $\varepsilon$ of $F^r$ with $L_\varepsilon(F^r) = C$, where $L_\varepsilon(F^r)$ is the $\varepsilon$-limit set of $F^r$ (see Nishimori [8]).

**LEMMA 5.4.** Suppose that $C$ is vertical.

(1) If $L$ is a leaf of $\mathcal{C} \mid C$ contained in $C - \bigcup \{Int |R| | R is a Reeb component contained in $\mathcal{C} \mid C\}$, then each connected component of $RP_x(L)$ intersects $H_i$ at exactly one point for all $i \in \mathbb{Z}$.

(2) For a Reeb component $|R|$ contained in $\mathcal{C} \mid C$, the real projection $RP_x(|R|)$ of $|R|$ intersects $H_i$ for all $i \in \mathbb{Z}$, and $\mathcal{C} \mid (RP_x(|R|) \cap F_i)$ is as in Figure 5.1.

**PROOF.** (1) is clear and (2) follows from Lemma 4.2.

**PROOF OF PROPOSITION 5.3 CONTINUED.** (3) Suppose that $L$ is compact and $C$ is vertical. Since $H \cap L$ is finite, so is the set $H_i \cap RP_x(L)$ for all $i \in \mathbb{Z}$. By Lemma 5.4 (1), we have $\#P^*_x(L) < \infty$.

![Figure 5.1](image-url)
Suppose that $\#P^*_x(L) < \infty$. Then $C$ is vertical. For, otherwise, we may suppose that $H = \{[0]\} \times \hat{C}$ is a compact leaf and it is easy to see that for each leaf $L'$ of $\mathcal{F}|C$ the intersection $P^*_x(L') \cap (F^i - F^i_{i-1})$ consists of exactly one leaf of $\mathcal{F}|F^s$. Therefore $\#P^*_x(L) = \infty$, which is a contradiction.

We see that $L$ is not a non-compact leaf contained in $C - \bigcup \{|R| | R$ is a Reeb component contained in $\mathcal{F}|C\}$, as follows. Suppose the contrary. Then $L$ intersects $H$ infinitely many times, and $\#(H \cap RP^*_x(L)) = \infty$ for each $i \in \mathbb{Z}$. Since the foliation $\mathcal{F}|C$ is orientable, the curves $K \cap RP^*_x(L)$ for $K \in P^*_x(L)$ cross the circle $H_0$ in the same direction when $\mathcal{F}|F^s$ is oriented. Furthermore the closure of each connected component of $RP^*_x(L)$ with respect to the topology of $F^u$ is non-compact by Lemma 5.4 (1). Therefore each $K \in P^*_x(L)$ intersects $H_0 \cap RP^*_x(L)$ at exactly one point. Thus we have $\#P^*_x(L) = \infty$, which is contradiction.

We see that $L$ is not a non-compact leaf of a positive Reeb component contained in $\mathcal{F}|C$, as follows. Suppose the contrary. Then $\#(H \cap RP^*_x(L)) = \infty$ for each $i \in \mathbb{Z}$. By Lemma 5.4 (2), the closure of each connected component of $RP^*_x(L)$ in $F^u$ is non-compact. Therefore each $K \in P^*_x(L)$ intersects $H_0 \cap RP^*_x(L)$ at at most two points. We have a contradiction as above. This completes the proof of Proposition 5.3.

6. Negative Reeb cycles. In this section we investigate negative Reeb cycles defined below, which can be regarded as a preparation for the next section. Let $\mathcal{F} \in t_0(\Gamma(h; \sigma))$.

**Definition 6.1.** A negative Reeb chain of $\mathcal{F}$ is a finite ordered set $C = (\mathcal{N}_0, \ldots, \mathcal{N}_n)$, $n \geq 1$, of negative Reeb components contained in $\mathcal{F}|\partial E(h)$ such that

1. $P^*_x(N^i_2) = P^*_x(N^i_{i+1})$ for $i = 1, \ldots, n - 1$, where $N^i_1$ and $N^i_2$ are the compact leaves of $\mathcal{N}_i$, and

2. $\text{Int} |\mathcal{N}_i|$ and $\text{Int} |\mathcal{N}_{i+1}|$ are in the same side of the compact leaf $G_i$ of $\mathcal{F}$ containing $N^i_1$ and $N^i_{i+1}$, for $i = 1, \ldots, n - 1$.

We denote $N^i_1$, $N^i_2$, $\mathcal{N}_i$ and $\mathcal{N}_n$ by $o(\mathcal{E})$, $e(\mathcal{E})$, $\mathcal{N}_0(\mathcal{E})$ and $\mathcal{N}_n(\mathcal{E})$, respectively.

**Definition 6.2.** A negative Reeb cycle of $\mathcal{F}$ is a negative Reeb chain $C$ of $\mathcal{F}$ with $e(\mathcal{E}) = o(\mathcal{E})$. A negative Reeb cycle $\mathcal{E}$ is called strange if there are a vertical $C \in \Gamma(h)$ and a compact leaf $L$ of $\mathcal{F}|C$ with $P^*_x(L) \cap RP^*_x(\text{Int} |\mathcal{N}_0(\mathcal{E})|) \neq \emptyset$.

Note that if $\mathcal{F}$ contains a gear component $\mathcal{F}_0$ then $\mathcal{F}_0$ contains a negative Reeb cycle.
First we have the following.

**Proposition 6.3.** Let $\mathcal{C} = (\mathcal{N}_1, \ldots, \mathcal{N}_n)$ be a negative Reeb chain. Then $\sigma(|\mathcal{N}_1| \cup \cdots \cup |\mathcal{N}_n|)$ is constant.

**Proof.** Let $G_i$ be the compact manifold of $\mathcal{C}$ containing $N_i^2$ and $N_{i+1}^2$, as in Definition 6.1. A consideration on the holonomy of $G_i$ on the side of $|\mathcal{N}_i|$ tells us that $\sigma(|\mathcal{N}_i| \cup |\mathcal{N}_{i+1}|)$ is constant. Then the proposition follows.

Let $\mathcal{C} = (\mathcal{N}_1, \ldots, \mathcal{N}_n)$ be a negative Reeb chain. We use the notations in Definition 6.1. Let us consider the holonomy of leaves in $\mathbb{P}^s(N_i^2)$ with respect to $\mathcal{C} | F^\varepsilon$. For each $i$, there is a unique bijection $\alpha_i: \mathbb{P}^s(N_i^2) \to \mathbb{P}^s(N_i^2)$ such that $M \in \mathbb{P}^s(N_i^2)$ and $\alpha_i(M)$ intersect the same connected component of $\mathbb{RP}_s(|\mathcal{N}_i|)$. For each $M \in \mathbb{P}^s(N_i^2)$, take a small line segment $T^i(M)$ in $\mathbb{RP}_s(|\mathcal{N}_i|)$ transverse to $\mathcal{C} | F^\varepsilon$ with an endpoint $z^i(M)$ in $M$. A local homeomorphism $\phi: (X, x_0) \to (Y, y_0)$ means a homeomorphism from a neighborhood of $x_0$ in $X$ to a neighborhood of $y_0$ in $Y$ with $\phi(x_0) = y_0$. For each $M \in \mathbb{P}^s(N_i^2)$, there is a local homeomorphism $h[M]: (T^i(M), z^i(M)) \to (T^i(\alpha_i(M)), z^i(\alpha_i(M)))$ such that $z \in \text{Dom}(h[M])$ and $h[M](z)$ are on the same leaf of $\mathcal{C} | \mathbb{RP}_s(|\mathcal{N}_i|)$, where $\text{Dom}(h[M])$ is the domain of $h[M]$. Furthermore there is a local homeomorphism $k[M]: (T^i(M), z^i(M)) \to (T^i(M), z^i(M))$ such that $w \in \text{Dom}(k[M])$ and $k[M](w)$ are on the same leaf of $\mathcal{C} | W$, where $W$ is a sufficiently small neighborhood of $M$ in $F^\varepsilon$. (See Figure 6.1.)
Suppose that $l$ is a negative Reeb cycle, that is, $N_2 = N_1$. Let $\alpha = \alpha_1 \circ \cdots \circ \alpha_n : P_\tau^*(N_1) \to P_\tau^*(N_2)$. Since $\#P_\tau^*(N_1) < \infty$ by Proposition 5.3, there is a minimal positive integer $\nu$ with $\alpha^\nu = \text{id}$. Let $\tilde{T}(M) = k[\alpha(M)] \circ h[\alpha_{\nu-1} \circ \cdots \circ \alpha(M)] \cdots \circ h[\alpha(M)] \circ h[M] : (T_1(M), z_1(M)) \to (T_1(M), z_1(M))$ and $\tilde{T}[M] = \tilde{T}[\alpha^\nu(M)] \circ \cdots \circ \tilde{T}[M] : (T_1(M), z_1(M)) \to (T_1(M), z_1(M))$.

The goal of this section is to prove the following.

**Proposition 6.4.** Let $\mathcal{C} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ be a strange negative Reeb cycle. Then there exists a torus $S(\mathcal{C})$ imbedded in $\operatorname{Int} E(h)$ satisfying the following conditions.

1. $S(\mathcal{C}) \cap F^x \subset \operatorname{RP}_x(|\mathcal{A}_1|) \cup \cdots \cup \operatorname{RP}_x(|\mathcal{A}_n|) \cup U$ for all $x \in S^1$, where $U$ is an arbitrarily small neighborhood of $P_x(N_1) \cup \cdots \cup P_x(N_n)$.
2. $S(\mathcal{C})$ is transverse to $\mathcal{C}$ and to $\mathcal{F}(h) \circ \sigma$.
3. $S(\mathcal{C}) \cap F^x$ consists of $a(C_i)/\nu$ circles for all $x \in S^1$, where $C_i \in \Gamma(h)$ contains $|\mathcal{A}_i|$ and $\nu$ is as above.

**Definition 6.5.** We call $S(\mathcal{C})$ in Proposition 6.4 a separating torus of $\mathcal{C}$.

**Remark 6.6.** In §11, we see that $\mathcal{C} | D$ is a gear component, where $D$ is the closure of the domain surrounded by $|\mathcal{A}_1|, \cdots, |\mathcal{A}_n|, G_1, \cdots, G_n$ and $S(\mathcal{C})$. In §14, we show that $S(\mathcal{C}) \cap F^x$ consists of exactly one circle, that is, $\nu = a(C_i)$.

**Proof of Proposition 6.4.** Since $\mathcal{C}$ is strange, there is a vertical cycle $C \in \Gamma(h)$ and a compact leaf $L$ of $\mathcal{C} | C$ with $P_\tau(L) \cap \operatorname{RP}_x(\operatorname{Int} |\mathcal{A}_i|) \neq \emptyset$. This is true for $x = [0]$. For simplicity, we omit the suffix $[0]$ from $F_\tau[0]$.

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**Figure 6.2**
We use the notations as above for $x = [0]$ and consider $M \in P^*(N_i)$. Then there is $K \in P^*(L)$ intersecting the connected component $B$ of $R^1(\mathcal{M}_i)$ containing $M$. Let $K_i$ be a connected component of $K \subset B$. Put $a_i = a(C_i)$. Since $\psi^a(M) = M$, the confocal parabolas $\psi^{an}(K_i)$ approach $M$ when $n$ moves to $\infty$ or $-\infty$. This implies that $K$ intersects $T^v(M)$ at infinitely many points converging to $z^v(M)$, since $\psi^{an}(\mathcal{C})(K) = K$ for all $n \in \mathbb{Z}$. Therefore the local homeomorphism $\eta_{M}: (T^v(M), z^v(M)) \rightarrow (T^v(M), z^v(M))$ has no fixed point. Take $z \in K \cap \text{Dom}(\eta_{M})$. Let $\Omega$ be the domain in $F$ surrounded by $M$, $\alpha_1(M)$, $\ldots$, $\alpha_{n-1} \circ \cdots \circ \alpha_1 \circ \alpha_{n-1}(M)$, the closed interval $T^u$ in $T^v(M)$ between $z$ and $\eta_{M}(z)$, and the closed interval $K^*$ in $K$ between $z$ and $\eta_{M}(z)$. (See Figure 6.2.)

We can take a closed transversal $S_c$ of $g|F$ in $\Omega$ with $S_c' = \psi^{a/\varepsilon}(S_c) \subset \Omega$.

Denote by $S_e$ the union of the intervals of orbits of $X$ between some point $y \in S_c$ and $\psi^{a/\varepsilon}(y)$. Then $S_e \cap F^e$ passes near each $M' \in P^*(N_i)$ exactly once for all $x \in S_e[[0]]$. We see that $S_c$ and $S_c'$ intersect each leaf of $g|\mathcal{C}$ at exactly one point. Therefore $S_c$ and $S_e$ intersect each leaf of $g|\mathcal{C}$ at exactly one point. Then there is a diffeomorphism $\xi: S_e \rightarrow S_e$ such that $y \in S_e$ and $\xi(y)$ on the same leaf of $g|\mathcal{C}$. Now we can modify $S_e$ in $\bigcup \{ F^e | 1 - \varepsilon < t \leq 1 \}$ for small $\varepsilon > 0$ by translating each $y \in S_e$ to $\xi(y)$ along a leaf of $g|F$, so that we obtain a torus $S(e)$ transverse to $g$ and $\mathcal{F}(h; \sigma)$. Clearly $S(e)$ has the desired property.

**7. An investigation of leaves of $g \in t(h; \sigma)$ containing compact leaves of $g|C$ for a vertical $C \subset \mathcal{C}$**

Let $g \in t(h; \sigma)$ and fix a vector field $X$ of $E(h)$ as in §5. Let $e_1, \ldots, e_p$ be the strange negative Reeb cycles of $g$. For each $e_i$, take a separating torus $S(e_i)$. We may suppose that $X$ is tangent to $S(e_i)$ for all $i$.

For a vertical $C \subset \mathcal{C}(h)$, we have the following proposition. Proposition 7.1 is a generalization of Lemma 3 in Tamura-Sato [15], and its proof can be regarded as a new proof of the lemma.

**PROPOSITION 7.1.** Let $C \subset \mathcal{C}(h)$ be vertical. Let $L$ be a compact leaf of $g|\partial E(h)$, and $G$ the leaf of $g$ containing $L$. Then one of the following occurs.

1. $G$ is a compact leaf diffeomorphic to $S^1 \times I$.
2. $G$ is a non-compact leaf diffeomorphic to $S^1 \times [0, \infty]$ and the limit set of $G$ consists of a compact leaf of $g$ diffeomorphic to $T^e$.
3. $G \cap S(e_i) \neq \emptyset$ for some $i$, and the closure of the connected component of $G - (S(e_1) \cup \cdots \cup S(e_p))$ containing $L$ is diffeomorphic to $S^1 \times I$.

**PROOF.** Let $A = E(h) - k(\partial E(h) \times [0, \varepsilon])$, $A^\varepsilon = F^e \cap A$, etc., be as in
§4. We fix \([0]\) as \(x \in S^1\) and omit the suffix \([0]\) except from \(A^{[0]}\) for simplicity. Thus \(F = F^{[0]}, \varphi = \varphi^{[0]}, P(L) = P^{[0]}(L)\) and so forth. We may suppose that \(S(\varphi^i) \subset \text{Int } A\) for all \(i\).

Take a leaf \(K \in P^*(L)\) and a point \(y_0 \in K \cap \text{RP}(L) \cap \partial A^{[0]}\). When the connected component \(K^*\) of \(K \cap A^{[0]}\) containing \(y_0\) is not compact, the limit set of \(K^*\) is a circle \(S\) contained in \(\text{Int } A^{[0]}\) by the Poincaré-Bendixson theorem. Since \(a = \#P^*(L) < \infty\) and \(\varphi(P(L)) = P(L)\), it follows that \(\varphi^a(S) = S\). It is easy to check that the case (2) occurs. (Consequently \(P(L) \cap \text{RP}(\text{Int } |A^\omega|) = \emptyset\) for all Reeb components \(A^\omega\) contained in \(\mathcal{F}|\partial E(h)\).)

When \(K^*\) is compact, the endpoint \(y_i\) of \(K^*\) with \(y_i \neq y_0\) belongs to \(\partial A^{[0]}\). Then there is a leaf \(L'\) of \(\mathcal{F}|\partial E(h)\) with \(L' \neq L\) and \(y_1 \in \text{RP}(L')\). Since \(\#P^*(L') = \#P^*(L) < \infty\), the leaf \(L'\) is

(i) a compact leaf,

or

(ii) a non-compact leaf of some negative Reeb component \(\mathcal{N}^\omega\) contained in \(\mathcal{F}|\partial E(h)\),

by Proposition 5.3. In the case (i), we see easily that the case (1) occurs. (Consequently \(P(L) \cap \text{RP}(\text{Int } |A^\omega|) = \emptyset\) for all Reeb components \(A^\omega\) of \(\mathcal{F}|\partial E(h)\), again.)

Now consider the case (ii). We need the following.

**Lemma 7.2.** In the case (ii), there is an infinite sequence \(L_1 = L, L_2, \ldots\) of compact leaves of \(\mathcal{F}|\partial E(h)\) such that for each \(i\) one of the following occurs.

(a) \(L_{i+1}\) is a compact leaf of a negative Reeb component \(\mathcal{N}^\omega\) contained in \(\mathcal{F}|\partial E(h)\) with \(\text{RP}(\text{Int } |\mathcal{N}^\omega|) \cap P(L_i) \neq \emptyset\).

(b) \(L_{i+1}\) is a compact leaf of a negative Reeb component \(\mathcal{N}^\omega\) such that \(L'_i \neq L_i\) and \(P(L'_i) = P(L_i)\), where \(L'_i\) is the other compact leaf of \(\mathcal{N}^\omega\).

**Proof.** Let \(L_1\) be a compact leaf of \(\mathcal{N}^\omega\). Then for \(i = 1\) the case (a) occurs. Suppose that we have already obtained \(L_1, \ldots, L_n (n \geq 2)\) satisfying (a) or (b). Note that \(P(L) \cap \text{RP}(\text{Int } |\mathcal{N}^\omega|) \neq \emptyset\) for all \(i = 1, \cdots, n - 1\) and that the limit set \(\mathcal{L}(K)\) of \(K\) in \(\mathcal{F}\) intersects \(P(L_i)\) for \(i = 2, \cdots, n\). Take \(K_n \in P^*(L_n)\) and \(y_n \in K_n \cap \text{RP}(L_n) \cap \partial A^{[0]}\). Let \(K_n^\star\) be the connected component of \(K_n \cap A^{[0]}\) containing \(y_n\). We see that \(K_n^\star\) is compact, as follows. Suppose the contrary. Then \(\mathcal{L}(K_n^\star)\) is a circle in \(\text{Int } A^{[0]}\). Since \(\mathcal{L}(K) \supset \mathcal{L}(K_n^\star)\), the case (2) occurs. Therefore \(P(L) \cap \text{RP}(\text{Int } |\mathcal{N}^\omega|) = \emptyset\) for all Reeb components \(\mathcal{N}^\omega\) contained in \(\mathcal{F}|\partial E(h)\), which is a contradiction.
Let $z_n$ be the other endpoint of $K_n$. Then $z_n \in \partial A^{(i)}$ and there is a leaf $L'_n$ of $\mathcal{F}|E(h)$ with $L'_n \neq L_n$ and $P(L'_n) = P(L_n)$. Therefore $\# P(L'_n) < \infty$. When $L'_n$ is a non-compact leaf of a negative Reeb component $\mathcal{N}_n$ contained in $\mathcal{F}|E(h)$, let $L_{n+1}$ be a compact leaf of $\mathcal{N}_n$. Then $L_{n+1}$ satisfies the condition (a).

Now suppose that $L'_n$ is a compact leaf. Since $P(L_n) = P(L'_n)$, there is a leaf $L''$ of $\mathcal{F}|E(h)$ passing arbitrarily near $L'_n$ with $P(L'') = P(L)$. Since $P(L) \cap \overline{P(\text{Int}|N_1|)} \neq \emptyset$, the leaf $L''$ cannot be compact by the remark before Lemma 7.2. (For, otherwise the case (1) occurs.) Therefore $L'_n$ is a compact leaf of a negative Reeb component $\mathcal{N}_n$ contained in $\mathcal{F}|E(h)$. Let $L_{n+1}$ be the other compact leaf of $\mathcal{N}_n$. Then $L_{n+1}$ satisfies the condition (b). This completes the proof of Lemma 7.2.

**Proof of Proposition 7.1 Continued.** We see that (a) in Lemma 7.2 occurs for only a finite number of $i$'s, as follows. Suppose the contrary. Since $\mathcal{F}|E(h)$ contains only a finite number of negative Reeb components, there is a sequence $i(1) < i(2) < \ldots$ with $L_{i(1)} = L_{i(2)} = \ldots$. Note that for $j, k$ with $j < k$ the limit set of each leaf $\in P*(L_{i(j)})$ contains a leaf $\in P*(L_{i(k)})$. Since $\# P*(L_{i(1)}) < \infty$, there is $M \in P*(L_{i(1)})$ with $\mathcal{L}(M) \supset M$. Since all leaves of $\mathcal{F}|F$ are proper by the Poincaré-Bendixson theorem, we get a contradiction.

Since $\mathcal{F}|E(h)$ contains only a finite number of negative Reeb components, there are $i, j$ with $i < j$ such that $\mathcal{N}_i = \mathcal{N}_j$. Thus we obtain a negative Reeb cycle $\mathcal{C} = (\mathcal{N}_i, \ldots, \mathcal{N}_{j-1})$. Easily we see that the case (3) occurs. (Consequently we see that (a) occurs only for $i = 1$.) This completes the proof of Proposition 7.1.

**8. An investigation of $\mathcal{F} \in \mathfrak{L}_0(\mathcal{F}(h; \sigma))$ near a horizontal $C \in \Gamma(h)$.**

Let $\mathcal{F} \in \mathfrak{L}_0(\mathcal{F}(h; \sigma))$ and fix a vector field $X$ on $E(h)$ as in §5. For a horizontal $C \in \Gamma(h)$, we have the following.

**Proposition 8.1.** Let $C_0 \in \Gamma(h)$ be horizontal. Then one of the following occurs.

1. $h = 2$, the other $C_i \in \Gamma(h) - \{C_0\}$ is horizontal, $\sigma(C_i) = -\sigma(C_0)$, and $\mathcal{F}$ is isomorphic to the product foliation $\mathcal{F}|C_0 \times [0, 1]$.

2. All leaves of $\mathcal{F}|C_0$ are compact, all leaves of $\mathcal{F}$ intersecting $C_0$ intersect no $C \in \Gamma(h) - C_0$, and $\mathcal{F}|\text{Cl}(\text{Sat}(C_0))$ is a tunnelled Reeb component, where Sat( ) means the saturation with respect to $\mathcal{F}$.

3. The foliation $\mathcal{F}|C_0$ has no Reeb component, all leaves of $\mathcal{F}$ intersecting $C_0$ intersect a vertical $C \in \Gamma(h)$, and if $C \cap \text{Sat}(C_0) \neq \emptyset$ for $C \in \Gamma(h) - C_0$, then $C$ is vertical and $\sigma(C) = -\sigma(C_0)$. Furthermore
\( \mathcal{G}|\text{Cl}(\text{Sat}(C_0)) \) is a gear component.

**Proof.** For simplicity, suppose that \( \sigma(C_0) = 1 \). Hence \( X \) is inward at \( C_0 \). We omit the suffix \([0]\) except from \( A[0] \), as in §7.

Take a compact leaf \( L \) of \( \mathcal{G}|C_0 \). Since \( C_0 \) is horizontal, we can number the leaves in \( P^*(L) \) so that \( P^*(L) = \{\cdots, L_{-2}, L_{-1}, L_0, L_1, \cdots\} \), and \( \psi(L_j) = L_{j+1} \) for all \( j \in \mathbb{Z} \). Note that \([L_j] = [L_k]\) in \( H_1(\text{RP}(C_0); \mathbb{Z}) \) for all \( j, k \in \mathbb{Z} \). Consider \( L = \bigcap_{n \in \mathbb{Z}} \text{Cl}_F(\bigcup_{j \geq n} L_j) \), where \( \text{Cl}_F(\cdots) \) means the closure with respect to the topology of \( F \). Note that \( L = \text{Cl}_F(\text{RP}(C_0)) - \text{RP}(C_0) \).

(i) Suppose that \( L \) is empty. Then \( F = \text{RP}(C_0) \) and \( h = 2 \). Let \( \Gamma(h) - \{C_0\} = \{C_1\} \). Since \( \text{RP}(C_0) \cap \text{RP}(C_1) \neq \emptyset \), it follows that \( \sigma(C_1) = -\sigma(C_0) \). It is easy to check that \( \mathcal{G} \) is isomorphic to \( (\mathcal{G}|C_0) \times I \). Thus we have the case (1).

(ii) Suppose that \( L \) is non-empty and compact. Then we may suppose that \( L \subset A[0] \). Since \( \text{Cl}_F(\bigcup_{j \geq n} L_j) \) is saturated with respect to \( \mathcal{G}|F \), so is \( L \). If \( L \) contains a non-compact leaf \( K \) of \( \mathcal{G}|F \), then the limit set \( L(K) \) consists of exactly two compact leaves in \( L \) because \( A[0] \) can be regarded as a subspace of \( D^2 \). If \( L \) contains at least two compact leaves \( K_i \) and \( K_2 \), then \( \text{RP}(C_0) \) must contain a one-sided neighborhood of \( K_i \) in \( F \) for \( i = 1, 2 \), and \( \text{RP}(C_0) \) has at least three isolated ends. This is a contradiction since \( \text{RP}(C_0) \) is homeomorphic to \( S^1 \times R \). Therefore \( L \) consists of exactly one compact leaf \( L^* \) of \( \mathcal{G}|F \). Since \( \psi(L^*) = L^* \), the leaf \( G^* \) of \( \mathcal{G} \) containing \( L^* \) is diffeomorphic to \( T^2 \). Clearly \( G^* \subset \text{Int} E(h) \).

We see that all leaves of \( \mathcal{G}|C_0 \) are compact, as follows. Suppose

![Figure 8.1](image.png)
the contrary. Then \( L^* \) has a non-trivial holonomy on the side from which the sequence \( L_1, L_2, \cdots \) converges to \( L^* \). Since \( G^* \) has a non-trivial holonomy in the direction of orbits of \( X \), too, and \( C^* \) is of class \( C^\infty \), we have a contradiction by the result of Kopell [7]. It is easy to check that \( \mathcal{F}|\text{Cl} (\text{Sat} (C_0)) \) is a tunneled Reeb component. Thus we have the case (2).

(iii) Suppose that \( \mathcal{L} \) is non-empty and non-compact. It follows that \( \mathcal{L} \) contains no compact leaf of \( \mathcal{F}|F \). Let \( F' = F - \text{Int} A^\infty \). For \( C \in \Gamma(h) \), let \( F'_C \) be the connected component of \( F' \) containing \( \partial_0 A^\infty \). Then \( F'_C \) is diffeomorphic to \( ]-\infty, 0[ \times \hat{C} \) (or \( [0, \infty[ \times \hat{C} \) if \( \sigma(C) = 1 \) (or \(-1\)), and \( \mathcal{F}|F'_C \) is isomorphic to the restriction of the covering foliation on \( R \times \hat{C} \) of \( \mathcal{F}|C \). (See Figure 8.1.)

Since \( \mathcal{L} \) is non-empty, it follows that \( L_j \cap F'_C = \emptyset \) for all horizontal \( C \in \Gamma(h) \) \(-\{C_0\} \). Since \( \mathcal{L} \) is non-compact, it follows that \( L_j \cap F'_C \neq \emptyset \) for a sufficiently large \( j \) and some \( C \in \Gamma(h) \) \(-\{C_0\} \). If \( L_j \cap F'_C \neq \emptyset \) for \( C \in \Gamma(h) \) \(-\{C_0\} \), then \( L_j \) is tangent to the curves in \( \text{RP}(\{x\} \times C) \), \( x \in S^1 \), at some point by (E5) in §4, and \( \sigma(C) = -\sigma(C_0) = -1 \) by the remark before Proposition 5.3. Furthermore in this case it follows that \( L_j' \cap F'_C \neq \emptyset \) for \( j' > j \), and \( L_j \in \text{P}^* (L') \) for a non-compact leaf \( L' \) of a negative Reeb component of \( \mathcal{F}|C \). We denote by \( \mathcal{L} \) the set of negative Reeb components \( \mathcal{N} \) contained in \( \mathcal{F}|\partial E(h) \) with \( \text{RP} (\text{Int} \mathcal{N}) \cap \text{P} (L) \neq \emptyset \). Then \( \mathcal{L} = \bigcup \{P(N) \mid N \text{ is a compact leaf of some } \mathcal{N} \in \mathcal{L}\} \) by the above arguments. Since \( \text{RP} (C_0) \) has exactly two ends, we can give an order to \( \mathcal{L} \) so that \( \mathcal{L} \) becomes a negative Reeb cycle. Let \( \mathcal{L} = (\mathcal{N}_1, \cdots, \mathcal{N}_n) \).

We use the notations in Definition 6.1.

We see that \( \mathcal{F}|C_0 \) contains no Reeb components, as follows. Suppose the contrary. Then the structure of \( \mathcal{F}|F \) as a foliation breaks near points in \( \mathcal{L} \), which is a contradiction.

Clearly \( \text{Cl} (\text{Sat} (C_0)) \) is homeomorphic to \( S^1 \times S^1 \times I \). And \( \mathcal{F}|(\text{Cl} (\text{Sat} (C_0)) - U) \) is \( C^0 \) equivalent to the product foliation \( (\mathcal{F}|C_0) \times I \), where \( U \) is an open tubular neighborhood of \( G_1 \cup \cdots \cup G_n \) in \( E(h) \). Since \( N_t \) is homotopic to a circle \( S \) in \( |\mathcal{N}_t| \cup \partial \) and \( S \) is transverse to \( \mathcal{F} \), the circle \( N_t \) is homotopic to a circle transverse to \( \mathcal{F}|C_0 \) in \( \text{Cl} (\text{Sat} (C_0)) \). Furthermore \( S \) intersects all the leaves of \( \mathcal{F}|(\text{Cl} (\text{Sat} (C_0)) - U) \). Now it is easy to see that \( \mathcal{F}|\text{Cl} (\text{Sat} (C_0)) \) is a gear component. Thus we have the case (3). This completes the proof of Proposition 8.1.

9. The decomposition theorem. The purpose of this section is to state one of the main theorems of Part I. Let \( \mathcal{F} \in \mathcal{C}^0 (\mathcal{F}(h_\sigma)) \). We suppose that \( \mathcal{F} \) is not \( C^0 \) isomorphic to \( (\mathcal{F}|C) \times I \) for any \( C \in \Gamma(h) \). When \( h \neq 2 \), this assumption is automatically satisfied.
Let \( \mathcal{L}_1, \ldots, \mathcal{L}_n \) be the strange negative Reeb cycles of \( \mathcal{L} \). For each \( \mathcal{L}_i \), take a separating torus \( S(\mathcal{L}_i) \). A subleaf of \( \mathcal{L} \) (or \( \mathcal{L}|A \)) is the closure of a connected component of \( G - (S(\mathcal{L}_1) \cup \cdots \cup S(\mathcal{L}_n)) \) for some leaf \( G \) of \( \mathcal{L} \) (or \( \mathcal{L}|A \)). Let \( \mathcal{L} \) be the set of compact manifolds obtained from the connected components of \( A - (S(\mathcal{L}_1) \cup \cdots \cup S(\mathcal{L}_n)) - \bigcup \{G|G \text{ is a compact subleaf of } \mathcal{L}|A\} \) by attaching the boundary. For \( D \in \mathcal{L} \), we denote by \(*D\) the image of canonical immersion \( \varphi_D: D \to A \). Let \(*\mathcal{L} = \{*D|D \in \mathcal{L}\} \).

Let \(*\mathcal{L}\) be the set of the closure of connected components of \( A - (S(\mathcal{L}_1) \cup \cdots \cup S(\mathcal{L}_n)) - \bigcup \{*D|D \in \mathcal{L}\} \). For \(*T \in *\mathcal{L}\), the foliation \( \mathcal{L}|*T \) consists of a compact subleaf, or is a bundle foliation over \( I \) or \( S^1 \). When \( \mathcal{L}|*T \) is a bundle foliation over \( S^1 \), let \( T = q(*T) \), where \( q:[0,1] \to S^1 = [0,1]/\{0,1\} \) is the quotient map, and denote by \( \varphi_T: T \to *T \) the canonical immersion (when we fix a bundle structure of \( *T \)). In the other cases, let \( T = *T \) and \( \varphi_T = \text{id}: T \to *T \). Let \( *\mathcal{L} = \{T|T \in *\mathcal{L}\} \).

For each \( *D \in *\mathcal{L} \) and \( x \in S^1 \), let \( *D^x = *D \cap A^x \). The number of connected components of \( *D^x \) is finite. Let \( *D_1, \ldots, *D_{a(D)} \) be the connected components of \( *D^x \). Let \( D_j = \varphi^{-1}(D_j) \). Furthermore let \( \mathcal{L}^x = \{D_j|D \in \mathcal{L}, \ j = 1, \ldots, a(D)\} \), \( \mathcal{L} = \{D_j|D \in \mathcal{L}, \ j = 1, \ldots, a(D)\} \), \( *\mathcal{L} = \{T_j|T \in \mathcal{L}, \ j = 1, \ldots, a(T)\} \) and \( \varphi^x = \{T_j|T \in \mathcal{L}, \ j = 1, \ldots, a(T)\} \).

**Definition 9.1.** We call \( \mathcal{L} \cup \mathcal{L} \) (or \( \mathcal{L}^x \) or \( \mathcal{L}^x \)) the TS decomposition of \( A \) (or \( A^x \)) with respect to \( \mathcal{L} \).

For each \( D \in \mathcal{L} \), we define six non-negative integers, as follows. A leaf of \( \mathcal{L}^x \) is a connected component of \( G \cap *D^x \) for some leaf \( G \) of \( \mathcal{L} \). We say a connected component \( J \) of \( \partial D^x \) is
- of type (l) if \( *J \) is equal to \( \partial_c A^x \) for some \( C \in \Gamma(h) \) or to a connected component of \( A^x \cap S(\mathcal{L}_i) \) for some \( i \),
- of type (m) if \( *J \) is equal to a connected component of \( A^x \cap \text{Int } A \) for some compact leaf \( G \) of \( \mathcal{L} \) or \( \mathcal{L}^x \), or
- of type (n) if \( *J \) contains a leaf of \( \mathcal{L}|*D^x \) homeomorphic to \( I \),

where \( *J = \varphi(D)(J) \). Let \( l(D) \) (or \( m(D) \), \( n(D) \)) be the number of connected components of \( \partial D^x \) of type (l) (or (m), (n)). Since each connected component of \( \partial D^x \) is of type (l), (m) or (n) by Propositions 7.1 and 8.1, we see that \( l(D) + m(D) + n(D) \) equals the number of the connected components of \( \partial D^x \).

Let \( J \) be a connected component of \( \partial D^x \) of type (n). By Propositions 7.1 and 8.1, \( *J \) contains a finite number of leaves \( J_1, \ldots, J_x \) of \( \mathcal{L}|*D^x \) homeomorphic to \( I \), and \( *J - (J_1 \cup \cdots \cup J_x) \) consists of open intervals contained in \( \text{Int } A^x \) for Reeb or slope components \( A^x \) contained in \( \mathcal{L}|\partial *D \). Furthermore \( *J \cap (S(\mathcal{L}_1) \cup \cdots \cup S(\mathcal{L}_n)) = \emptyset \), and \( C \in \Gamma(h) \)
intersecting \( *J \) is vertical by (E4)' in §4.

Let \( p(D) \) (or \( q(D), s(D) \)) be the number of positive Reeb (or negative Reeb, slope) components contained in \( \mathcal{F} | \partial *D \) intersecting \( \partial_n *D|^\circ \), where \( \partial_n *D|^\circ \) is the union of \( *J \) for connected components \( J \) of \( D|^\circ \) of type \( (n) \). Note that \( p(D) + q(D) + s(D) \) equals the number of leaves of \( \mathcal{F} | D|^\circ \) contained in the connected components of \( \partial D|^\circ \) of type \( (n) \), where \( \mathcal{F} | D|^\circ = (\delta D) | D|^\circ \).

**DEFINITION 9.2.** We call \( \text{ch} (D) = (l(D), m(D), n(D); p(D), q(D), s(D)) \) the characteristic hexad of \( D \).

Now we can state the following.

**THEOREM 3 (The decomposition theorem).** Let \( \mathcal{F} \in t_0(\mathcal{F}(h; \sigma)) \). Suppose that \( \mathcal{F} \) is not \( C^0 \) isomorphic to \( (\mathcal{F} | C) \times I \) for any \( C \in \Gamma(h) \). Let \( \Omega \cup \Theta \) be the TS decomposition of \( \mathcal{A} \) with respect to \( \mathcal{F} \). Then for each \( D \in \Omega \) the possibilities for \( \mathcal{F} | D \) are the cases in the following table, and these cases can occur for some \( \mathcal{F} \).

| type | \( ch (D) \) | \( \mathcal{F} | D \) |
|------|-------------|------------------|
| I    | \( (0, 0, 1; 1, 0, 0) \) | a half Reeb component |
| II   | \( (0, 0, 1; 0, 0, 2) \) | an \( I \)-times slope component\(^(*)\) |
| III  | \( (0, 0, 1; 0, q > 0, 2) \) | a TS' component \([\sigma]\) |
| IV   | \( (0, 0, 1; 1, q > 0, 0) \) | an arcade component \([\sigma]\) |
| V    | \( (0, 0, 2; 0, q > 1, 2) \) | a double gear component |
| VI   | \( (1, 0, 1; 0, q > 0, 0) \) | a gear component \([\sigma]\) |
| VII  | \( (0, 1, 1; 0, q > 0, 0) \) | (1) a turbulent gear component \([\sigma]\) |
|      |             | (2) a perturbed gear component |
| VIII | \( (1, 1, 0; 0, 0, 0) \) | (1) a tunneled Reeb component \([\sigma]\) |
|      |             | (2) a rational rifle component\(^(*)\) |
|      |             | (3) an irrational rifle component\(^(*)\) |
| IX   | \( (0, 2, 0; 0, 0, 0) \) | (1) an \( S^1 \)-times slope component\(^(*)\) |
|      |             | (2) an \( S^1 \)-times Reeb component\(^(*)\) |
|      |             | (3) a twisted \( S^1 \)-times Reeb component\(^(*)\) |

The terms with \( (*) \) in the table will be defined in the next section.

The mark \([\sigma]\) in the table means the existence of the following restrictions to \( \sigma(C) \) for \( C \in \Gamma(h) \) concerned.

(III) \( \sigma(C^\circ) = \sigma(C) \) (or \( -\sigma(C) \)) if \( \mathcal{F} | (*D \cap \partial_c A) \) and \( \mathcal{F} | (*D \cap \partial_c A) \) contain negative Reeb components, and \( *D \cap \partial_c A \) and \( *D \cap \partial_c A \) belong to the same (or different) connected components of \( \partial *D - \text{Int} (| \mathcal{F}_1 | \cup | \mathcal{F}_2 |) \), where \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are the slope components contained in \( \mathcal{F} | \partial *D \).

(IV) \( \sigma(C^\circ) = -\sigma(C) \) if \( \mathcal{F} | (*D \cap \partial_c A) \) contains a positive Reeb component and \( \mathcal{F} | (*D \cap \partial_c A) \) contains a negative Reeb component.
(VI) \( \sigma(C') = -\sigma(C) \) if \( C \in \Gamma(h) \) with \( \partial_v A \subset *D \) is horizontal, \( \partial_v A \neq \emptyset \) and \( C' \neq C \).

(VII) \( \sigma(C') = \sigma(C) \) if \( \partial_v A \neq \emptyset \) and \( D \cap \partial_v A \neq \emptyset \).

**Remark 9.3.** In the case \( h = 1 \), Theorem 3 corresponds to Theorem 1 in Tamura-Sato [15], where the possibilities for \( \mathcal{F} |D \) are types I, II and III, and there exist no negative Reeb cycles.

10. **Several components II.** We give the definition for the components appearing in the decomposition theorem (Theorem 3) and not yet defined. Recall Definition 2.6.

**Definition 10.1.** A standard \( I \)-times (or \( S^1 \)-times) slope component is the product foliation \( I \times \mathcal{G}_0 \) (or \( S^1 \times \mathcal{G}_0 \)), where \( \mathcal{G}_0 \) is a standard slope component of \( S^1 \times I \). A standard \( S^1 \)-times Reeb component is the product foliation \( S^1 \times \mathcal{F}_{\mathbb{H}}(\pm 1) \), where \( \mathcal{F}_{\mathbb{H}}(\pm 1) \) is a standard Reeb component of \( S^1 \times D^1 \).

![An I-times slope component](image1)
![An S^1-times slope component](image2)
![An S^1-times Reeb component](image3)

![A twisted S^1-times Reeb component](image4)
![A rational (or irrational) rifle component](image5)

**Figure 10.1**
Consider $I \times \mathcal{F}_R^{\pm}(\pm 1)$ and take a diffeomorphism $\phi: S^1 \times D^1 \to S^1 \times D^1$ such that $\phi$ maps each non-compact leaf of $\mathcal{F}_R^{\pm}(\pm 1)$ to a different leaf of $\mathcal{F}_R^{\pm}(\pm 1)$, as in §3. Then we have a foliation $\mathcal{F}_\phi$ of a compact manifold diffeomorphic to $S^1 \times S^1 \times D^1$ from $I \times \mathcal{F}_R^{\pm}(\pm 1)$ by attaching the top and bottom of $I \times S^1 \times D^1$ by $\phi$.

**Definition 10.2.** The foliation $\mathcal{F}_\phi$ constructed above is called a standard twisted $S^1$-times Reeb component.

Take $\alpha \in R$. Let $\mathcal{F}_\alpha$ be the foliation of $R^2 \times I$ consisting of a leaf $R \times \{0\}$ and leaves
\[
\{(x + f(t), -\alpha f(t), t) | t \in I\}
\]
for $x \in R$, where we use the function $f: [0, 1] \to [-\infty, 0]$ introduced in §2. Since the canonical foliation of $Z \oplus Z$ to $R^2 \times I$ preserves $\mathcal{F}_\alpha$, we obtain the quotient foliation $\mathcal{F}_\alpha/ Z \oplus Z$ of $S^1 \times S^1 \times I = R^2 \times I/ Z \oplus Z$.

**Definition 10.3.** We call $\mathcal{F}_\alpha/ Z \oplus Z$ a standard rational (or irrational) rifle component if $\alpha$ is rational (or irrational).

We give some figures. (See Figure 10.1.)

### 11. The proof of the decomposition theorem.

The purpose of this section is to prove Theorem 3. Let $D \in D$ and $\text{ch}(D) = (l, m, n; p, q, s)$. Construct the double $W$ of $D^{[0]}$ by pasting two copies of $D^{[0]}_1$ along $\partial_s D^{[0]}_1 - \bigcup \{K | K$ is a compact leaf of $\mathcal{F} | D^{[0]}_1 \}$ contained in $\partial_s D^{[0]}_1$, where $\partial_s D^{[0]}$ is the union of connected components of $\partial D^{[0]}$ of type $(n)$. Then we have a vector field $Y$ on $W$ whose orbits are the leaves of $(\mathcal{F} | D^{[0]}_1 \cup \mathcal{F} | D^{[0]}_1)$. The index of $Y$ equals $p - q$, as in §4. Since $W$ is obtained from a closed surface of genus $n - 1$ by deleting $2(l + m) + p + q + s$ open two disks, we have
\[
p - q = 4 - 2(l + m + n) - p - q - s.
\]
Therefore we have an equation
\[
(*) \quad 2(l + m + n + p) + s = 4.
\]
The solutions of $(*)$ are
\[
(0, 0, 1; 1, q, 0), \quad (0, 0, 1; 0, q, 2), \quad (0, 0, 2; 0, q', 0), \quad (1, 0, 1; 0, q', 0),
\]
\[
(0, 1, 1; 0, q', 0), \quad (2, 0, 0; 0, 0, 0), \quad (1, 1, 0; 0, 0, 0), \quad (0, 2, 0; 0, 0, 0),
\]
where $q \geq 0$, $q' \geq 1$ and $q'' \geq 2$. Now let us examine the solutions of $(*)$ one by one.

The case $\text{ch}(D) = (0, 0, 1; 1, q, 0)$ where $q \geq 0$. Suppose that $q = 0$. Then $\mathcal{F} | D^{[0]}_1$ consists of concentric half circles with center $z_0$ at $\partial D^{[0]}_1$. 

\[
\]
We denote by $\overline{D}$ the compact manifold obtained from $D - D_{1}^{[0]}$ by attaching two copies of $D_{1}^{[0]}$ as the boundary. Then $\overline{D}$ is diffeomorphic to $I \times D_{1}^{[0]}$. (See Figure 11.1.)

Regard the center $z_{0}$ as a point in the bottom of $\overline{D}$. Then the leaves of $\mathcal{G}|\overline{D}$ passing sufficiently near $z_{0}$ are all homeomorphic to $D^{2}$. Using the local stability theorem for simply connected compact leaves (see Reeb [13], Haefliger [4]), we see that all the leaves of $\mathcal{G}|\overline{D}$ except $\{z_{0}\}$ are homeomorphic to $D^{2}$. Now it is easy to construct an orientation preserving homeomorphism $f_{\overline{D}}: \overline{D} \rightarrow S^{1} \times D^{2}$ with $\mathcal{G}|D = f^{*}T[f_{S^{1}}S^{1} \times D^{2}, D^{1} \times \{0\}, \sigma(C)]$, where $C \in \Gamma(h)$ intersects $*D$. Therefore $\mathcal{G}|D$ is a half Reeb component. Thus we have the case (I).

Suppose that $q > 0$. By a consideration on the holonomy of compact leaves of $\mathcal{G}$ contained in $\partial^{*}D$, we see that $\sigma(C') = -\sigma(C)$, where $\mathcal{G}|(\partial^{*}D \cap \partial_{c}A)$ has a positive Reeb component and $\mathcal{G}|(\partial^{*}D \cap \partial_{c}A)$ has a negative Reeb component. By the same arguments as above, we see that all the leaves of $\mathcal{G}|\overline{D}$ except the one point leaves are homeomorphic to $D^{2}$. Then it is easy to show that $\mathcal{G}|D$ is an arcade component. We omit the details. Thus we have the case (IV).

The case $\text{ch}(D) = (0, 0, 1; 0, q, 2)$ where $q \geq 0$. If $q = 0$, then $\mathcal{G}|D_{1}^{[0]}$ is isomorphic to the foliation $\{(x) \times I\}_{x \in I}$ of $I \times I$, and $\mathcal{G}|D$ is an $I$-times slope component, which is the case (II). Suppose that $q > 0$. Let $\mathcal{K}_{i}$, $\mathcal{K}_{j}$ be the slope component of $\mathcal{G}|(\partial^{*}D \cap \partial A)$, and $\mathcal{N}_{1}, \ldots, \mathcal{N}_{q}$ the negative Reeb components of $\mathcal{G}|(\partial^{*}D \cap \partial A)$. Considering the holonomy of compact leaves of $\mathcal{G}$ contained in $\partial^{*}D$, we see the following: a connected component $\partial(1)$ of $\partial^{*}D - \text{Int}(|\mathcal{K}_{i}| \cup |\mathcal{K}_{j}|)$ has the property that if $|\mathcal{N}_{j}| \subset \partial(1)$ and $|\mathcal{N}_{j}| \subset \partial_{c}A$ then $\sigma(C) = 1$. On the other hand, the other component $\partial(-1)$ has the property that if $|\mathcal{N}_{j}| \subset \partial(-1)$ and $|\mathcal{N}_{j}| \subset \partial_{c}A$ then $\sigma(C) = -1$. Using the arguments on $\mathcal{G}|\overline{D}$ as above,
we see that \( \Sigma | D \) is a TS' component, which is the case (III).

The case \( \text{ch}(D) = (0, 0, 2; 0, q, 0) \) where \( q \geq 2 \). Let \( \mathcal{N}_1', \ldots, \mathcal{N}_n' \) be the negative Reeb components contained in \( \Sigma | \partial A \) intersecting a connected component \( \tilde{\partial}(1) \) of \( \tilde{\partial}^* D^{[0]} \) and \( \mathcal{N}_{n+1}', \ldots, \mathcal{N}_{n+n}' \), the ones intersecting the other connected component \( \tilde{\partial}(-1) \). Let \( \mathcal{N}_j' = \gamma^* \mathcal{N}_j \), where \( \gamma:\tilde{\partial} E(h) \to A \) is as in \( \S 4 \). Then \( \mathcal{C} = (\mathcal{N}_1, \ldots, \mathcal{N}_n) \) and \( \mathcal{C}' = (\mathcal{N}_{n+1}, \ldots, \mathcal{N}_{n+n}) \) are negative Reeb cycles. We use the notations for \( \mathcal{C} \) in \( \S 6 \). Take \( M \in P^*(\mathcal{N}_1) \) and consider the local homeomorphism \( \eta(M):(T^1(M), z^1(M)) \to (T^1(M), z^1(M)) \).

Suppose that \( \eta[M] \) has a fixed point \( z \). Let \( K \) be the leaf of \( \Sigma | F \) passing through \( z \). Then \( K \) is a circle. Let \( K_i = \psi(i)K \) for \( i \in \mathbb{Z} \). For simplicity, suppose that \( \sigma(y) = 1 \) for \( y \in |\mathcal{N}_1| \cup \cdots \cup |\mathcal{N}_n| \). Since \( \Sigma | \text{Int} * D \) has no compact leaf, the set \( (\bigcap_{i \in \mathbb{Z}} \text{Cl}_F (\bigcup_{i \geq j} K_i)) \cap A^{[0]} \) must be the union of leaves of \( \Sigma | * D^{[0]} \) contained in \( \tilde{\partial}(-1) \). This implies that the vector field \( X \) is outward at \( |N_i| \) for \( i = n+1, \ldots, n+n' \). Therefore \( \sigma(y) = -1 \) for \( y \in |\mathcal{N}_{n+1}| \cup \cdots \cup |\mathcal{N}_{n+n}'| \). We see that \( \Sigma | D \) is \( C^0 \) isomorphic to a standard double gear component constructed from \( S_0 \) of \( S^1 \times S^1 \) containing a compact leaf homologous to \( \{\ast\} \times S^1 \). Thus we have the case (V).

When \( \eta[M] \) has no fixed point, we can take a torus imbedded in \( \text{Int} A \) transverse to \( \Sigma \) and to \( \mathcal{F}(h; \sigma) \) as in Proposition 6.4. Then we see that \( \Sigma | D \) is a double gear component, and have the case (V) again. We omit the details.

The case \( \text{ch}(D) = (1, 0, 1; 0, q, 0) \) where \( q > 0 \). By similar arguments, we see that \( \Sigma | D \) is a gear component, and can check the condition on \( \sigma \). Thus we have the case (VI).

The case \( \text{ch}(D) = (0, 1, 1; 0, q, 0) \) where \( q > 0 \). Let \( G \) be the compact leaf of \( \Sigma | \text{Int} A \) contained in \( \tilde{\partial}^* D \). Then \( K = G \cap * D^{[0]} \) is diffeomorphic to \( S^1 \), and there is \( a \in N \) with \( \psi(a)K = K \). If \( \Sigma | \text{Int} * D^{[0]} \) has a compact leaf \( K' \), then \( \psi^j(K') \) converges to \( K \) as \( j \) moves to \( \infty \) or \( -\infty \), and the leaves of \( \Sigma | * D^{[0]} \) are all compact by the usual arguments by means of the theorem of Kopell [7]. In this case, we see that \( \Sigma | D \) is a turbulized gear component, and have the case (VII-1). When \( \Sigma | \text{Int} * D^{[0]} \) has no compact leaf, we see that \( \Sigma | D \) is a perturbed gear component, and have the case (VII-2). The condition on \( \sigma \) is easily checked.

The case \( \text{ch}(D) = (2, 0, 0; 0, 0, 0) \). It follows that \( \tilde{\partial}^* D \subset \tilde{\partial} A \). Therefore \( * D \) is a non-empty closed open subset of \( A \). Since \( A \) is connected, it follows that \( * D = A \) and \( h = 2 \). Furthermore \( \Sigma \) is \( C^0 \) isomorphic to \( (\Sigma | C) \times I \) for some \( C \in \Gamma(h) \), which is a contradiction. Therefore this case does not occur.
Finally we see that the cases \( \text{ch} (D) = (1, 1, 0; 0, 0, 0), (0, 2, 0; 0, 0, 0) \) imply the cases (VIII), (IX) respectively. We omit the details. The construction of several components in the decomposition theorem is indicated in §§2, 3, 10. This completes the proof of Theorem 3.

12. Regular TS pieces. In this and next sections, we define a regular TS diagram as a generalization of a TS diagram introduced in Tamura-Sato [15] for \( \mathcal{F} \in t_t((\mathcal{F}(h; \sigma)) \) not \( C \) isomorphic to \( (\mathcal{F}|C) \times I \) for any \( C \in \Gamma(h) \). The construction of a regular TS diagram is like a jigsaw puzzle or a tangram. The pieces admitted in our puzzle are regular TS pieces defined below. In order to classify \( \mathcal{F}(\mathcal{F}(h; \sigma)) \), we will attach a regular TS piece to each \( D_j^{[n]} \in \Omega^{[n]} \cup \Theta^{[n]} \), where \( \Omega^{[n]} \cup \Theta^{[n]} \) is the TS decomposition of \( A^{[n]} \) with respect to \( \mathcal{F} \). For \( \mathcal{F} \) isomorphic to \( (\mathcal{F}|C) \times I \) for some \( C \in \Gamma(h) \), we will define a singular TS piece and a singular TS diagram in §20.

We make some preparations.

DEFINITION 12.1. A TS block \( \Delta \) is a compact oriented \( C^\infty \) manifold homeomorphic to \( D^2 \) or \( S^1 \times I \) and possibly with an even number of corner points on each connected component of \( \partial \Delta \).

DEFINITION 12.2. Let \( \mathcal{G} \) be a TS block. When \( \mathcal{G} \) has no corner, let \( J(\mathcal{G}) = \emptyset \). When \( \mathcal{G} \) has corner points, take and fix a set \( J(\mathcal{G}) \) of disjoint closed intervals of \( \partial \mathcal{G} \), whose endpoints are in the corner \( \partial \mathcal{G} \) of \( \mathcal{G} \), such that \( 2 \# J(\mathcal{G}) = \# \partial \mathcal{G} \). We denote by \( K(\mathcal{G}) \) the set of connected components of the closure of \( \partial \mathcal{G} - \bigcup \{ J \mid J \subseteq J(\mathcal{G}) \} \).

DEFINITION 12.3. An orientation of \( K \in K(\mathcal{G}) \) is sympathetic (or antipathetic) if it coincides with that of \( \partial \mathcal{G} \) as the boundary.

DEFINITION 12.4. Let \( \mathcal{J} \) be the set of five symbols \( \bigcirc, \bullet, \vee, \wedge, \| \). Let \( \text{TYPE} = \{ I, II, III, IV, V, VI, VII, VIII, IX \} \).

Now we can define TS pieces for \( \mathcal{F} \) not isomorphic to \( (\mathcal{F}|C) \times I \) for any \( C \in \Gamma(h) \), as follows.

DEFINITION 12.5. A regular TS piece is a quadruplet \( P = (\Delta, \nu, s; J, \omega) \): \( \mathcal{J}(\Delta) \rightarrow \mathcal{J}, \omega: K \rightarrow \{ 1, -1 \} \), where \( \Delta \) is a TS block and \( \nu \) belongs to \( \text{TYPE} \), and \( K \) is a subset of \( K(\mathcal{G}) \), satisfying the following conditions.

(P0) If \( \nu \in \{ VI, VIII \} \), then \( \mathcal{K} = K(\mathcal{G}) \).

(P1) If \( \nu = I \), then \( \Delta \simeq D^2 \), \( \# \mathcal{J}(\Delta) = 1 \) and \( s(J) = \bigcirc \), where \( \mathcal{J}(\Delta) = \{ J \} \).

(P2) If \( \nu = II \), then \( \Delta \simeq D^2 \), \( \# \mathcal{J}(\Delta) = 2 \), \( s(\mathcal{J}(\Delta)) = \{ \| \} \) or \( \{ \vee, \wedge \} \), and \( \omega(K(\mathcal{G})) = \{ 1, -1 \} \).

(P3) If \( \nu = III \), then \( \Delta \simeq D^2 \), \( \# \mathcal{J}(\Delta) > 2 \), \( s(J_1) = \vee \) and \( s(J_2) = \wedge \).
Figure 12.1 Regular TS pieces
for some $J_1, J_2 \in \mathcal{J}(\Delta)$, $s(J) = \bullet$ for all $J \in \mathcal{J}(\Delta) - \{J_1, J_2\}$, and $\omega(K) = -\omega(K)$ if $K, K_0 \in \mathcal{K}$ are contained in different connected components of $\partial \Delta - \text{Int}(J_1 \cup J_2)$.

(P4) If $\nu = IV$, then $\Delta \simeq D^2, \# \mathcal{J}(\Delta) > 1$, $s(J) = \circ$ for some $J \in \mathcal{J}(\Delta)$, and $\omega$ is constant.

(P5) If $\nu = V$, then $\Delta \simeq S^1 \times I, s(\mathcal{J}(\Delta)) = \{\bullet\}$, $\mathcal{K}(\Delta)$ contains no circle, and $\omega(K) = -\omega(K)$ if $K, K_0 \in \mathcal{K}$ are contained in different connected components of $\partial \Delta$.

(P6) If $\nu = VI$ or VII, then $\Delta \simeq S^1 \times I, s(\mathcal{J}(\Delta)) = \{\bullet\}$, $\mathcal{K}(\Delta)$ contains exactly one circle $K_0$, and $\omega$ is constant. If $\nu = VI$, then $\mathcal{K} = \mathcal{K}(\Delta) - \{K_0\}$.

(P7) If $\nu = VIII$ or IX, then $\Delta \simeq S^1 \times I$ and $\mathcal{J}(\Delta) = \emptyset$. If $\nu = VIII$, then $\# \mathcal{K} = 1$.

We call $\Delta$, $\nu$, $s$, $\omega$ the underlying block, type, symbol map, orienting map of $P$ respectively. We denote $\Delta$ (or $\nu$) by $|P|$ (or type $(P)$) sometimes.

REMARK 12.6. A regular TS piece corresponds to a component of the same type in Theorem 3, and the symbols $\circ$ (or $\bullet$) to a positive (or negative) Reeb component of $\mathcal{G}|\partial \mathcal{E}(h)$. The symbols $\triangledown$ and $\wedge$ correspond to a slope component, and the symbol $\parallel$ to a trivial component.

REMARK 12.7. The map $\omega: \mathcal{K} \rightarrow \{1, -1\}$ means the choice of orientations of $K \in \mathcal{K}$ (cf. the proof of Theorem 4 in §14). Precisely we give $K \in \mathcal{K}$ the sympathetic (or antipathetic) orientation if $\omega(K) = 1$ (or $-1$).

In order to make regular TS pieces more understandable, we picturize them in Figure 12.1. In figures, we use the convention that $\triangledown$ (or $\wedge$) is put inward (or outward) to underlying TS blocks (see Figure 12.1, II, III for example). When it is not necessary to distinguish between $\triangledown$ and $\wedge$, we will use $\times$ in their place. The bold (or fine) lines mean the elements of $\mathcal{J}(\Delta) \cup (\mathcal{K}(\Delta) - \mathcal{K})$ (or $\mathcal{K}$). All the TS blocks are oriented as $\circ$. The mark $[\sigma]$ means the existence of conditions on $\sigma$ for constructing regular TS diagrams in the next section.

13. Regular TS diagrams. For the future use, we define regular TS diagrams in a more general setting. Let $\Sigma_g(h)$ be the compact oriented manifold obtained from the closed surface of genus $g$ by deleting $h(>0)$ small open two disks, and denote by $\Gamma_g(h)$ the set of connected components of $\partial \Sigma_g(h)$. Clearly $E(h) = \Sigma_g(h)$ and $\hat{E}(h) = \Gamma_g(h)$. Take a continuous map $\sigma: \partial \Sigma_g(h) \rightarrow \{1, -1\}$. As before we regard $\sigma$ as a map from $\Gamma_g(h)$ sometimes.
DEFINITION 13.1. A pre TS diagram of $\Sigma_f(h)$ is a triad $(S, \{P_f\}_{f \in \Lambda}, \{\gamma_f \mid P_f \to \Sigma_f(h)\}_{f \in T})$ satisfying the following conditions.

(PR1) $S$ is the union of a finite number of disjoint circles $S_1, \ldots, S_n$ contained in $\text{Int} \Sigma_f(h)$.

(PR2) $P_f = (\gamma_f, \delta_f, s: J \to \mathbb{R}, w: K \to \{-1, 1\})$ is a regular TS piece, $\#\{\lambda \in \Lambda \mid \gamma_f \in \{\Pi, IX\}\} < \infty$, and $\gamma_f: \Delta \to \Sigma_f(h)$ is an orientation preserving $C^\infty$ immersion such that $\gamma_f|\text{Int} \Delta$ is an imbedding.

(PR3) $(\text{Int} \Delta_f)'$'s are disjoint, and $\Sigma_f(h)$ is the closure of $\bigcup \{\gamma_f \mid \lambda \in \Lambda\}$.

For each $S_i$, there are $\delta_i, \delta_i', S_i' \in \Sigma_f(h)$ and $C, C' \in \Sigma_f(h)$ such that

1. $\delta_i = VI, \delta_i' = \delta_i, C \neq \emptyset$,
2. $\delta_i' \cap S_i \neq \emptyset, \delta_i' \cap C' \neq \emptyset$.

(We call $P_f$ the TS piece of Type VI separated by $S_i$.)

(PR4) $J \subset S \cup \partial \Sigma_f(h)$ for $J \in \Delta_f, \delta_i \subset \partial \Sigma_f(h)$ for $K \in \mathcal{K}(\Delta) - \mathcal{N}$, and $\text{Int} \delta_i \subset \text{Int} \Sigma_f(h) - S$ for $K \in \mathcal{K}$.

(PR5) If $K \neq K'$ and $\delta_i = \delta_i'$ for $K \in \mathcal{K}$ and $K' \in \mathcal{K}'$, then $w(K') = -w(K)$. (Hence we can give $\delta_i$ an orientation such that $\gamma_f|K$ and $\gamma_f'|K'$ are orientation preserving.)

DEFINITION 13.2. Let $\mathcal{F} = (S, \{P_f\}_{f \in T}, \{\gamma_f\}_{f \in T})$ and $\mathcal{F}' = (S', \{P'_f\}_{f' \in T'}, \{\gamma'_f\}_{f' \in T'})$ be pre TS diagrams of $\Sigma_f(h)$. A homeomorphism $\phi: \Sigma_f(h) \to \Sigma_f(h)$ is called an isomorphism from $\mathcal{F}$ to $\mathcal{F}'$ if $\phi(S) = S'$ and if there are bijections

$$\phi: A \to A', \gamma_f \to \gamma'_f; (\mathcal{F}_f(\gamma)) \text{ and } : \mathcal{K} \to \mathcal{K'}$$

such that

1. $\gamma_f' = \gamma_f, \phi(\gamma_f) = \gamma_f'$,
2. $\phi(\delta_i) = \delta_i'$ and $s(\delta_i) = s(\delta_i')$ for $J \in \Delta_f$,
3. $\phi(\delta_i') = \delta_i'$ and $w(K') = w(K)$ for $K \in \mathcal{K}$.

where $P_f = (\Delta_f, \gamma_f, s: \Delta_f \to \mathbb{R}, \omega: \Delta_f \to \{-1, 1\})$ and $P'_f = (\Delta_f', \gamma'_f, s: \Delta_f' \to \mathbb{R}, \omega: \Delta_f' \to \{-1, 1\})$.

Now we can define regular TS diagrams as follows.

DEFINITION 13.3. A regular TS diagram of $(\Sigma_f(h); \sigma)$ is a triad $\mathcal{F} = (\mathcal{F}, \{\phi_i; \Sigma_f(h) \to \Sigma_f(h)\}_{i \in I}, (a, b; \sigma): \Gamma_f(h) \to (N \times Z)^* \times 2Z)$ satisfying the following conditions.

(R1) $\mathcal{F} = (S, \{P_f\}_{f \in T}, \{\gamma_f\}_{f \in T})$ is a pre TS diagram of $\Sigma_f(h)$, and $\{\phi_i\}_{i \in I}$ is a $C^\infty$ isometry of diffeomorphisms such that $\phi_0$ is the identity and $\phi_i$ is an isomorphism from $\mathcal{F}$ to $\mathcal{F}$.

(R2) Let $C \in \Gamma_f(h)$ and put $p(C)$ (or $q(C)$) = $\#\{J \mid J \in \Delta_f \}$ for some $\lambda \in \Lambda, \delta_i \subset C, s(\Delta_f) = \emptyset$ (or $\emptyset$).
(i) If \((a(C), b(C)) = (0, 1)\) or \((\varphi^*, \varphi^*)\), then \(r(C) = 0\) and there are \(\varphi \in A\) and \(K \in \mathcal{K}(A) - \mathcal{K}_1\) with \(*K = C\).

(ii) If \((a(C), b(C)) = (a, b) \in \mathbb{N} \times \mathbb{Z}\) coprime, then \(r(C) = (p(C) - q(C))/a\), there are \(\varphi \in A\) and \(J \in \mathcal{J}_1\) with \(*J = C\), the map \((\varphi_1|C)^*\) is the identity, \((\varphi_1|C)^*\) has no fixed point for \(0 < a' < a\), and the degree of \(\eta: [0, a]/[0, a] \to C\) equals \(b\), where \(\eta\) is defined by \(\eta([t]) = \varphi_t(\varphi_0(y_0))\) for \(t = k + t', k \in \mathbb{Z}, 0 \leq t < 1\) and a fixed point \(y_0 \in C\).

(s) Let \(S_i\) be a circle in \(S\), and \(C, C' \in \Gamma_\sigma(h)\) be as in Definition 13.1 (PR3). Then \((a(C'), b(C')) = (a(C), -b(C))\).

(R3) (The conditions on \(\sigma\)). Below, \(J\) and \(J'\) are the elements of \(\mathcal{J}_1\) with \(*J, *J' \in \mathcal{J}(h)\).

(iii) If \(\nu_1 = III\), then \(\sigma(*y') = \sigma(*y)\) (or \(-\sigma(*y)\)) for \(y \in J\) and \(y' \in J'\) such that \(J\) and \(J'\) are contained in the same (or different) connected component of \(\partial \mathcal{J}_1 - \bigcup \{J''|J'' \in \mathcal{J}_1, s(J'') = \lor\ or\ \land\}\).

(iv) If \(\nu_1 = IV\), then \(\sigma(*y') = -\sigma(*y)\) for \(y \in J\) and \(y' \in J'\) such that \(s(J) = \lor\) and \(s(J') = \land\).

(vi) If \(\nu_1 = VI\) and \((a(C), b(C)) = (0, 1)\) for \(C \supseteq K \in \mathcal{K}(A) - \mathcal{K}_1\), then \(\sigma(*y') = -\sigma(*y)\) for \(y \in K\) and \(y' \in J\).

(vii) If \(\nu_1 = VII\), then \(\sigma(*y') = \sigma(*y)\) for \(y \in J\) and \(y' \in J'\).

In (R2) and (R3), we used the description \(P_1 = (\mathcal{J}_1, \nu_1, s\mathcal{J}_1 \to \mathcal{I}, \omega: \mathcal{J}_1 \to \{1, -1\})\).

**Definition 13.4.** For a regular TS piece \(P = (\mathcal{J}, \nu, s: \mathcal{J} \to \mathcal{I}, \omega: \mathcal{J} \to \{1, -1\})\), let \(-P = (\mathcal{J}, \nu, s, -\omega)\). For a pre TS diagram \(\mathcal{T} = (S, \{P_{z \in I}\}, \{\phi_z\}_{z \in I})\), let \(-\mathcal{T} = (S, \{-P_{z \in I}\}, \{\phi_z\}_{z \in I})\). For a regular TS diagram \(\mathcal{T} = (\mathcal{T}, \{\phi_z\}_{z \in I}, (a, b; r))\), let \(-\mathcal{T} = (-\mathcal{T}, \{\phi_z\}_{z \in I}, (a, b; r))\).

We introduce an equivalence relation on regular TS diagrams of \((\Sigma_\sigma(h); \sigma)\) as follows.

**Definition 13.5.** Let \(\mathcal{T} = (\mathcal{T}, \{\phi_z\}_{z \in I}, (a, b; r))\) and \(\mathcal{T}' = (\mathcal{T}', \{\phi_z\}_{z \in I}, (a', b'; r'))\) be regular TS diagrams of \((\Sigma_\sigma(h); \sigma)\). Then \(\mathcal{T}\) is isomorphic to \(\mathcal{T}'\) if there exists a \(C^0\) isotopy of homeomorphisms \(\{h_t: \Sigma_\sigma(h) \to \Sigma_\sigma(h)\}_{t \in I}\), such that \(\phi_t' = h_t \circ \phi_t\) for \(t \in I\), \(h_0\) is an isomorphism from \(\mathcal{T}\) to \(\mathcal{T}'\) or \(-\mathcal{T}'\), \(h_0\) is an isotopic to the identity, and \((a', b'; r') = (a, b; r)\).

We denote by \(\text{RTS}(\Sigma_\sigma(h); \sigma)\) the set of isomorphism classes of regular TS diagrams of \((\Sigma_\sigma(h); \sigma)\).

14. The classification theorem. Using regular TS diagrams, we can generalize the classification theorem of Tamura-Sato [15]. In this paper we give only the following.

**Theorem 4.** Let \(\mathcal{F} \in \tau_1^0(\mathcal{F}(h; \sigma))\). If \(\mathcal{F}\) is not \(C^0\) isomorphic to
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(g|C) Í I for any horizontal C Í Γ(h), then an element of RTS (Ẽ(h); σ) is canonically attached to $\mathcal{G}$.

PROOF. Take an orientation of $\mathcal{G}|F^{[0]}$. Let $\mathcal{G}_1, \ldots, \mathcal{G}_n$ be the strange negative Reeb cycles of $\mathcal{G}$ and $S^*_i$ a separating torus of $\mathcal{G}_i$. Define a map $(a, b; \bar{r}): \Gamma(h) \to (N \times Z)^* \times 2Z$ as in §4, and let $(\bar{a}, \bar{b}; \bar{r}) = (a, b; r) \circ \zeta$, where $\zeta: \hat{\Gamma}(h) \to \Gamma(h)$ is defined by $\zeta(C) = S^i \times C$.

We construct a $C^\infty$ vector field $Z$ on $A$ transverse to $A^*$ for all $x \in S^i$ and tangent to $S_1^*, \ldots, S_n^*$ and $\partial A$. First we define $Z$ on $\partial A$ as follows. Let $C \in \Gamma(h)$. When $C$ is horizontal or $\mathcal{G}|\partial_c A$ has no compact leaf, let $Z = \partial/\partial t$ on $\partial_c A$, where $t$ is the coordinate of the factor $S^i$. When $C$ is vertical and $\mathcal{G}|C$ has a compact leaf, take as $Z|\partial_c A$ a vector field transverse to $A^*$ for all $x \in S^i$, tangent to the compact leaves of $\mathcal{G}|\partial_c A$, and having no non-closed orbit. By Proposition 7.1, the foliation $\mathcal{G}|S^*_i$ has a compact leaf. Take as $Z|S^*_i$ a vector field transverse to $A^*$ for all $x \in S^i$, tangent to the compact leaves of $\mathcal{G}|S^*_i$, and having no non-closed orbits. Then we can take as $Z|G$ a vector field on $G$ for each compact subleaf $G$ of $\mathcal{G}|A$ in a consistent way by Propositions 7.1 and 8.1. Furthermore we can extend the vector field thus obtained over all $A$ by the decomposition theorem.

We define a $C^\infty$ isotopy $\{\phi_t: A^{[0]} \to A^{[0]}\}_{t \in I}$ as follows. Let $C(A, A^{[0]})$ be the compact manifold obtained from $A$ by cutting along $A^{[0]}$. Denote by $A^b$ (or $A^t$) the bottom (or top) of $C(A, A^{[0]})$. Let $A^t = A^{[0]}$ for $t \in [0, 1]$. Now define $\phi_t(z)$ for $z \in A^{[0]}$ as the intersection point of $A^t$ and the orbit of $Z$ passing through $z$, and define $\phi_t(z)$ as the intersection point of $A^{[0]}$ and the fiber of the projection: $S^1 \times \hat{E}(h) \to S^1$ passing through $\phi_t(z)$.

In a canonical way, we can construct a pre TS diagram $\mathcal{J} = (S, \{P_1\}_{i \in \lambda}, \{\ell_i\}_{i \in \lambda})$ of $A^{[0]}$ satisfying the following conditions.

1. $S = A^{[0]} \cap (S^*_1 \cup \cdots \cup S^*_n)$.
2. $\{P_1\}_{1 \leq i \leq t}$ (or $\{\ell_i\}_{1 \leq i \leq t}$) coincides with $\mathcal{Q}^{[0]} \cup \Theta^{[0]}$ (or $\ast \mathcal{Q}^{[0]} \cup \ast \Theta^{[0]}$) except some compact subleaf $\in \Theta^{[0]}$ (or $\ast \Theta^{[0]}$).
3. To an element $D^{[0]}_j \in \mathcal{Q}^{[0]}$ corresponds a regular TS piece of the same type as $D$.
4. To an element of $\Theta^{[0]}$ corresponds a regular TS piece of type II with symbol $\|$ or a regular TS piece of type IX.
5. The symbols $\bigcirc, \bullet$ and $\|$ correspond to the components stated in Remark 12.7.
6. The symbol $\vee$ (or $\wedge$) for $J \subset \partial A^{[0]}$ corresponds to a slope component $\mathcal{G}$ contained in $\mathcal{G}|\partial A^{[0]}$ such that a connected component of $\partial |\mathcal{G}|$ has an expanding holonomy with respect to $\mathcal{G}$ in the same (or
opposite) direction as the orientation of $\partial|S|$. (See Figure 14.1.)

(7) For $K \in \mathcal{K}$, the orientation of $*K$ determined by $\omega$ coincides with that of $E|F^{[0]}$.

Now take a diffeomorphism $\xi: A^{[0]} \to \hat{E}(h)$ isotopic to the identity. Transforming $(\mathcal{J}, \{\phi_t\}_{t \in I})$ by $\xi$, we have a triad $\mathcal{J}' = (\mathcal{J}', \{\phi'_t: \hat{E}(h) \to \hat{E}(h)\}_{t \in I}, (\bar{a}, \bar{b}; \bar{r}))$. Then $\mathcal{J}'$ is a regular TS diagram of $(\hat{E}(h); \sigma)$ and the isomorphism class of $\mathcal{J}'$ depends only on $\mathcal{J}$. Since the check of the details is tedious, we omit it except for the condition (R2-s) in Definition 13.3. For checking (R2-s), we need the following.

**Proposition 14.1.** Let $\mathcal{J} = (S, \{P_{\mathcal{J}}\}_{\mathcal{L} \in \mathcal{L}}, \{Q_{\mathcal{J}}\}_{\mathcal{L} \in \mathcal{L}})$ be a pre TS diagram and $\{\phi_t: \Sigma(h) \to \Sigma(h)\}_{t \in I}$ an isotopy satisfying the condition (R1) in Definition 13.3. Let $S_i$ be a circle in $S$. Then $\phi_1(S_i) = S_i$.

**Proof.** Suppose that $\phi_1(S_i) \neq S_i$. Since $\phi_1(S) = S$, it follows that $S_j = \phi_1(S_i) \subset S$. Take $\lambda, \lambda' \in \Lambda, \mathcal{K} \in \mathcal{K}$, $\mathcal{K}' \in \mathcal{K}$, and $C, C' \in \Gamma(h)$ satisfying the conditions in Definition 13.1 (RP3). Since $\phi_1(C) = C$ and $\phi_1(C') = C'$, it follows that $\phi_1(*A) \cap C \neq \emptyset, \phi_1(*K' \cap S_j \neq \emptyset$ and $\phi_1(*K') \cap C' \neq \emptyset$. Take a path $\pi: I \to *A$ with $\pi(0) \in C$ and $\pi(1) \in *K' \cap S_i$. Consider the circle

$$L = \pi(I) \cup *K' \cup \{\phi_t(\pi(0))|t \in I\} \cup \phi_1(\pi(I)) \cup *K' \cup \{\phi_t(y)|t \in I\},$$

where $\{y\} = *K' \cap C'$. Then the intersection number $L \cdot S_i$ equals $\pm 1$. On the other hand, $L$ is the boundary of the degenerate disk $\{\phi_t(\pi(I)) \cup *K'|t \in I\}$ as a singular chain. Hence $[L] = 0$ in $H_1(\Sigma(h); \mathbb{Z})$, which is a contradiction. This completes the proof of Proposition 14.1.

**The Check of (R2-s).** By Proposition 14.1, we see that $S_i^* \cap A^{[0]}$ consists of exactly one circle. Furthermore a compact leaf of $\mathcal{J}|S_i^*$ is isotopic to a compact leaf in $\mathcal{J}|C'$ if for $C' \in \Gamma(h)$ there is a compact
subleaf of $\mathcal{F}$ intersecting $S^*_i$ and $C'$. The same argument as in the proof of Proposition 8.1 implies that $(a(C'), b(C')) \neq (a(C), -b(C))$, where $C \supset |\mathcal{N}|$ for some negative Reeb component $\mathcal{N}$ in $\mathcal{F}_i$. This completes the proof of Theorem 4.

**PART II**

Existence problem of transverse foliations

In order to check the conditions (A2) and (A3) for $\mathcal{F} \in \mathfrak{T}_1(\mathcal{F}(h); \sigma)$, we investigate regular TS diagrams thoroughly in this and next sections. This is the most essential part of the proof of Theorem 1 and can be regarded also as an addendum to Part I.

Let $J = (J, \{f_t\}_{t \in I}, \{a, b; r\})$ be a regular TS diagram of $(E(h); f^\sigma)$. Let $J = (S, \{P_{\partial J}\}_{\partial J}, \{f_{\partial J}\}_{\partial J})$ and $P_{\partial J} = (\partial J, \nu_{\partial J}, s: J_{\partial J} \rightarrow \mathcal{J}_2, \omega: \mathcal{J}_2 \rightarrow \{1, -1\})$. Transforming $(a, b; r)$ by the canonical bijection $\gamma^*: \Gamma(h) \rightarrow \hat{\Gamma}(h)$, we can regard it as a map from $\Gamma(h)$ to $(N \times Z)^* \times 2Z$.

Note the following, which is a direct consequence of Definition 13.3 (R2). We omit the proof.

**LEMMA 15.1.** Let $(a, b; r): \Gamma(h) \rightarrow (N \times Z)^* \times 2Z$ be in a regular TS diagram.

1. If $r(C) \neq 0$, then $(a(C), b(C)) \subset (N \times Z)^{coprime}$.
2. If $a(C)r(C) = 0$, then $r(C) = 0$.

Let $E_0$ be the closure of a connected component of $\hat{E}(h) - S$. Clearly $E_0$ is diffeomorphic to $\hat{E}(h_0)$ for some $h_0 \in N$. Let $\Gamma_0 = \{C \in \Gamma(h) | \hat{C} \subset \partial \hat{E}_0\}$. Then we have the following.

**PROPOSITION 15.2** (The TS formula for regular TS diagrams).

1. $\sum_{C \in \Gamma(h)} a(C)r(C) = 4 - 2h_0$.
2. $\sum_{C \in \Gamma(h)} a(C)r(C) = 4 - 2h$.

**PROOF.** We can construct a vector field $Y$ on $\hat{E}(h)$ satisfying the following conditions.

1. For each $J \in \mathcal{J}_2$, $\lambda \in A$, with $s(J) = \bigcirc$ (or $\bullet$), there is a singular point $z(J) \in \text{Int} *J$ of $Y$ such that the orbits of $Y$ near $z(J)$ are concentric half circles (or confocal parabolas). (See Figure 4.2.) Furthermore $Y$ has no other singular point.
2. $Y$ is tangent to $*K$ for each $K \in \mathcal{K}_2$, $\lambda \in A$, and the direction of $Y$ coincides with the orientation of $*K$ determined by $\omega(K)$.

Now we can prove Proposition 15.2 in the same way as Proposition 4.3.
Hereafter we suppose that $h = 2$. The goal of this section is the following, which corresponds to (A2).

**THEOREM 5.** Let $\mathcal{S} = (\mathcal{S}, \{\phi_t\}_{t \in I}, (a, b; r))$ be a regular TS diagram of $(\hat{E}(2); a)$. Let $\Gamma(h) = \{C, C'\}$. Then

1. $r(C') = -r(C)$.
2. If $r(C) \neq 0$, then $(a(C'), b(C')) = (a(C), -b(C))$.

In order to prove Theorem 5, we prove three lemmas. Let $(a(C), b(C); r(C)) = (a, b; r)$ and $(a(C'), b(C'); r(C)) = (a', b'; r')$. The first lemma is the following.

**LEMMA 15.3.** If there are $\lambda \in A$ and $K \in \mathcal{S}^r$, such that $*K \cap \hat{C} \neq \emptyset$ and $*K \cap \hat{C}' \neq \emptyset$, then $(a', b'; r') = (a, -b; -r)$.

**PROOF.** Since $*K \cap \hat{C} \neq \emptyset$ and $*K \cap \hat{C}' \neq \emptyset$, it follows that $(a, b), (a', b') \in (N \times \mathbb{Z})^{\text{coprime}}$ by Definition 13.3 (R2-i). Let $*K \cap \hat{C} = \{y\}$ and $*K \cap \hat{C}' = \{y'\}$. Then $\phi_t^\lambda(y) = y$ and $\phi_t^\lambda(y') \neq y'$ for $0 < k < a$ by (R2-ii). Similarly $\phi_t^\lambda(y) = y'$ and $\phi_t^\lambda(y') \neq y'$ for $0 < k < a'$. Since $\phi_t^\lambda$ is an isomorphism from $\mathcal{S}$ to $\mathcal{S}$ by (R1), it follows that $\phi_t^\lambda(*K) = \phi_t^\lambda(*K) = *K$ and $\phi_t^\lambda(*K) \neq *K$ for $0 < k < a$ or $0 < k < a'$. This implies that $a' = a$.

Modifying $\{\phi_t\}_{t \in I}$ if necessary, we may suppose that $(\phi_t^\lambda(*K))^* = \text{id}$ since $*K$ is diffeomorphic to $I$. Define a map $F: \mathcal{S}^r \times *K \to \hat{E}(h)$ by

$$F([t], z) = \phi_{t'}^\lambda(\phi_t^\lambda(z))$$

for $[t] \in S^r = [0, a]/\{0, a\}, t = k + t', k \in \mathbb{Z}, 0 \leq t' < 1$ and $z \in *K$. Then $F$ can be regarded as a homotopy from $F|\mathcal{S}^r \times \{y\}$ to $F|\mathcal{S}^r \times \{y'\}$. By (R2-ii), it follows that

$$b[\hat{C}] = F_*(\mathcal{S}^r \times \{y\}) = F_*(\mathcal{S}^r \times \{y'\}) = b'[\hat{C}']$$

in $H_1(\hat{E}(2); Z)$. Since $[\hat{C}'] = -[\hat{C}] \neq 0$, we have $b' = -b$. Since $ar + ar' = 0$ by Proposition 15.2, it follows that $r' = -r$, which completes the proof of Lemma 15.3.

The second is the following.

**LEMMA 15.4.** If $S \neq \emptyset$, then $(a', b') \neq (a, -b)$ and $r' = r = 0$.

**PROOF.** By the condition (R2-s) in Definition 13.3, it follows that $(a', b') \neq (a, -b)$. Since $\hat{E}(2)$ is an annulus, a circle in $\hat{E}(2)$ bounds a disk or is isotopic to $\hat{C}$ in $\hat{E}(2)$. By the condition (PR1) in Definition 13.1, a circle in $S$ bounds no disk. Furthermore we see that $S$ is connected. By Proposition 15.2 (1), it follows that $ar = a'r' = 0$. By Lemma 15.1 (2), we have $r = r' = 0$. This completes the proof of Lemma 15.4.

Now Theorem 5 follows directly from Lemmas 15.3 and 15.4 and the following.
LEMMA 15.5. Suppose that $S = \emptyset$ and that for any $\lambda \in \Lambda$ and $K \in \mathcal{X}_2$ with $*K \cap \hat{C} \neq \emptyset$ it holds that $*K \cap \hat{C'} = \emptyset$. Then $r = r' = 0$.

Proof. Let $V = \bigcup \{^*D_2 \mid *D_2 \cap \hat{C} \neq \emptyset\}$. When $V \cap \hat{C'} = \emptyset$, there is a circle in $\partial V$ separating $\hat{C}$ and $\hat{C'}$. Then we can show that $ar = a'r' = 0$ in the same way as in the proof of Proposition 15.2. Hence $r = r' = 0$. When $V \cap \hat{C'} \neq \emptyset$, there is exactly one regular TS piece $P_3$ of type $V$ or VI. Taking a circle in $\text{Int} *D_2$ separating $\hat{C}$ and $\hat{C'}$, we see that $ar = a'r' = 0$ as above. Hence $r = r' = 0$. This completes the proof of Lemma 15.5 and Theorem 5.


The purpose of this section is to prove the following, which corresponds to (A3).

THEOREM 6. Let $\mathcal{T} = (\hat{\mathcal{T}}, \{\hat{\phi}_t\}_{t \in I}, (a, b; r))$ be a regular TS diagram of $(\hat{E}(h); \sigma)$ and suppose that $h > 2$. If $r(C) \neq 0$ for $C \in \Gamma(h)$, then $(a(C), b(C)) = (1, 0)$.

We use the same notations as in §15. First we prove the following.

LEMMA 16.1. Suppose that $h > 2$ and let $C \in \Gamma(h)$. If there are $\lambda \in \Lambda$ and $K \in \mathcal{X}_2$ such that $*K \cap \hat{C} \neq \emptyset$ and $*K \cap \hat{C'} \neq \emptyset$ for some $C' \in \Gamma(h) - \{C\}$, then $(a(C), b(C)) = (1, 0)$.

Proof. Let $(a(C), b(C)) = (a, b)$ and $(a(C'), b(C')) = (a', b')$. Then we see that $a = a'$ and $b[\hat{C}] = b'[\hat{C'}]$ in $H_1(\hat{E}(h); \mathbb{Z})$ as in the proof of Lemma 15.3. Since $h > 2$, the homology classes $[\hat{C}]$ and $[\hat{C'}]$ have no linear relation. Therefore $b = b' = 0$. Since $(a, b) \in (N \times \mathbb{Z})^{\text{coprime}}$, it follows that $a = 1$. This completes the proof of Lemma 16.1.

Now Theorem 6 follows directly from Lemma 16.1 and the following.

LEMMA 16.2. Suppose that $h > 2$. Let $C \in \Gamma(h)$ and $K(C) = \bigcup \{*K \mid \lambda \in \Lambda, K \in \mathcal{X}_2, *K \cap \hat{C} \neq \emptyset\}$. If $K(C) \cap \hat{C'} = \emptyset$ for all $C' \in \Gamma(h) - \{C\}$, then $(a(C), b(C)) = (1, 0)$ or $r(C) = 0$.

Proof. Suppose that $r(C) \neq 0$. Let $(a(C), b(C)) = (a, b)$. We are going to prove that $(a, b) = (1, 0)$. By Lemma 15.1 (1), it follows that $(a, b) \in (N \times \mathbb{Z})^{\text{coprime}}$.

When $K(C) \cap S \neq \emptyset$, we see that $(a, b) = (1, 0)$, as follows. Since $r(C) \neq 0$, the connected component of $\hat{E}(h) - S$ containing $\hat{C}$ is homeomorphic to $\hat{E}(h')$ for some $h' > 2$ by Proposition 15.2 (1). Then the arguments in the proof of Lemma 16.1 implies that $(a, b) = (1, 0)$.

Hereafter suppose that $K(C) \cap S = \emptyset$. Let $V = \bigcup \{^*D_2 \mid *D_2 \cap \hat{C} \neq \emptyset\}$
and \( \partial V = \partial V - \{ \hat{C} \} \). Denote by \( B \) the set of connected components of \( \partial V \). Then we can write

\[
B = \{ \hat{C}_1, \ldots, \hat{C}_\alpha, S_1, \ldots, S_\beta, *K_1, \ldots, *K_\gamma \},
\]

where \( C_j \in \mathcal{G}(h), S_j \subset S \) and \( K_j \in \mathcal{K}(j) \) for \( \lambda(j) \in \Lambda \). Clearly \( P_{\lambda(j)} \) is of type \( VII \) and \( *A_{\lambda(j)} \subset V \). For each \( C_j \), there is \( \mu(j) \in \Lambda \) such that \( *A_{\mu(j)} \subset V, \nu_{\mu(j)} = VI \) and \( \hat{C}_j = *L_j \) for \( L_j \in \mathcal{K}(A_{\mu(j)}) - \mathcal{K}(j) \). For each \( S_j \), there is \( \rho(j) \in \Lambda \) such that \( *A_{\rho(j)} \subset V, \nu_{\rho(j)} = VI \) and \( \hat{S}_j = *M_j \) for \( M_j \in \mathcal{K}(A_{\rho(j)}) - \mathcal{K}(j) \).

We see that \( \#B = \alpha + \beta + \gamma > 1 \) as follows. Suppose that \( \#B = 1 \). Then \( V \) is homeomorphic to \( E(2) \). Since the TS formula can be obtained for \( V \), we have \( r(C) = 0 \) from it. Thus we have a contradiction.

Applying the arguments in the proof of Proposition 14.1, we see that \( \phi_1(*K_j) = *K_j \). Therefore \( \phi_1 \) fixes all the elements of \( B \). Since \( \#B > 1 \), the map \( \phi_1 \) must fix all \( J \in \mathcal{J}_{\lambda(1)} \cup \cdots \cup \mathcal{J}_{\lambda(\beta)} \cup \mathcal{J}_{\mu(1)} \cup \cdots \cup \mathcal{J}_{\mu(\gamma)} \). This implies that \( a = 1 \) and \( \phi_1 | \hat{C} = id \).

Since \( \#B > 1 \), there are \( \lambda \in \Lambda \) and \( K \in \mathcal{K} \) such that \( *K \subset K(C) \) and \( *[K] \neq 0 \) in \( H_1(V, \hat{C}; \mathbb{Z}) \). Let \( L \) be the union of \( *K \) and a connected component of \( \hat{C} - *K \). Let \( y_0 \in \hat{C} \cap *K \). Then \( \hat{C} \) and \( L \) determine elements \( \xi_i \) and \( \xi_j \) of \( \pi_1(V, y_0) \), respectively. Adding adequate elements \( \xi_i, \ldots, \xi_{i+\beta+\gamma}, \xi_{a+\beta+\gamma} \), we can regard \( \pi_1(V, y_0) \) as the free group generated by \( \xi_i, \ldots, \xi_{a+\beta+\gamma} \), \( \xi_{i+\beta+\gamma}, \xi_{a+\beta+\gamma} \). Modifying \( \phi_1 | t \) in a neighborhood of \( *K \), we may suppose that \( \phi_i | L = id \). Then we can define a map \( \eta: S^1 \times L \rightarrow V \) by \( \eta([t], y) = \phi_i(y) \) for \( t \in I \) and \( y \in L \). The paths \( \eta | S^1 \times \{ y_0 \} \) and \( \eta | ([0]) \times L \) represent \( \xi_i \) and \( \xi_j \), respectively. Since \( \pi_1(S^1 \times L, ([0], y_0)) \) is abelian, it follows that \( \xi_i \cdot \xi_j = \xi_j \cdot \xi_i \). Therefore \( b = 0 \). This completes the proof of Lemma 16.2 and Theorem 6.

17. The proof of Theorem 1. Let \( \mathcal{F} \in \mathcal{T}_+(\Phi, \mathcal{F}(h, \mathcal{F}; \sigma)) \), where \( \Phi, \mathcal{F} \) and \( \sigma \) be in \( \mathcal{S} \). We use the notations in \( \mathcal{S} \). For each vertex \( v \in V(\Phi) \), consider \( \mathcal{E} | E[v] \) and regard it as an element of \( \mathcal{T}_+(\Phi, \mathcal{F}(h, \mathcal{F}; \sigma) | E[v]) \). For each \( C \in \Gamma[v] = \Gamma(h(v)) \), define \( (a(C), b(C); r(C)) \) by using \( \mathcal{E} | E[v] \) as in \( \mathcal{S} \). Then we have a map \( (a, b; r): \Gamma[\Phi] \rightarrow (N \times \mathbb{Z})^{* \times \mathbb{Z}} \). We are going to show that \( (a, b; r) \) is an arithmetic model transverse to \( \mathcal{F}(h, \mathcal{F}; \sigma) \).

The transverse orientability of \( \mathcal{E} | E[v] \) for each \( v \in V(\Phi) \) implies that \( \text{Image } r \in 2\mathbb{Z} \). The condition \( \text{A4} \) in Definition 1.2 holds by Proposition 4.3. The condition \( \text{A1} \) follows from \( \text{A4} \).

When \( \mathcal{E} | E[v] \) is not \( C^0 \) isomorphic to \( \mathcal{E} | C \times I \) for any \( C \in \Gamma[v] \), we can attach to \( \mathcal{E} | E[v] \) a regular TS diagram with \( (a, b; r) | \Gamma[v] \) by Theorem 4. Then the conditions \( \text{A2} \) and \( \text{A3} \) are guaranteed by
Theorems 5 and 6, respectively. When $\mathcal{E}|\Gamma[v]$ is $C^0$ isomorphic to $(\mathcal{E}|C) \times I$ for some $C \in \Gamma[v]$, we have $h(v) = 2$. Let $\Gamma[v] = \{C, C'\}$. Then we see the following.

(i) If $(a(C), b(C)) = (\infty, \infty)$, then $(a(C'), b(C')) = (\infty, \infty)$ and $r(C) = r(C') = 0$.

(ii) If $(a(C), b(C)) = (0, 1)$, then $(a(C'), b(C')) = (0, 1)$ and $r(C') = -r(C)$.

(iii) If $(a(C), b(C)) \notin (N \times Z)_{\text{coprime}}$, then $(a(C'), b(C')) = (a(C), -b(C))$ and $r(C') = -r(C)$.

Therefore the condition (A2) holds.

Finally we check the condition (A5), as follows. Let $s \in S(\Phi)$ with $\partial(s) = (v_1) - (v_2)$, $v_1, v_2 \in V(\Phi)$ and $C_i = C[v_i](s)$. Since $(\mathcal{V}^*[s])*(\mathcal{E}|C_i) = \mathcal{E}|C_i$, we see that if $a(C_i) = \infty$ then $a(C_2) = \infty$. When $a(C_i) \neq \infty$, the foliation $\mathcal{E}|C_i$ has a compact leaf $L$ and we have

$$(\mathcal{V}^*[s])_*[L] = (ka(C_i) + lb(C_i))[S^i \times \{\ast\}] + (ma(C_i) + nb(C_i))[\{\ast\} \times \tilde{C}_i].$$

Then it follows that

$$\begin{pmatrix} a(C_2) \\ b(C_2) \end{pmatrix} = \gamma_1 \begin{pmatrix} k & l \\ m & n \end{pmatrix} \begin{pmatrix} a(C_1) \\ b(C_1) \end{pmatrix} \quad \text{and} \quad r(C_2) = \gamma_2 r(C_1),$$

where $\gamma_1$ and $\gamma_2$ are as in Definition 1.2. This completes the proof of Theorem 1.

18. Some remarks on arithmetic models. In this section, we investigate the properties of arithmetic models. Let $\Phi, \mathcal{V}$ and $\sigma$ be as in §1. First we obtain some informations on a side $s \in S(\Phi)$ such that $h(v_1) > 2$ and $h(v_2) > 2$, where $\partial(s) = (v_1) - (v_2)$, $v_1, v_2 \in V(\Phi)$. The following is useful.

**Definition 18.1.**

1. A side $s \in S(\Phi)$ is called longitude preserving if $\mathcal{V}(s) = \begin{pmatrix} 1 & l \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & l \\ 0 & 1 \end{pmatrix}$ for some $l \in \mathbb{Z}$, and otherwise longitude twisting.

2. Let $s \in S(\Phi)$ with $\partial(s) = (v_1) - (v_2)$ and $\mathcal{V}(s) = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$. For $s$ with $k = 1$ or $-1$, let

$$\xi(s) = k \cdot \sigma(C[v_1](s)) \cdot \sigma(C[v_2](s)).$$

We call $\xi(s)$ the glueing sign of $s$.

Now we have the following.

**Proposition 18.2.** Let $\mathcal{X} = (a, b; r) \in am(\Phi, \mathcal{V}; \sigma)$. Let $s \in S(\Phi)$ with $\partial(s) = (v_1) - (v_2)$ and $C_i = C[v_i](s)$, $j = 1, 2$. Suppose that $h(v_1) > 2$ and $h(v_2) > 2$. 
(1) If $s$ is longitude twisting, then $r(C_1) = r(C_2) = 0$.

(2) If $s$ is longitude preserving, then $\mathcal{A}'$ obtained from $\mathcal{A}$ by changing $(a(C_1), b(C_1))$ and $(a(C_2), b(C_2))$ for $(1, 0)$ is also an arithmetic model transverse to $\mathcal{T}(\Phi, \mathcal{F}; \sigma)$.

**Proof.** (1) Suppose that $r(C_1) \neq 0$. Then $r(C_2) \neq 0$ by (A5) in Definition 1.3. By (A3), it follows that $(a(C_j), b(C_j)) = (1, 0)$ for $j = 1, 2$. Let $\mathcal{F}(s) = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$. By (A5), we have $\begin{pmatrix} k & l \\ m & n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{sgn}(k) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore $m = 0$ and $k = -n = 1$ or $-1$.

(2) When $r(C_1) \neq 0$, we have $r(C_2) \neq 0$ and $\mathcal{A}' = \mathcal{A}$ by (A5) and (A3). Suppose that $r(C_1) = 0$. Then $r(C_2) = 0$ by (A5). It is easy to see that $\mathcal{A}' \in \text{am}(\Phi, \mathcal{F}; \sigma)$. This completes the proof of Proposition 18.2.

By Proposition 18.2, we have simpler equations to determine whether $\text{am}(\Phi, \mathcal{F}; \sigma)$ is empty or not in the case $h(v) > 2$ for all $v \in V(\Phi)$, as follows. We omit the proof.

**Theorem 7.** Suppose that $h(v) > 2$ for all $v \in V(\Phi)$. Then $\text{am}(\Phi, \mathcal{F}; \sigma) \neq \emptyset$ if and only if there is a map $r: I[\Phi] \to 2\mathbb{Z}$ satisfying the following conditions.

(A3)' If $s$ is longitude twisting, then $r(C[v_1](s)) = r(C[v_2](s)) = 0$, where $\partial(s) = (v_1) - (v_2)$.

(A4)' $\sum_{v \in r[v]} r(C) = 4 - 2h(v)$ for all $v \in V(\Phi)$.

(A5)' If $s$ is longitude preserving, then $r(C[v_1](s)) = \xi(s) r(C[v_2](s))$, where $\partial(s) = (v_1) - (v_2)$.

For $v \in V(\Phi)$ with $h(v) = 1$, we have the following.

**Proposition 18.3.** Let $s \in S(\Phi)$ with $\partial(s) = (v_1) - (v_2)$. If $h(v_1) > 2$, $h(v_2) = 1$ and $\text{am}(\Phi, \mathcal{F}; \sigma) \neq \emptyset$, then $\mathcal{F}(s) = \begin{pmatrix} 1 & l \\ m & n \end{pmatrix}$ or $\begin{pmatrix} -1 & l \\ m & n \end{pmatrix}$ for some $l, m, n \in \mathbb{Z}$.

**Proof.** Let $(a, b; r) \in \text{am}(\Phi, \mathcal{F}; \sigma)$. By (A1) and (A4), we have $a(C_2) = 1$ and $r(C_2) = 2$. Then $r(C_1) \neq 0$ by (A5) and $(a(C_1), b(C_1)) = (1, 0)$ by (A3). Let $\mathcal{F}(s) = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$. Since $\begin{pmatrix} k & l \\ m & n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{sgn}(k) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, it follows that $k = 1$ or $-1$ and $m = kb(C_2)$.

When the graph $\Phi$ is a tree, we have the following.

**Proposition 18.4.** Suppose that $\Phi$ is a tree and that $h(v) \neq 2$ for all $v \in V(\Phi)$. If $(a_1, b_1; r_1), (a_2, b_2; r_2) \in \text{am}(\Phi, \mathcal{F}; \sigma)$, then $r_1 = r_2$.

In order to prove Proposition 18.4, we need the following lemma. The proof is easy and we omit it.
**Lemma 18.5.** Suppose that $\Phi$ is a tree. Then there exists a sequence $\Gamma_0, \ldots, \Gamma_\rho$, $\rho = \# S(\Phi)$, of subsets of $\Gamma[\Phi]$ satisfying the following conditions.

1. $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_\rho = \Gamma[\Phi]$.
2. For each $j = 1, \ldots, \rho$, there is $s_j \in S(\Phi)$ with $\Gamma_j - \Gamma_{j-1} = \{C[v_j](s_j), C[v'_j](s_j)\}$, where $\partial(s_j) = \pm((v_j) - (v'_j))$.
3. For each $j = 1, \ldots, \rho - 1$, define a subgraph $\Psi_j$ of $\Phi$ by

$$V(\Psi_j) = \{v \in V(\Phi) \mid \Gamma[v] \not\subset \Gamma_j\} \quad \text{and} \quad S(\Psi_j) = S(\Phi) - \{s_1, \ldots, s_j\},$$

where $V(\Psi_j)$ (or $S(\Psi_j)$) is the set of vertices (or sides) of $\Psi_j$. Then $\Psi_j$ is a tree, and $\#(\Gamma[v_j] - \Gamma_{j-1}) = 1$.

**Proof of Proposition 18.4.** We use the sequence $\Gamma_0, \ldots, \Gamma_\rho$ in Lemma 18.5. Let $C_j = C[v_j](s_j)$ and $C'_j = C[v'_j](s_j)$. We prove $r_1(C) = r_2(C)$ for $C \in \Gamma_j$, by induction on $j$. First we see that $h(v_1) = 1$. Then we have $a_1(C_1) = a_2(C_1) = 1$ and $r_1(C_1) = r_2(C_1) = 2$. Since $r_1(C'_1) \neq 0$ and $r_2(C'_1) \neq 0$ by (A5), it follows that $(a_1(C'_1), b_1(C'_1)) = (a_2(C'_1), b_2(C'_1)) = (1, 0)$ by (A3). Then we have $b_1(C_1) = b_2(C_1)$ by (A5). Using (A5) once more, we get $r_1(C'_1) = r_2(C'_1)$. Therefore $(1)$ holds.

Now suppose $(1)$. Since $\#(\Gamma[v_{j+1}] - \Gamma_j) = 1$, we see that $a_1(C_{j+1})r_1(C_{j+1}) = a_2(C_{j+1})r_2(C_{j+1})$ by (A4) and the fact that $a_1(C) = a_2(C) = 1$ if $r_1(C) \neq 0$ for $C \in \Gamma_j$. When $h(v_{j+1}) = 1$, we verify $(1 + 1)$ as above. Suppose that $h(v_{j+1}) > 2$. If $a_1(C_{j+1})r_1(C_{j+1}) = 0$, we have $r_1(C_{j+1}) = r_2(C_{j+1}) = 0$ by Lemma 15.1 (2). By (A5), it follows that $r_1(C_{j+1}) = r_2(C_{j+1}) = 0$. Now consider the case $a_1(C_{j+1})r_1(C_{j+1}) \neq 0$. Since $r_1(C_{j+1}) \neq 0$ and $r_2(C_{j+1}) \neq 0$, it follows that $r_1(C'_{j+1}) \neq 0$ and $r_2(C'_{j+1}) \neq 0$ by (A5). Then we have $(a_1(C), b_1(C)) = (a_2(C), b_2(C)) = (1, 0)$ for $C = C_{j+1}, C'_{j+1}$ by (A3). Therefore $r_1(C) = r_2(C)$ for $C = C_{j+1}, C'_{j+1}$. Thus $(1 + 1)$ holds. This completes the proof of Proposition 18.4.

**19. Some application of the arithmetic criterion.** The purpose of this section is to determine whether $am(\Phi, \Psi; \sigma)$ is empty or not for some $\mathcal{F}(\Phi, \Psi; \sigma)$. First consider the graphs in Figure 19.1.

**Proposition 19.1.** Suppose that $V(\Phi) = \{v\}$ and $\# S(\Phi) > 1$. (See Figure 19.1 (a).) Then $am(\Phi, \Psi; \sigma) \neq \emptyset$ if and only if there is a longitude preserving side $s \in S(\Phi)$ with $\xi(s) = 1$.

**Proof.** Suppose that $am(\Phi, \Psi; \sigma) \neq \emptyset$. By Proposition 18.1, we have an arithmetic model $(a, b; r)$ transverse to $\mathcal{F}(\Phi, \Psi; \sigma)$ satisfying the following.
(i) If \( s \in S(\Phi) \) is longitude twisting, then \( r(C[v](s^+)) = r(C[v](s^-)) = 0 \).

(ii) If \( s \in S(\Phi) \) is longitude preserving, then \((a(C[v](s^+)), b(C[v](s^+))) = (a(C[v](s^-)), b(C[v](s^-))) = (1, 0)\).

Since \( r(C[v](s^-)) = \xi(s) r(C[v](s^+)) \) for a longitude preserving side \( s \in S(\Phi) \), it follows that

\[
\sum_{C \in \Gamma} a(C) r(C) = 2 \sum_{C \in \Gamma} r(C) = 4 - 2h(v),
\]

where \( \Gamma = \{C[v](s^+) \mid s \in S(\Phi) \text{ is longitude preserving and } \xi(s) = 1\} \). Since \( 4 - 2h(v) = 4(1 - \# S(\Phi)) \neq 0 \), it follows that \( \Gamma \neq \emptyset \). Therefore there is a longitude preserving side \( s \in S(\Phi) \) with \( \xi(s) = 1 \).

Conversely suppose that there is a longitude preserving side \( s \in S(\Phi) \) with \( \xi(s) = 1 \). Let \( r(C) = 0 \) for \( C \in \Gamma[v] - \{C[v](s^+), C[v](s^-)\} \), and \( r(C[v](s^+)) = r(C[v](s^-)) = 2 - h(v) \). Then \( \rho : \Gamma[v] \to 2\mathbb{Z} \) satisfies (A3)', (A4)', and (A5)' in Theorem 7. By Theorem 7, it follows that \( \text{am}(\Phi, \Psi; \sigma) \neq \emptyset \). This completes the proof of Proposition 19.1.

**Proposition 19.2.** Suppose that \( V(\Phi) = \{v_0, \ldots, v_\mu\}, \mu > 2 \), and that \( S(\Phi) = \{s_1, \ldots, s_\mu\} \) and \( \partial(s_j) = (v_0) - (v_j) \) for all \( j \). (See Figure 19.1 (b).)

Then \( \text{am}(\Phi, \Psi; \sigma) \neq \emptyset \) if and only if \( \Psi[s_j] = \begin{pmatrix} 1 & l \\ m & n \end{pmatrix} \) or \( \begin{pmatrix} -1 & l \\ m & n \end{pmatrix} \) for all \( j \) and \( \# \{s \in S(\Phi) \mid \xi(s) = 1\} = 1 \).

**Proof.** Suppose that \( \text{am}(\Phi, \Psi; \sigma) \neq \emptyset \) and let \( (a, b; r) \) be an arithmetic model. By the proof of Proposition 18.2, we have

(i) \( \Psi[s_j] = \begin{pmatrix} 1 & l \\ m & n \end{pmatrix} \) or \( \begin{pmatrix} -1 & l \\ m & n \end{pmatrix} \) for all \( j \),

(ii) \( a(C) = 1 \) for all \( C \in \Gamma[v] \),

(iii) \( r(C) = 2 \) for all \( C \in \Gamma[v] \cup \cdots \cup \Gamma[v_\mu] \).

Therefore we see that \( r(C[v_j](s_j)) = 2\xi(s_j) \). Since \( 2(\xi(s_1) + \cdots + \xi(s_\mu)) = 4 - 2h(v_0) = 2(2 - \mu) \) and \( \xi(s_j) = 1 \) or \( -1 \), it follows that \( \# \{s \in S(\Phi) \mid \xi(s) = 1\} = 1 \).

Conversely suppose that \( \Psi[s_j] = \begin{pmatrix} 1 & l \\ m & n \end{pmatrix} \) or \( \begin{pmatrix} -1 & l \\ m & n \end{pmatrix} \) for all \( j \) and \( \# \{s \in S(\Phi) \mid \xi(s) = 1\} = 1 \). We may suppose that \( \xi(s_1) = 1 \) and \( \xi(s_\mu) = \cdots = \)
\[ \xi(s_j) = -1. \] Let \( r(C[v_i](s_i)) = 2, r(C[v_i](s_j)) = -2 \) for \( j = 2, \ldots, \mu, \) and \( r(C[v_i](s_j)) = -2 \) for \( j = 1, \ldots, \mu. \) Let \( a(C) = 1 \) for all \( C \in \Gamma[\varphi] \) and \( b(C[v_i](s_j)) = 0 \) for all \( j. \) Determining \( b(C[v_i](s_j)) \) by \((A5)\) for all \( j, \) we have an arithmetic model \((a, b; r)\) transverse to \( \mathcal{F}(\varphi; \mathcal{V}; \varrho) \). This completes the proof of Proposition 19.2.

**REMARK 19.3.** Consider the graphs \( \varphi \) in Proposition 19.2. Let \( \mathcal{V}[s] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( \mathcal{V}[s_j] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) for \( j = 2, \ldots, \mu. \) Then \( M(\varphi, \mathcal{V}) \) is diffeomorphic to \( S^3. \) By Proposition 19.2, it follows that \( \text{am}(\varphi, \mathcal{V}; \varrho) = 0 \) for an arbitrary \( \varrho. \) This implies that \( t(C(\varphi, \mathcal{V}; \varrho)) = 0 \) by Theorem 1. On the other hand, the foliation \( \mathcal{F}(\varphi, \mathcal{V}; \varrho) \) admits a transverse 2-plane field as proved in Tamura-Sato [15]. Since \((A1)\) does not depend on the integrability of transverse foliations, this means that \((A4)\) reflects the integrability.

Hereafter we consider \( \mathcal{F}(\varphi, \varphi; \varrho) \) such that \( h(v) > 2 \) for all \( v \in V(\varphi). \) Let \( \varphi' \) be the subgraph of \( \varphi \) such that \( V(\varphi') = V(\varphi) \) and \( S(\varphi') = \{ s \in S(\varphi) | s \text{ is longitude preserving} \}. \) The following is the direct consequence of Theorem 7 and we omit the proof.

**PROPOSITION 19.4.** If \( \varphi' \) has an isolated vertex, then \( \text{am}(\varphi, \mathcal{V}; \varrho) = 0. \)

Suppose that \( \varphi' \) is a tree. Take a vertex \( v_0 \in V(\varphi) \) and fix it. For each \( v \in V(\varphi) - \{ v_0 \}, \) there are a unique sequence \( S(v, v_0) = (s_i, \ldots, s_{l(v)}) \) in \( S(\varphi') \) and a unique sequence \( V(v, v_0) = (v_1 = v, v_2, \ldots, v_{l(v)} = v_0) \) in \( V(\varphi) \) such that \( \partial(s_j) = \pm ((v_j) - (v_{j+1})) \) for \( j = 1, \ldots, l(v). \) Let \( \xi(v, v_0) = (-1)^{l(v)} \xi(s_1) \cdots \xi(s_{l(v)}) \) and \( \xi(v_0, v_0) = 1. \) Then we have the following.

**PROPOSITION 19.5.** Suppose that \( \varphi' \) is a tree. Then \( \text{am}(\varphi, \mathcal{V}; \varrho) \neq 0 \) if and only if \( \sum_{v \in V(\varphi)} \xi(v, v_0)(4 - 2h(v)) = 0 \) for some \( v_0 \in V(\varphi). \)

**PROOF.** Suppose that \( \text{am}(\varphi, \mathcal{V}; \varrho) \neq 0. \) Then there is a map \( r: \Gamma[\varphi] \to 2\mathbb{Z} \) satisfying \((A3)', (A4)'\) and \((A5)'\) by Theorem 7. By Lemma 18.5, we have a sequence \( \varnothing = \Gamma_0, \Gamma_1, \ldots, \Gamma_\rho, \rho = \# S(\varphi'), \) satisfying the conditions corresponding to (1), (2) and (3) in Lemma 18.5. We use the notations in Lemma 18.5. Let \( C_j = C[v_j](s_j) \) and \( C'_j = C[v'_j](s_j). \) By induction on \( j, \) we prove

\[ [(j)] \]
\[ r(C_j) = 4 - 2h(v_j) - \sum_{v \in V(j)} \xi(v, v_j)(4 - 2h(v)) \]

where \( V(j) = \{ v \in V(\varphi) - \{ v_0 \} | S(v, v_j) \subseteq \{ s_i, \ldots, s_{j-1} \} \}. \)

Since \( V(1) = \varnothing \) and \( r(C) = 0 \) for \( C \in \Gamma[v_1] - \{ C_i \}, \) the condition \([(1)]\) follows from \((A4)'\). Now suppose that \([(i)]\) holds for \( i \leq j. \) For each \( C \in \Gamma[v_{j+1}], \) there is a unique side \( s \in S(\varphi) \) with \( C = C[v_{j+1}](s). \) If \( s \) is...
longitude twisting, then $r(C) = 0$ by $(A3)'$. If $s$ is longitude preserving, then there is $j(C) \in \{1, \ldots, j+1\}$ with $s = s_{j(C)}$. When $j(C) < j+1$, we have

$$r(C) = \xi(s_{j(C)})\left\{4 - 2h(v_s) - \sum_{v \in V_{j(C)}} \xi(v, v_{j(C)})(4 - 2h(v))\right\}.$$  

Since $V(j+1) = \bigcup \{V(j(C)) \cup \{v_{j(C)}\} \mid C \in \Gamma[j+1] \}$ and $j(C) < j+1$, the condition $[[j+1]]$ follows from $(A4)'$ and the above formula. This completes the induction.

Since $V(\Phi) = V(\rho) \cup \{v_{j(C)}\}$ and $r(C) = 0$ for $C \in \Gamma[v_{j(C)}] - \{C_{j(C)}\}$, we have

$$\sum_{v \in V(\Phi)} \xi(v, v_{j(C)})(4 - 2h(v)) = 0$$

by $[[\rho]]$ and $(A4)'$.

Conversely suppose that $\sum_{v \in V(\Phi)} \xi(v, v_{j(C)})(4 - 2h(v)) = 0$ for some $v_{j(C)} \in V(\Phi)$. We can take a sequence $\Gamma_0, \ldots, \Gamma_{\rho}$ as above and we may suppose that $v_{j(C)} = v_{j+1}$. For $s \in S(\Phi) - S(\Phi')$ with $s = (v) = (v')$, let $r(C[v])(s) = r(C[v']) = 0$. Using induction on $j$, define $r(C[v_j(s)])$ by the formula $[[j]]$. It is easy to check that $r: \Gamma[\Phi] \to 2\mathbb{Z}$ satisfies $(A3)'$, $(A4)'$ and $(A5)'$. This completes the proof of Proposition 19.5.

**Remark 19.6.** Suppose that $\Phi'$ is not connected but consists of trees containing at least two vertices. Then $am(\Phi, \Phi'; \sigma) \neq \emptyset$ if and only if the corresponding formula holds for each tree contained in $\Phi'$.

**20. TS models transverse to $\mathcal{F}(\Phi, \Phi'; \sigma)$**. Using TS diagrams, we describe a necessary and sufficient condition under which $t^*_1(\Phi, \Phi; \sigma)$ becomes non-empty in this and the next section.

First we must define a TS diagram for $G_{\tau_01}(F(\Phi; \sigma)) C_0$ isomorphic to $(G|C)_{\tau_12}[I]$ for some $C \in \Gamma(h)$. Note that the existence of such $G$ implies that $h = 2$ and that if $C_1$ is horizontal with respect to $G$ then $s(C_1) = -s(C_2)$.

**Definition 20.1.** A singular TS diagram of $(E(2); \sigma)$ is a quadruplet $F = (\{J(\lambda)\}_{\lambda \in \Lambda}, \{l(\lambda) : J(\lambda) \to S^1\}_{\lambda \in \Lambda}, \{s(\lambda) \circ F\}_{\lambda \in \Lambda}, (a, b; r) \in (N \times Z)^* \times 2Z)$ satisfying the following conditions.

(S1) $J(\lambda)$ is a copy of $I$ for all $\lambda \in \Lambda$.

(S2) If $\# \Lambda > 1$, then $\int: J(\lambda) \to S^1$ is an imbedding for each $\lambda \in \Lambda$.

Int $J_1 \cap \text{Int} \ast J_2 = \emptyset$ for $\lambda \neq \lambda'$, and $S^1$ is the closure of $\bigcup \{\ast J_\lambda \mid \lambda \in \Lambda\}$, where $\ast J(\lambda)$ means $J(\lambda)$ for an appropriate $\lambda \in \Lambda$. If $\# \Lambda = 1$, then $\int: J_1 \to S^1$ is an imbedding and $\ast J_1 = S^1$, where $\lambda = \{\lambda\}$.

(S3) $\# \{\lambda \in \Lambda \mid s(\lambda) = 0 \text{ or } \bullet\} < \infty$.

(S4) If $\sigma$ is constant, then $(a, b) \in (N \times Z)^\text{cyclic} \cup \{(\infty, \infty)\}$. If $(a, b) = \cdots$
(∞, ∞), then #A = 1 and r = # {λ ∈ A | s(J) = ∅ or •} = 0. If (a, b) ∈ (N × Z)_{prime} ∪ {(0, 1)}, then r = # {λ ∈ A | s(J) = ∅} − # {λ ∈ A | s(J) = •}.

We call a triad (J, J, s(J)), in a singular TS diagram, a singular TS piece.

We introduce an equivalence relation on the set of singular TS diagrams as follows.

**DEFINITION 20.2.** Let F = (1_{J {i ∈ A}}, 1_{l {i ∈ A}}, s(J)) = (1_{a, b; r}) and F' = (1_{J' {i ∈ M}}, 1_{l' {i ∈ M}}, s(J')) = (1_{a', b'; r'}). Then F and F' are isomorphic if (a, b; r) = (a', b'; r') and there are a homeomorphism φ: S1 → S1 and a bijection ρ: A → M such that φ(Ji) = Ji and s(Ji) = s(J) for all i ∈ A.

We denote by STS(E(2); σ) the set of isomorphism classes of singular TS diagrams of (E(2); σ).

**DEFINITION 20.3.** We call P a TS piece if P is a regular TS piece or a singular TS piece. We call F a TS diagram if F is a regular TS diagram or a singular TS diagram. Let TS(E(h); σ) = RTS(E(h); σ) if h = 2, and TS(E(2); σ) = RTS(E(2); σ) ∪ STS(E(2); σ).

As a generalization of Theorem 4, we have the following and we omit the proof.

**THEOREM 4**. There exists a canonical map

τ: t^*(Φ, τ; σ) → TS(E(h); σ).

Now let us consider F(Φ, τ; σ) as in §1. For each v ∈ V(Φ), we obtain a set TS[v] = TS(E[v]; σ | [v]). We describe the compatibility conditions for a family {}(F(v)) ∈ TS[v]_{v ∈ V(Φ)} to correspond to some F ∈ t^*(Φ, τ; σ) as follows.

**DEFINITION 20.4.** Let F = (F, φ, (a, b; r)) be a regular TS diagram of (E(h); σ). For each C ∈ Γ(h), the C boundary diagram d_C of F is a quadruplet (J {i ∈ M}, s(J) {i ∈ M}, (a(C), b(C); r(C))) satisfying the following conditions.

(B0) If a(C) = 0 or ∞, then L(C) = S^1 × {*}. If 0 < a(C) < ∞, then L(C) = C/ ~, where y ~ y' for y, y' ∈ C if and only if y' = φ_k(y) for some k ∈ Z. (Note that (φ_k | C)^a(C) = id.)

(B1) J_i is a copy of I for all i ∈ M.

(B2) If a(C) = 0 or ∞, then #A = 1 and τ_i | Int J_i is an imbedding and *J_i = L(C) for i ∈ M, where * = τ_i| Int J_i as before. If 0 < a(C) < ∞, then τ_i | Int J_i is an imbedding for i ∈ M and there is a surjection ξ: F(C) = {J | A ∈ A, J ∈ (X(A) − Y(A)), *J ⊂ C} → M such that *J_{τ_C(K)} =
\[\pi(\lambda)\] and that \(s(J_{\lambda}(\lambda)) = s(\lambda)\) if \(K \in \mathcal{A}\), for some \(\lambda\), and otherwise \(s(J_{\lambda}(\lambda)) = \|\), where \(\mathcal{A}\) contains \(\{P_1 = (J_{1, \omega} : s : J_{1} \rightarrow \mathcal{A}, \omega : \mathcal{A} \rightarrow \{1, -1\})\}_{\omega \in \mathcal{A}}\) and \(\pi : C \rightarrow L(C)\) is the projection.

**Definition 20.5.** Suppose that \(h = 2\) and let \(\mathcal{T} = (\{J_\lambda\}_{\lambda \in \Sigma}, \{\epsilon_\lambda\}_{\lambda \in \Sigma}, \{s(J_\lambda)\}_{\lambda \in \Sigma}, (a, b, r))\) be a singular TS diagram. For \(C \in \Gamma(2)\), the \(C\) boundary diagram \(\partial_C\mathcal{T}\) is defined as follows.

1. When \(\sigma\) is not constant, let \(\sigma(C^+) = 1\) and \(\sigma(C^-) = -1\), where \(\Gamma(2) = [C^+, C^-]\). Then \(\partial_{C^+}\mathcal{T}\) is equal to \(\mathcal{T}\), and \(\partial_{C^-}\mathcal{T}\) to \(\{\{J_\lambda\}_{\lambda \in \Sigma}, \{\epsilon_\lambda\}_{\lambda \in \Sigma}, \{s'(J_\lambda)\}_{\lambda \in \Sigma}, (a', b'; -r)\}\) such that
   - \(s'(J_\lambda) = \|, \bigcirc, \wedge, \bigvee, \|\) if \(s(J_\lambda) = \|, \bigcirc, \epsilon, \bigvee, \|\) respectively.
   - \((a', b') = (a, b)\) if \((a, b) = (0, 1)\) or \((\epsilon, \epsilon, \epsilon, \epsilon, \epsilon)\), and \((a', b') = (a, -b)\) if \((a, b) \in (N \times Z)^{\text{coprime}}\).

2. When \(\sigma\) is constant, we fix an order \(\prec\) on \(\Gamma(2)\) and let \(C \prec C', C, C' \in \Gamma(2)\). Then \(\partial_{C^+}\mathcal{T}\) is equal to \(\mathcal{T}\), and \(\partial_{C^-}\mathcal{T}\) to \(\{\{J'_\lambda\}_{\lambda \in \Sigma}, \{\epsilon'_\lambda\}_{\lambda \in \Sigma}, \{s(J'_\lambda)\}_{\lambda \in \Sigma}, (a', b'; r')\}\) such that
   - \(s'(J_\lambda) = \|, \bigcirc, \bigodot, \bigvee, \|\) if \(s(J_\lambda) = \|, \bigcirc, \epsilon, \bigvee, \|\) or \((a, b) = (0, 1)\) or \((\epsilon, \epsilon, \epsilon, \epsilon, \epsilon)\), and \((a', b'; r') = (a, -b); r')\) if \((a, b) \in (N \times Z)^{\text{coprime}}\).

**Definition 20.6.** Given a map \((a, b) : \Gamma \rightarrow (N \times Z)^{\ast}\), for each \(C \in \Gamma\) let \(v(C) = \left(\frac{-b(C)}{a(C)}\right)\) if \(0 < a(C) < \epsilon\), and \(v(C) = \left(\frac{1}{0}\right)\) if \(a(C) = \epsilon\).

**Definition 20.7.** A TS model transverse to \(\mathcal{T}(\Phi, \mathcal{T}; \varphi)\) is a family \(\{[v] : \mathcal{T} \rightarrow \mathcal{T}\}_{v \in \mathcal{T}(v)}\) satisfying the following conditions.

- Let \(s \in \mathcal{S}(\Phi)\) with \(\delta(s) = (v) - (v')\). Let \(C = C[v](s), C' = C[v'](s), \partial_C\mathcal{T}(v) = \{(J_\lambda)_{\lambda \in \Sigma}, \{\epsilon_\lambda\}_{\lambda \in \Sigma}, \{s(J_\lambda)\}_{\lambda \in \Sigma}, (a, b; r)\}\) and \(\partial_{C'}\mathcal{T}(v') = \{(J'_\lambda)_{\lambda \in \Sigma}, \{\epsilon'_\lambda\}_{\lambda \in \Sigma}, \{s(J'_\lambda)\}_{\lambda \in \Sigma}, (a', b'; r')\}\).

1. \((a, b; r)\) and \((a', b'; r')\) satisfies the condition corresponding to \((A5)\) in Definition 1.2.

2. There is a homeomorphism \(\phi[s] : L(C) \rightarrow L(C')\) such that
   - \(a(C) = \infty\), then \(\phi[s]\) is orientation preserving,
   - \(a(C) \neq \infty\) and \(\varepsilon > 0\) (or \(\varepsilon < 0\)), then \(\phi[s]\) is orientation preserving (or reversing), where \(\varepsilon = v(C') \cdot \mathcal{T}[s] \cdot v(C)\) (product as matrices).

3. Furthermore there is a bijection \(\rho : A \rightarrow M\) such that for each \(\lambda \in A,\)
   - \(\phi[s]([J_\lambda]) = [J'_{\rho(\lambda)}]\), and
   - \(s(J'_{\rho(\lambda)}) = \bigcirc, \bigodot, \bigvee, \|\) (or \(\bigodot, \bigcirc, \bigvee, \|\)) if \(s(J_\lambda) = \bigcirc, \bigodot, \bigvee, \|\) and \(\gamma > 0\) (or \(\gamma < 0\)), where \(\gamma\) is as in Definition 1.2.

We denote by \(\mathcal{TS}(\Phi, \mathcal{T}; \varphi)\) the set of TS models transverse to \(\mathcal{T}(\Phi, \mathcal{T}; \varphi)\).
21. The geometric criterion. We formulate the geometric criterion precisely. First we have the following. Since the proof is long and tedious, we omit it.

**Theorem 8.** There exists a canonical commutative diagram

\[
\begin{array}{ccc}
t^*(\mathcal{F}(\Phi, \Psi; \sigma)) & \xrightarrow{\xi} & \text{TS}(\Phi, \Psi; \sigma) \\
\alpha & & \alpha^* \\
& \downarrow a & \\
& \text{am}(\Phi, \Psi; \sigma) & 
\end{array}
\]

When a TS model \( \mathcal{M} \) contains an infinite number of TS pieces, the construction of a foliation \( G \) transverse to \( \mathcal{F}(\Phi, \Psi; \sigma) \) corresponding to \( \mathcal{M} \) has troubles concerning the differentiability of \( G \). In order to get a better formulation, we need the following.

**Definition 21.1.** A TS model \( \mathcal{M} \) is called *finite* if it contains at most a finite number of TS pieces.

**Definition 21.2.** A TS model \( \mathcal{M} = \{[\mathcal{F}(v)]_{v \in V(\Phi)}\} \) is called *irreducible* if the following conditions are satisfied.

1. For each \( v \in V(\Phi) \), a representative \( \mathcal{F}(v) \) contains no regular TS piece \( P = (\mathcal{J}, \nu, \sigma; \mathcal{J} \to \mathcal{K}; \omega; K \to \{1, -1\}) \) such that \( \nu = 1X \) and \( \omega \) is not constant.

2. Let \( \mathcal{P} \) be the set of TS pieces in fixed representatives \( \{\mathcal{F}(v)\}_{v \in V(\Phi)} \). On \( \mathcal{P} \), we consider an equivalence relation \( \sim \) determined by

\[
P \sim P' \text{ if there is } s \in S(\Phi) \text{ with } \partial(s) = (v) - (v')
\]

such that \( P \) (or \( P' \)) belongs to \( \mathcal{F}(v) \) (or \( \mathcal{F}(v') \)) and \( \phi[s] \) in Definition 20.7 maps \( \pi(J) \) to \( \pi'(J') \) for some \( J, J' \in \mathcal{J}(|P|) \) and \( J', J' \in \mathcal{J}(|P'|) \), where \( \pi \) (or \( \pi' \)) is the projection to \( L(C[v](s)) \) (or \( L(C[v'](s)) \)).

Then there is no equivalence class \( \mathcal{C} \) with respect to \( \sim \) such that if \( P \subset \mathcal{C} \) then the symbols attached to \( P \) are \( \vee, \wedge \) or \( \parallel \).

We denote by \( \text{ts}(\Phi, \Psi; \sigma) \) the set of finite irreducible TS models transverse to \( \mathcal{F}(\Phi, \Psi; \sigma) \).

Now we have the following.

**Theorem 8*.** (1) There exists a canonical commutative diagram

\[
\begin{array}{ccc}
t^*(\mathcal{F}(\Phi, \Psi; \sigma)) & \xrightarrow{\tau^*} & \text{ts}(\Phi, \Psi; \sigma) \\
& \alpha & \\
\downarrow a & \downarrow a^* & \\
\text{am}(\Phi, \Psi; \sigma) & 
\end{array}
\]

(2) $\tau^*$ is surjective.

The following follows directly from Theorem 8$^*$. 

**Theorem 8$^{**}$ (The geometric criterion).** $t^*_n(\mathcal{F}(\Phi, \Psi; \sigma)) \neq \emptyset$ if and only if $t_s(\Phi, \Psi; \sigma) \neq \emptyset$.

**Proof of Theorem 8$^*$.** Let $\alpha^* = \overline{\alpha}|t_s(\Phi, \Psi; \sigma)$. We define $\tau^*$ as follows. Let $E \in t^*_n(\mathcal{F}(\Phi, \Psi; \sigma))$ and $M = \tau(E)$. Using the theorem of Kopell [7] as in Nishimori [9], we can show that if $M$ is irreducible then $M$ is finite, since $M \subseteq \text{Image} \overline{\tau}$. In this case, let $\tau^*(E) = M$. Suppose that $M$ is not irreducible. Then $E$ contains foliated $I$-bundles corresponding to regular TS pieces of type IX or to equivalence classes of $P$ in Definition 21.2 (2). Collapsing such foliated $I$-bundles along fibers, we obtain a foliation $\mathcal{G}'$ such that $\overline{\tau}(\mathcal{G}')$ is irreducible. Then $\overline{\tau}(\mathcal{G}')$ is finite as above. Let $\tau^*(E) = \overline{\tau}(\mathcal{G}')$. Since $\alpha(\mathcal{G}') = \alpha(E)$, we have $\alpha^* \circ \tau^* = \alpha$.

(2) Let $M = \{[F(v)]\}_{v \in \mathcal{F}(\Phi; \sigma) \in t_s(\Phi, \Psi; \sigma)}$. For each regular TS piece $P$ of $\mathcal{F}(v)$, we take an appropriate component of the same type as $P$ in Theorem 3 if $\mathcal{F}(v)$ is regular. When $\mathcal{F}(v)$ is singular, take $C^+$ in Definition 20.5 (1) or $C$ in Definition 20.5 (2) and denote it by $C$. We construct a foliation $\mathcal{G}$ of $S^1 \times S^1$ such that

(i) $\mathcal{G} \mid *J_j \times S^1$ is a Reeb component such that the connected components of $\partial(\mathcal{G}) \times S^1$ have expanding holonomy in the same (or opposite) direction as the orientation of $S^1$ if $s(J_j) = \emptyset$ (or $\bullet$),

(ii) $\mathcal{G} \mid *J_j \times S^1$ is a slope component such that the connected components of $\partial(\mathcal{G}) \times S^1$ have expanding holonomy in the same (or opposite) direction as the orientation of $\partial(\mathcal{G}) \times S^1$ as the boundary of $*J_j \times S^1$ if $s(J_j) = \emptyset$ (or $\wedge$),

(iii) $\mathcal{G} \mid *J_j \times S^1$ consists of leaves $\{x\} \times S^1$ for $x \in *J_j$ if $s(J_j) = \emptyset$, where $\mathcal{F}(v) = ([J_j], \{t_j\}, \{s(J_j)\}, (a, b; r))$. Now take a foliation $\mathcal{G}(v)$ of $E[v]$ with $\mathcal{G}(v)|C = \mathcal{G}$ such that $\mathcal{G}(v)$ is $C^\infty$ isomorphic to $\mathcal{G} \times I$.

Since $M$ satisfies the condition in Definition 20.7 and $M$ is finite, we have a $C^\infty$ foliation $\mathcal{G}$ of $M(\Phi, \Psi)$ transverse to $\mathcal{F}(\Phi, \Psi; \sigma)$ with $\tau^*(\mathcal{G}) = M$. We omit the details. This completes the proof of Theorem 8$^*$.

The proof of Theorem 8$^*$ implies the following.

**Theorem 9.** (1) For each $M \in \text{TS}(\Phi, \Psi; \sigma)$, there is cannonically a $C^0$ foliation of $M(\Phi, \Psi)$ transverse to $\mathcal{F}(\Phi, \Psi; \sigma)$.

(2) If $\text{TS}(\Phi, \Psi; \sigma) \neq \emptyset$, there is a $C^\infty$ 2-plane field of $M(\Phi, \Psi)$ transverse to $\mathcal{F}(\Phi, \Psi; \sigma)$. 
PROOF. (1) is clear. As for (2) there is a transverse $C^0$ foliation $\mathcal{F}$ by (1). It suffices to take a $C^\infty$ approximation of $T\mathcal{F}$.

22. Some applications of the geometric criterion. We treat $\mathcal{F}(\Phi, \Psi; \sigma)$ considered already in §19. For such $\mathcal{F}(\Phi, \Psi; \sigma)$, we obtained a necessary and sufficient condition under which $\text{am}(\Phi, \Psi; \sigma) \neq \emptyset$ by Propositions 19.1, 19.2, 19.5 and Remark 19.6. First consider $\mathcal{F}(\Phi, \Psi; \sigma)$ such that $\Phi$ is a graph in Figure 19.1. We show that for such $\mathcal{F}(\Phi, \Psi; \sigma)$ the arithmetic criterion is complete. Precisely we have the following.

THEOREM 10. Let $\mathcal{F}(\Phi, \Psi; \sigma)$ be as in §1. Suppose that

(a) $V(\Phi) = \{v\}$ and $\# S(\Phi) > 1$, or

(b) $V(\Phi) = \{v_0, \ldots, v_\mu\}$, $\mu > 2$, $S(\Phi) = \{s_1, \ldots, s_\mu\}$ and $\partial(s_j) = (v_0) - (v_j)$

for all $j$.

Then the following conditions are equivalent.

(1) $t^*(\mathcal{F}(\Phi, \Psi; \sigma)) \neq \emptyset$.
(2) $ts(\Phi, \Psi; \sigma) \neq \emptyset$.
(3) $\text{am}(\Phi, \Psi; \sigma) \neq \emptyset$.

PROOF. Note that (1)$\Rightarrow$(2) and (2)$\Rightarrow$(3) are already known for general $\mathcal{F}(\Phi, \Psi; \sigma)$'s by Theorems 8** and 1*. Therefore it is sufficient to prove that (3) implies (2). Suppose that $\text{am}(\Phi, \Psi; \sigma) \neq \emptyset$.

Case (a). By Proposition 19.1, there is a longitude preserving side $s \in S(\Phi)$ with $\xi(s) = 1$. Furthermore we have an arithmetic model $(a, b; r)$: $\Gamma[\Phi] = \Gamma[v] \to (N \times \mathbb{Z})^* \times 2\mathbb{Z}$ such that

(i) $\langle a(C_1), b(C_1); r(C_1) \rangle = \langle a(C_2), b(C_2); r(C_2) \rangle = (1, 0; 2 - \mu)$,

(ii) $r(C) = 0$ for $C \in \Gamma[v] \setminus \{C_1, C_2\}$,

where $C_1 = C[v](s^+)$ and $C_2 = C[v](s^-)$. Let $\Gamma[v] \setminus \{C_1, C_2\} = \{C_3, \ldots, C_\mu\}$, where $\mu = 2 \cdot \# S(\Phi)$. Now we find a regular TS diagram $\mathcal{T}$ of $(\hat{E}(\mu); \sigma)$ indicated by Figure 22.1.

The right and left vertical segments are to be glued

Figure 22.1
Then it is easy to check that \([\mathcal{T}] \in \text{ts}(\Phi, \Psi; \sigma)\) and \(\alpha^*(\mathcal{T}) = (a, b; r)\).

Case (b). By Proposition 19.2, there is \(s_j \in S(\Phi)\) with \(\xi(s_j) = 1\), and \(\xi(s) = -1\) for all \(s \in S(\Phi) - \{s_j\}\). We may suppose that \(s_j = s_1\). Furthermore we have an arithmetic model \((a, b; r); \Gamma[\Phi] \to (N \times \mathbb{Z})^* \times \mathbb{Z}\) such that

\[
\begin{align*}
(i) & \quad (a(C), b(C)) = (1, 0) \text{ for all } C \in \Gamma[\Phi], \\
(ii) & \quad r(C_j) = 2 \text{ and } r(C'_j) = 2 \xi(s_j) \text{ for } j > 0,
\end{align*}
\]

where \(C_j = C[v_0](s_j)\) and \(C'_j = C[v_j](s_j)\). Now we find regular TS diagrams \(\mathcal{T}_0, \cdots, \mathcal{T}_p\) indicated by Figure 22.2 in the case \(\mu = 4\).

**Figure 22.2**

Then it is easy to check that \(\mathcal{M} = \{[\mathcal{T}_i]\}_{i=1}^p\) is a finite TS model and \(\alpha^*(\mathcal{M}) = (a, b; r)\). We have \(\mathcal{M}' \in \text{ts}(\Phi, \Psi; \sigma)\) from \(\mathcal{M}\) by reduction as in the proof of Theorem 8*. This completes the proof of Theorem 10.

Consider \(\mathcal{T}(\Phi, \Psi; \sigma)\) such that \(\Phi\) is a graph in Figure 22.3, where

**Figure 22.3**
longitude preserving (or twisting) sides are represented by solid (or dotted) lines.

The arithmetic criterion is complete in this case, too, and we have the following.

**Theorem 11.** Let $\mathcal{F}(\Phi, \Psi; \sigma)$ be as in §1. Suppose that

1. $V(\Phi) = \{v_0, \ldots, v_n\}$, $h(v_0) = \mu > 1$, $h(v_j) = 3$ for $j > 0$,
2. $S(\Phi) = \{s_1, \ldots, s_n, s'_1, \ldots, s'_n\}$,

where $s_j$ (or $s'_j$) is longitude preserving (or twisting). Then $t_*(\mathcal{F}(\Phi, \Psi; \sigma)) \neq \emptyset$ if and only if $\text{am}(\Phi, \Psi; \sigma) \neq 0$.

**Proof.** As above it suffices to prove that $\text{am}(\Phi, \Psi; \sigma) \neq 0$ implies $t_*(\mathcal{F}(\Phi, \Psi; \sigma)) \neq \emptyset$. Suppose that $\text{am}(\Phi, \Psi; \sigma) \neq 0$. By Proposition 19.5, we have

$$\sum_{j=0}^n \xi(v_j, v_0)(4 - 2h(v_j)) = 2(2 - \mu + \sum_{j=1}^n \xi(s_j)) = 0.$$ 

Therefore $\xi(s_{j*}) = -1$ for some $j*$ and $\xi(s_j) = 1$ for all $j \in \{1, \ldots, \mu\} - \{j^*\}$. We may suppose that $j^* = 1$. Now we find regular TS diagrams $\mathcal{I}_0', \ldots, \mathcal{I}_{n'}$, where $\mathcal{I}_0'$ equals $\mathcal{I}_0$ in Case (b) in the proof of Theorem 9, and $\mathcal{I}_1', \ldots, \mathcal{I}_{n'}$ are indicated by Figure 22.4.

Then we see that $t_*(\mathcal{F}(\Phi, \Psi; \sigma)) \neq \emptyset$ as above. This completes the proof of Theorem 11.

**Remark 22.2.** Let $\mathcal{F}(\Phi, \Psi; \sigma)$ satisfy the condition of Theorem 9
(b) or Theorem 10. Then \( t_0^*(\mathcal{F}(\Phi, \Psi; \sigma)) = \emptyset \) if and only if \( \text{am}(\Phi, \Psi; \sigma) \neq \emptyset \), since \( t_0^*(\mathcal{F}(\Phi, \Psi; \sigma)) = t^*(\mathcal{F}(\Phi, \Psi; \sigma)) \) in the case Theorem 9 (b). Consider the case of Theorem 10. In order to obtain transversely orientable foliation \( \mathcal{F} \) transverse to \( \mathcal{F}(\Phi, \Psi; \sigma) \), it suffices to insert, for each \( j > 0 \), a regular TS piece \( P = (\Phi, \Psi, s: F \to \mathcal{F}, \omega: H \to \{-1, 1\}) \), such that \( \nu = IX \) and \( \omega \) is constant, between the TS pieces of type VI and VIII in \( \mathcal{F}_j \) if necessary.

23. A construction of regular TS diagrams of \((\hat{E}(h); \sigma)\) with given \((a, b; r)\) in the case \( h > 2 \). The purpose of this section is to make preparations for the proof of Theorem 2. We prove the following.

**Theorem 12.** Let \( \mathcal{F}(h; \sigma) \) be as in §1 and suppose that \( h > 2 \). Let \( (a, b; r): \Gamma(h) \to (N \times \mathbb{Z})^* \times 2\mathbb{Z} \) be a map such that

(i) if \( r(C) = 0 \) for \( C \in \Gamma(h) \), then \( (a(C), b(C)) = (1, 0) \).

(ii) \( \sum_{C \in \Gamma(h)} a(C)r(C) = 4 - 2h \).

Then there is a regular TS diagram \( \mathcal{F} = (\hat{\mathcal{F}}, \{\phi_i\}_{i \in I}, (a', b'; r')) \) of \((\hat{E}(h); \sigma)\) with \((a', b'; r') = (a, b; r)\).

**Remark 23.1.** When \( h = 1 \) or 2, we obtain results similar to Theorem 12 more easily.

**Proof of Theorem 12.** Denote by \( \Gamma^+ \) (or \( \Gamma^- \), \( \Gamma^0 \)) the set of \( C \in \Gamma(h) \) with \( r(C) > 0 \) (or \( < 0 \), \( = 0 \)), and let \( \Gamma^+ = \{C^+_1, \ldots, C^+_k(+)\}, \Gamma^- = \{C^-_1, \ldots, C^-_k(-)\}, \) and \( \Gamma^0 = \{C^0_1, \ldots, C^0_\kappa(0)\} \). For each \( C \in \Gamma(h) \), take a set \( \Pi(C) \) of \( |r(C)| \) points of \( \hat{C} \). Let \( \Pi^+ = \bigcup \{\Pi(C) | C \in \Gamma^+\} \) and \( \Pi^- = \bigcup \{\Pi(C) | C \in \Gamma^-\} \). Number the elements of \( \Pi^+ \) in such a way that

\[
\pi^+\left(\sum_{j=1}^{k+1} |r(C^+_j)| + 1\right), \ldots, \pi^+\left(\sum_{j=1}^{k} |r(C^+_j)| \right) \in \Pi^+
\]

are on \( \hat{C}^+_k \) in the order opposite to the orientation of \( \hat{C}^+_k \) for \( k = 1, \ldots, k(+) \). Number the elements of \( \Pi^- \) in such a way that

\[
\pi^-(\sum_{j=1}^{k+1} |r(C^-_j)| + 1), \ldots, \pi^-(\sum_{j=1}^{k} |r(C^-_j)|) \in \Pi^-
\]

are on \( \hat{C}^-_k \) in the same order as the orientation of \( \hat{C}^-_k \) for \( k = 1, \ldots, k(-) \). Take an orientation preserving imbedding \( \iota: \hat{E}(h) \to \mathbb{R}^2 \) such that

1. \( \iota(\hat{C}^-_{k(-)}) = C_N(N, 0) \) for large \( N \),
2. \( \iota(\hat{C}^-_j) = C_i(9, -9) \) for \( j = 1, \ldots, k(-) - 1 \),
3. \( \iota(\hat{C}^-_j) = C_i(9, 9) \) for \( j = 1, \ldots, k(+) \),
4. \( \iota(\hat{C}^-_j) = C_i(9(j + k(+) + 1), 9) \) for \( j = 1, \ldots, k(0) \),

where \( C_\rho(x, y) \) is the circle of radius \( \rho \) with center \((x, y)\). Identifying
\( \hat{E}(h) \) and \( e(\hat{E}(h)) \), we regard \( \hat{E}(h) \) as a subspace of \( \mathbb{R}^2 \).

Since \( \sum_{C \in r} r(C) = \sum_{C \in r} a(C) r(C) = 4 - 2h < 0 \), we have \( k(-) > 0 \).

Since \( h - 1 \leq |4 - 2h| \) for \( h \geq 3 \) and \( k(0) \leq h - 1 \), it follows that \( k(0) + r^+ \leq r^- \), where \( r^+ = |II^+| \) and \( r^- = |II^-| \). Sliding the points of \( II^+ \) and \( II^- \) if necessary, we may take disjoint line segments \( L(1), \ldots, L(r^+) \) such that \( \partial L(j) = (\pi^+(j), \pi^-(j)) \). Furthermore we may take disjoint line segments \( L(r^+ + 1), \ldots, L(r^+ + k(0)) \) in such a way that an endpoint of \( L(r^+ + j) \) equals \( \pi^-(r^+ + j) \) and the other one belongs to \( \hat{C}_0 \). In addition to these, take line segments \( K(1), \ldots, K(\kappa_i) \) satisfying the following conditions (1)-(3).

1. An endpoint of \( k(j) \) belongs to \( \hat{C}_j^- \) and the other one belongs to \( \hat{C}_j^+ \), for some \( j' \) and \( j'' \).
2. The set \( B_i = \hat{C}_i^+ \cup \cdots \cup \hat{C}_{k(\kappa_i)}^+ \cup \hat{C}_{k^*}^- \cup \cdots \cup \hat{C}_{k^*}^- \cup L(1) \cup \cdots \cup L(r^+) \cup K(1) \cup \cdots \cup K(\kappa_i) \) is connected, where \( \hat{C}_{k^*}^- \) contains \( \pi^-(r^+) \).
3. For each \( j \in \{1, \ldots, \kappa_i\} \), the set \( B_i - K(j) \) is not connected.

Finally take line segments \( K(\kappa_1 + 1), \ldots, K(\kappa_i) \) satisfying the following conditions (4)-(6).

4. An endpoint of \( K(j) \) belongs to \( \hat{C}_j^- \) and the other one belongs to \( \hat{C}_{j' + 1}^- \), for some \( j' \).
5. The set \( B_2 = B_1 \cup \hat{C}_{k^* + 1}^- \cup \cdots \cup \hat{C}_{k^* - 1}^- \cup K(\kappa_1 + 1) \cup \cdots \cup K(\kappa_2) \) is connected.
6. For each \( j \in \{\kappa_1 + 1, \ldots, \kappa_2\} \), the set \( B_2 - K(j) \) is not connected.

(See Figure 23.1.)

Let \( H \), be the connected component of

\[
H = \hat{E}(h) - (L(1) \cup \cdots \cup L(r^+ + k(0)) \cup K(1) \cup \cdots \cup K(\kappa_i))
\]
containing the point \((-N,0)\) $\in \mathbb{R}^2$. We denote by $\overline{H}_0$ the compact manifold with corner obtained from $H_0$ by attaching the boundary. Then $\overline{H}_0$ is homeomorphic to $D^2$. Each other connected component $H_j$ of $H$ is surrounded by $\hat{C}_j', \hat{C}_j'', L(j^*)$ and $L(j^* + 1)$ for some $j', j''$ and $j^*$. The closure of $H_j$ is homeomorphic to $D^2$.

Take a non-singular vector field $Z$ on a neighborhood of $E(h) - \text{Int } H_0 = B_1 \cup (H - H_0)$ satisfying the following conditions (1)-(4).

1. $Z$ is tangent to $\partial E(h)$ at and only at $\Pi^+ \cup \Pi^- \cup \hat{C}_1' \cup \cdots \cup \hat{C}_k(0)$.

2. The orbits of $Z$ make concentric half circles (or confocal parabolas) near a point of $\Pi^+$ (or $\Pi^-$).

3. The line segments $K(1), \ldots, K(k_2)$ are orbits of $Z$.

4. For each $j = 1, \ldots, r^+$ the orbits of $Z$ make figures as in Figure 23.2 (a) in a closed neighborhood $U_j$ of $L(j)$, and for each $j = 1, \ldots, k(0)$ they do so as in Figure 23.2 (b) in a closed neighborhood $V_j$ of $L(r^+ + j) \cup \hat{C}_j$.

![Figure 23.2](a) (b)

![Figure 23.3](a) (b)
Since \( \sum_{C \in F(h)} r(C) = 4 - 2h \), we can extend \( Z \) to a non-singular vector field \( Z^* \) on \( \tilde{E}(h) \). For each \( j = r^+ + k(0) + 1, \ldots, r^- \), take a small closed interval \( J(j) \) in \( \partial \tilde{E}(h) \) containing \( \pi^-(j) \). Since \( J(j) \subset H_0 \), the saturation \( J(j)^* \) of \( J(j) \) with respect to \( Z^* \) is contained in \( H_0 \). Modifying \( Z^* \) if necessary, we may suppose that \( J(j)^* \cap (\pi^- - \{ \pi^-(j) \}) = \emptyset \) for all \( j \).

For a small closed neighborhood \( W_j \) of \( \pi^-(j) \) the saturation \( W^*_j \) of \( W_j \) is as in Figure 23.3 (a).

It is easy to see that for each connected component \( X_i \) of

\[
\tilde{E}(h) - \left( \bigcup_{j=1}^{r^-} U_j \right) \cup \left( \bigcup_{j=1}^{k(0)} V_j \right) \cup \left( \bigcup_{j=r^+}^{r^-} W_j \right)
\]

where \( r^+ = r^+ + k(0) + 1 \), the orbits of \( Z^* \) passing through \( X_j \) are as in Figure 23.3 (b).

Now we take a regular TS piece for each of \( U_j, V_j, W_j \) and \( X_j \), as in Figure 23.4.

Then it is easy to construct a regular TS diagram containing the above TS pieces and satisfying the condition of Theorem 12. This completes the proof of Theorem 12.

24. The proof of Theorem 2. Theorem 2 follows from Theorem 9 (2) and the following.

**THEOREM 13.** The map \( \alpha: \text{TS}(\Phi, \mathcal{F}; \sigma) \rightarrow \text{am}(\Phi, \mathcal{F}; \sigma) \) is surjective.

**Proof.** Let \( (a, b; r) \in \text{am}(\Phi, \mathcal{F}; \sigma) \). For each \( v \in V(\Phi) \) with \( h(v) > 2 \), we take the regular TS diagram \( \mathcal{F}(v) \) constructed in §23. For \( v \in V(\Phi) \) with \( h(v) = 1 \), we have \( a(C) = 1 \) and \( r(C) = 2 \), where \( \{ C \} = I[v] \). Take a regular TS diagram \( \mathcal{F}(v) \) containing exactly two TS pieces of type I. Consider \( v \in V(\Phi) \) with \( h(v) = 2 \). Let \( I[v] = \{ C, C' \} \). When \( r(C) = 0 \), we take a regular TS diagram \( \mathcal{F}(v) \) containing exactly two TS pieces of type VIII. When \( r(C) \neq 0 \) and \( a(C) = 0 \), we may suppose that \( \sigma(C) = 1 \).
and we take a singular TS diagram $\mathcal{F}(v)$ as in Figure 24.1. When $r(C) \neq 0$ and $a(C) > 0$, we take a regular TS diagram $\mathcal{F}(v)$ indicated by Figure 24.2.

Unfortunately $\{\mathcal{F}(v)\}_{v \in V(\Phi)}$ constructed above does not satisfy the compatibility condition described in Definition 20.7. For each $s \in S(\Phi)$ with $\delta(s) = (v) - (v')$, we see that $C = C[v](s)$ and $C' = C[v'](s)$ satisfy the compatibility condition on the symbols $\bigcirc$, $\bullet$ attached to $L(C)$ and $L(C')$, while we have trouble with the symbols $\vee$, $\wedge$, $\|$. We can overcome this trouble by using the following trick.

Suppose that $r(C) \neq 0$. First take a homeomorphism $\phi: L(C) \to L(C')$ satisfying the following conditions (1)-(3).

(1) For each $J$ with $s(J) = \circ$ or $\bullet$, the image $\phi(\text{Int } J)$ intersects only one $J'$ with $s(J') = \circ$ or $\bullet$.

(2) For each $J$ with $s(J) = \vee$, $\wedge$ or $\|$, the image $\phi(J)$ is contained in some $J'$.

(3) For each $J'$ with $s(J') = \vee$, $\wedge$ or $\|$, there is $J$ with $\phi(J) \supseteq J'$.

Inserting regular TS pieces of type II or singular TS pieces with symbols $\vee$, $\wedge$, $\|$ into $\mathcal{F}(v')$ for each $J$ with $s(J) = \vee$, $\wedge$, $\|$, we can modify $\phi$ to $\phi_1$ in such a way that $\phi_1$ satisfies the conditions corresponding...
to (1), (2) and (3) above and $\phi$, maps each $J$ with $s(J) = \vee, \wedge, \|$, to some $J'$ with the same symbol. (See Figure 24.3.)

Performing this for all $s \in S(\Phi)$, we have a family $\{T(v)^{(1)}\}_{v \in V(\mathcal{F})}$ of TS diagrams. Now make similar modifications in the opposite direction of $s$ for all $s \in S(\Phi)$ and get $\{T(v)^{(2)}\}_{v \in V(\mathcal{F})}$. Then $L(C[v](s))$ has possibly new $J'$s with $s(J') = \vee, \wedge$ or $\|$ for $s \in S(\Phi)$ with $\partial(s) = (v) - (v')$. Repeating the process of inserting TS pieces with symbols $\vee, \wedge, \|$ infinitely many times, we have a limit family $\{T(v)^{(\infty)}\}_{v \in V(\mathcal{F})}$ of TS diagrams. By construction, this limit family $M$ is a TS model transverse to $T(\Phi, \Psi; \sigma)$ with $\alpha(M) = (a, b; r)$. This completes the proof of Theorem 13.

REFERENCES