INDIVIDUAL ERGODIC THEOREMS FOR COMMUTING OPERATORS

Dedicated to Professor Tamotsu Tauchikura on his sixtieth birthday

RYOTARO SATO

(Received May 13, 1982)

Introduction. The main purpose of this paper is to prove the following theorem: If $T_1, \ldots, T_d$ are commuting positive contradictions on $L_1$ of a $\sigma$-finite measure space such that each operator $T_i$ satisfies the $L_1$-mean ergodic theorem, then the multiple ergodic average

$$(1/n)^d \sum_{i=0}^{n-1} \cdots \sum_{d=0}^{n-1} T_i \cdots T_d f(x)$$

converges to a finite limit almost everywhere as $n \to \infty$ for all $f \in L_1$.

Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and let $L_p(\mu)$, $1 \leq p \leq \infty$, denote the usual Banach spaces of (real or complex) functions on $(X, \mathcal{F}, \mu)$. A linear operator $T$ on $L_p(\mu)$ is called positive if $f \geq 0$ implies $Tf \geq 0$, and a contraction if $\|T\|_p \leq 1$, $\|T\|_p$ denoting the operator norm of $T$ on $L_p(\mu)$. We shall say that $T$ satisfies the $L_p$-mean ergodic theorem if the average $(1/n) \sum_{i=0}^{n-1} T_i f$ converges in $L_p$-norm as $n \to \infty$ for all $f \in L_p(\mu)$. Ito [9] proved that if $T$ is a positive contradiction on $L_1(\mu)$ satisfying the $L_1$-mean ergodic theorem, then the average $(1/n) \sum_{i=0}^{n-1} T_i f(x)$ converges to a finite limit a.e. on $X$ as $n \to \infty$ for all $f \in L_1(\mu)$. In the present paper we intend to extend his result to the case of multiple ergodic averages of $d$ commuting positive contradictions on $L_1(\mu)$. To do this, we use Brunel's theory [2] concerning a maximal ergodic inequality for commuting (not necessarily positive) contradictions on $L_1(\mu)$. As a corollary to the proof, it follows that if $T_1, \ldots, T_d$ are commuting (not necessarily positive) contradictions on $L_1(\mu)$ such that for some $1 < p \leq \infty$, $\|\tau_i\|_p \leq 1$ for all $1 \leq i \leq d$, $\tau_i$ denoting the linear modulus [3] of $T_i$, then the above multiple average converges to a finite limit a.e. on $X$ as $n \to \infty$ for all $f \in L_1(\mu)$. This is a generalization of McGrath's ergodic theorem [8], who treated the positive operator case. See also Emilion [5].

The continuous versions of these results are obtained by using a standard approximation argument.
2. Ergodic theorems for the discrete case.

THEOREM 1. Let \( T_1, \cdots, T_d \) be positive contractions on \( L_1(\mu) \) such that \( T_i T_j = T_j T_i \) for all \( 1 \leq i, j \leq d \). Suppose each \( T_i \) satisfies the \( L_1 \)-mean ergodic theorem. Then the limit

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} T_i^1 \cdots T_i^d f(x)
\]

exists and is finite a.e. on \( X \) for all \( f \in L_1(\mu) \).

PROOF. For simplicity we shall consider the case \( d = 2 \). (The general case follows similarly.) Since \( T_i \) satisfies the \( L_1 \)-mean ergodic theorem, \( \{h + (f - T_i f) : T_i h = h\} \) is a dense subset of \( L_1(\mu) \) by a well-known mean ergodic theorem (cf. e.g. [4, VIII, 5.2]). It follows that

\[
\{h + (g + f - T_i f) - T_i (g + f - T_i f) : T_i h = h, T_i g = g\}
\]

is a dense subset of \( L_1(\mu) \). Suppose \( T_i h = h \). Then Ito's ergodic theorem [9] shows that

\[
\frac{1}{n} \sum_{i=0}^{n-1} T_i^1 T_i^2 h(x) = \frac{1}{n} \sum_{i=0}^{n-1} T_i^2 h(x)
\]

converges to a finite limit a.e. on \( X \) as \( n \to \infty \). Next suppose \( k = g + f - T_i f \) with \( T_i g = g \). Then we get

\[
\left(\frac{1}{n}\right)^2 \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} T_i^1 T_i^2 (k - T_i^2 k) = \left(\frac{1}{n}\right)^2 \sum_{i=0}^{n-1} T_i^2 (k - T_i^2 k)
\]

\[
= \left(\frac{1}{n}\right)^2 \sum_{i=0}^{n-1} T_i^2 k - \left(\frac{1}{n}\right)^2 \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} T_i^2 k,
\]

where

\[
\lim_{n \to \infty} \left(\frac{1}{n}\right)^2 \sum_{i=0}^{n-1} T_i^2 k(x) = 0 \quad \text{a.e. on } X
\]

by Ito's theorem, and where

\[
\left(\frac{1}{n}\right)^2 T_i^2 \left(\sum_{i=0}^{n-1} T_i^2 \right) = \left(\frac{1}{n}\right)^2 T_i^2 \left(\sum_{i=0}^{n-1} [g + f - T_i f] \right)
\]

\[
= \left(\frac{1}{n}\right)^2 T_i^2 g + \left(\frac{1}{n}\right)^2 T_i^2 (f - T_i f).
\]

Ito's theorem shows that \( \lim_{n \to \infty} \left(\frac{1}{n}\right)^2 T_i^2 g(x) = 0 \) a.e. on \( X \). On the other hand, since \( \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 \|T_i^2 (f - T_i f)\|_1 < \infty \), we must have

\[
\lim_{n \to \infty} \left(\frac{1}{n}\right)^2 T_i^2 (f - T_i f)(x) = 0 \quad \text{a.e. on } X.
\]

Thus we have proved that the limit

\[
\lim_{n \to \infty} \left(\frac{1}{n}\right)^2 \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} T_i^1 T_i^2 f(x)
\]
exists and is finite a.e. on $X$ for every $f$ in a dense subset of $L_1(\mu)$. Hence the proof will be completed by Banach's convergence theorem (cf. e.g. [4, Theorem IV. 11.3]), if the following lemma is proved.

**Lemma.** If $T_1, \cdots, T_d$ are commuting positive contractions on $L_1(\mu)$ such that each $T_i$ satisfies the $L_1$-mean ergodic theorem, then for every $f \in L_1(\mu)$

$$\sup_{n \geq 1} \left( \frac{1}{n} \right)^d \sum_{i_1=0}^{n-1} \cdots \sum_{i_d=0}^{n-1} |T_{i_1}^{i_1} \cdots T_{i_d}^{i_d}f(x)| < \infty \text{ a.e. on } X.$$

To prove this lemma we need the following theorem due to Brunel [2]. (A slightly different form may be seen in [2].)

**Theorem A.** If $T_1, \cdots, T_d$ are commuting (not necessarily positive) contractions on $L_1(\mu)$, then there exists a constant $C_d > 0$ and a positive contraction $U$ on $L_1(\mu)$ of the form

$$U = \sum_{i_1=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} a(i_1, \cdots, i_d) \tau_{i_1}^{i_1} \cdots \tau_{i_d}^{i_d},$$

where $a(i_1, \cdots, i_d) \geq 0$, $\sum_{i_1=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} a(i_1, \cdots, i_d) = 1$, and $\tau_i$ denotes the linear modulus of $T_i$, such that for every $f \in L_1(\mu)$

$$\sup_{n \geq 1} \left( \frac{1}{n} \right)^d \sum_{i_1=0}^{n-1} \cdots \sum_{i_d=0}^{n-1} \tau(i_1, \cdots, i_d)|f|(x) \leq C_d \cdot \sup_{n \geq 1} \left( \frac{1}{n} \right)^d \sum_{i=0}^{n-1} U^i |f|(x)$$

a.e. on $X$, where $\tau(i_1, \cdots, i_d)$ denotes the linear modulus of $T_{i_1}^{i_1} \cdots T_{i_d}^{i_d}$.

**Proof of Lemma.** Let $U$ be as in Theorem A. We shall prove that $U$ satisfies the $L_1$-mean ergodic theorem, which, in turn, implies the lemma by virtue of Ito's theorem. To do this, we first show that for any $0 \leq h \in L_1(\mu)$, the set $\{T_i h : i \geq 0\}$ is weakly sequentially compact in $L_1(\mu)$. In fact, let $C$ and $D$ denote the conservative and dissipative parts (cf. e.g. [6]) of $T_1$, respectively. Then, since $T_1$ satisfies the $L_1$-mean ergodic theorem, there exists a function $0 \leq g \in L_1(\mu)$ such that $T_1 g = g$ and $\{g > 0\} = C$ ([9]). Further we have $\lim_{n \to \infty} \int_{D} (1/n) \sum_{i=0}^{n-1} T_i h d\mu = 0$; hence $\lim_{n \to \infty} \int_{D} T_i h d\mu = 0$. Let $E_n \in \mathcal{F}$, $E_{n+1} \subseteq E_n$ and $\bigcap_{n=1}^{\infty} E_n = \emptyset$. Given an $\varepsilon > 0$, take an $N \geq 1$ so that $\|(T_{i_1}^{i_1} h) 1_D\|_1 < \varepsilon$. Write $g_N = (T_{i_1}^{i_1} h) 1_D$ and $h_N = (T_{i_1}^{i_1} h) 1_{\mathcal{C}}$. Since $h_N \in L_1(C, \mu)$, an approximation argument implies that $\lim_{n \to \infty} \left( \sup_{t \geq 0} \int_{E_n} T_i h_N d\mu \right) = 0$. Thus

$$\lim_{n \to \infty} \left( \sup_{t \geq 0} \int_{E_n} T_i h d\mu \right) = \lim_{n \to \infty} \left( \sup_{t \geq 0} \int_{E_n} T_i (g_N + h_N) d\mu \right) \leq \|g_N\|_1 < \varepsilon;$$

since $\varepsilon > 0$ was arbitrary, the first expression equals zero. This shows
the weak sequential compactness of \( \{T_i^h: i \geq 0\} \). (See also [7, Theorem 3.2].)

Now, an induction argument implies easily that for any \( 0 \leq h \in L_1(\mu) \), the set \( \{T_{i_1}^h \cdots T_{i_d}^h: i_1, \ldots, i_d \geq 0\} \) is weakly sequentially compact, and thus \( \{U_i^h: i \geq 0\} \) is also weakly sequentially compact. By this and a mean ergodic theorem, \( U \) satisfies the \( L_1 \)-mean ergodic theorem. The proof is completed.

The following proposition is needed for the proof of Theorem 3 below. This proposition follows, as in Theorem 1, from an ergodic theorem of Akcoglu and Chacon [1] and a slight modification of McGrath's ergodic theorem ([8, Theorem 3]). Here it should be interesting to note that, when the author was typing the manuscript, he learned from Dr. Emilion that he also proved this proposition by using Brunel's theory [2]. See [5]. Hence we omit the details.

**Proposition.** Let \( T_1, \ldots, T_d \) be commuting (not necessarily positive) contractions on \( L_1(\mu) \) such that for some \( 1 < p \leq \infty \), \( \|\tau_i\|_p \leq 1 \) for each \( 1 \leq i \leq d \), where \( \tau_i \) denotes the linear modulus of \( T_i \). Then for any \( f \in L_1(\mu) \) the limit

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i_1=0}^{n-1} \sum_{i_d=0}^{n-1} T_{i_1}^h \cdots T_{i_d}^h f(x)
\]

exists and is finite a.e. on \( X \).

3. Ergodic theorems for the continuous case. By a strongly continuous semigroup \( \{T(t): t > 0\} \) of contractions on \( L_p(\mu) \), we mean that \( \|T(t)\|_p \leq 1 \), \( T(t)T(s) = T(t+s) \) and \( \lim_{s \to t} \|T(s)f - T(t)f\|_p = 0 \) for all \( t, s > 0 \) and \( f \in L_p(\mu) \). Such a semigroup \( \{T(t): t > 0\} \) is said to satisfy the \( L_p \)-mean ergodic theorem if \( (1/a) \int_0^a T(t)f dt \) converges in \( L_p \)-norm as \( a \to \infty \) for all \( f \in L_p(\mu) \).

**Theorem 2.** Let \( \{T_i(t): t > 0\}, i = 1, \ldots, d \), be strongly continuous semigroups of positive contractions on \( L_1(\mu) \) such that \( T_i(t)T_j(s) = T_j(s)T_i(t) \) for all \( 1 \leq i, j \leq d \) and \( t, s > 0 \). Suppose each semigroup \( \{T_i(t): t > 0\} \) satisfies the \( L_1 \)-mean ergodic theorem. Then the limit

\[
\lim_{a \to \infty} \frac{1}{a^d} \int_0^a \cdots \int_0^a T_1(t_1) \cdots T_d(t_d) f(x) dt_1 \cdots dt_d
\]

exists and is finite a.e. on \( X \) for all \( f \in L_1(\mu) \).

**Proof.** We consider the case \( d = 2 \). First we prove that each single operator \( T_i(1) \) satisfies the \( L_1 \)-mean ergodic theorem. To do this,
INDIVIDUAL ERGODIC THEOREMS 133

take \( h \in L_1(\mu) \) such that \( h > 0 \) a.e. on \( X \), and write \( h_0 = \int_0^1 T_i(t)h \, dt \). Since \( \{T_i(t): t > 0\} \) satisfies the \( L_1 \)-mean ergodic theorem,

\[
(1/n) \sum_{j=0}^{n-1} T_i(1)h_0 = (1/n) \int_0^n T_i(t)h \, dt
\]

converges in \( L_1 \)-norm as \( n \to \infty \). Therefore the set \( \{(1/n) \sum_{j=0}^{n-1} T_i(1)h_0: n \geq 1\} \) is weakly sequentially compact in \( L_1(\mu) \).

Now, let \( 0 \leq f \in L_1(\mu) \) be given. Then the strong continuity of \( \{T_i(t): t > 0\} \) implies that \( \{T_i(1)f > 0\} \subset \{T_i(1)h > 0\} \subset \{h_0 > 0\} \), and therefore by an approximation argument, the set \( \{(1/n) \sum_{j=0}^{n-1} T_i(1)f: n \geq 1\} \) is also weakly sequentially compact in \( L_1(\mu) \). By this and a mean ergodic theorem, \( T_i(1) \) satisfies the \( L_1 \)-mean ergodic theorem.

Next, to finish the proof, write \( f_0 = \int_0^1 \int_0^1 T_1(t_1)T_2(t_2)f \, dt_1 \, dt_2 \) for \( 0 \leq f \in L_1(\mu) \), and \( n = \lfloor a \rfloor \) for \( a > 1 \), where \( \lfloor a \rfloor \) denotes the integral part of \( a \). Then we obtain

\[
\left| (1/n)^2 \int_0^n \int_0^n T_i(t_1)T_j(t_2)f(x) \, dt_1 \, dt_2 - (1/n)^2 \int_0^n \int_0^n T_i(t_1)T_j(t_2)f(x) \, dt_1 \, dt_2 \right|
\leq (1/n)^2 \sum_{i_1=0}^n \sum_{i_2=0}^n T_i(1)T_j(1)f_0(x) \leq (1/n)^2 \sum_{i_1=0}^{n-1} \sum_{i_2=0}^{n-1} T_i(1)T_j(1)f_0(x),
\]

and the second expression converges to zero a.e. on \( X \) as \( n \to \infty \), by Theorem 1. This and Theorem 1 complete the proof.

**THEOREM 3.** Let \( \{T_i(t): t > 0\}, \ i = 1, \ldots, d \), be commuting strongly continuous semigroups of (not necessarily positive) contractions on \( L_1(\mu) \) such that for some \( 1 < p \leq \infty \), \( \|T_i(t)\|_p \leq 1 \) for all \( 1 \leq i \leq d \) and \( t > 0 \), where \( \tau_i(t) \) denotes the linear modulus of \( T_i(t) \). Then for any \( f \in L_1(\mu) \) the limit

\[
\lim_{a \to \infty} (1/a)^d \int_0^a \cdots \int_0^a T_i(t) \cdots T_d(t) f(x) \, dt_i \cdots dt_d
\]

exists and is finite a.e. on \( X \).

**PROOF.** We consider the case \( d = 2 \). By the Riesz convexity theorem we may assume \( p < \infty \). First suppose \( f \in L_1(\mu) \cap L_p(\mu) \). Write

\[
\tilde{f} = \int_0^1 \int_0^1 \tau_i(t_1)\tau_j(t_2) |f| \, dt_1 \, dt_2 \quad (\in L_1(\mu) \cap L_p(\mu)) .
\]

Here we note that the Bochner integral \( \int_0^1 \int_0^1 \tau_i(t_1)\tau_j(t_2) |f| \, dt_1 \, dt_2 \) exists, because \( \|\tau_i(s)\tau_j(t) - \tau_i(t_1)\tau_j(t_2)\|_1 \to 0 \) as \( s \to t_1 \to 0 \) and \( t \to t_1 \to 0 \), independently (cf. Sato [10]). Write \( n = \lfloor a \rfloor \) for \( a > 1 \). Then we obtain
and the second expression converges to zero a.e. on $X$ as $n \to \infty$, by McGrath’s ergodic theorem ([8, Theorem 3]). This and Proposition show that

$$\lim_{n \to \infty} \frac{1}{n^2} \int_0^1 \int_0^1 T_i(t_1)T_j(t_2)f(x)dt_1dt_2$$

exists and is finite a.e. on $X$.

Next, suppose $f \in L_1(\mu)$. If we denote by $\tau(i_1, i_2)$ the linear modulus of $T_i(i_1) T_j(i_2)$, then

$$\left(\frac{1}{n^2}\right) \int_0^1 \int_0^1 T_i(t_1)T_j(t_2)f(x)dt_1dt_2 \leq \left(\frac{1}{n^2}\right) \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \tau(i_1, i_2)\tilde{f}(x).$$

By virtue of Theorem A there exists a constant $C > 0$ and a positive contraction $U$ on $L_1(\mu)$ such that

$$\sup_{n \geq 1} \frac{1}{n^2} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \tau(i_1, i_2)\tilde{f}(x) \leq C \cdot \sup_{n \geq 1} \frac{1}{n} \sum_{i=0}^{n} U^i\tilde{f}(x) \ a.e. \ on \ X.$$

Since $\|\tau(1)\|_p \leq 1$ and $\|\tau(1)\|_p \leq 1$, we have $\|U\|_p \leq 1$, and hence by an ergodic theorem of Akcoglu and Chacon [1], $\frac{1}{n} \sum_{i=0}^{n-1} U^i\tilde{f}(x)$ converges to a finite limit a.e. on $X$ as $n \to \infty$. Therefore

$$\sup_{n \geq 1} \frac{1}{n} \sum_{i=0}^{n-1} U^i\tilde{f}(x) < \infty \ a.e. \ on \ X.$$

Thus Banach’s convergence theorem completes the proof.

REFERENCES


