CLOSED DERIVATIONS IN $C(I)$

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Introduction. Closed derivations in $C^*$-algebras have been studied by many authors motivated by mathematical physics. In commutative case closed derivations are also of great interest in connection with differentiations. In this paper we will discuss closed derivations in $C(I)$, where $C(I)$ is the algebra of all real valued continuous functions on the unit interval $I = [0, 1]$.

Let $\delta$ be a derivations in $C(I)$. Throughout this paper the domain $\mathcal{D}(\delta)$ of $\delta$ will be always assumed to be a dense subalgebra in $C(I)$ and we put $W_\delta = \{x \in I; \delta(f)(x) = 0 \text{ for every } f \in \mathcal{D}(\delta) \text{ with } \|f\| = |f(x)|\}$. $\delta$ is said to be quasi well-behaved if the interior $W_\delta^o$ of $W_\delta$ is dense in $I$. Batty [2], Goodman [3], and Sakai [6] have shown that a closed derivation $\delta$ in $C(I)$ is quasi well-behaved if and only if there exist $\lambda \in C(I)$ and an automorphism $\alpha$ of $C(I)$ such that $\delta \supset \lambda \alpha(d/dx)\alpha^{-1}$. But in [4] it has been shown that there exist non quasi well-behaved closed derivations in $C(I)$, those induced by non-atomic signed measures on $I$.

Let $\delta$ be a closed derivation in $C(I)$ and put $A_\delta = \{x \in I, \delta(f)(x) \neq 0 \text{ for some } f \in \mathcal{D}(\delta)\}$. In this paper we shall show that there exists an open dense set $U$ in $A_\delta$ and a continuous function $\mu$ on $U$ such that the restriction $\delta|_E$ of $\delta$ to any closed interval $E$ contained in $U$ is the derivation induced by a non-atomic signed measure $\mu|_E$ on $E$.

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Closed derivation in $C(I)$. We first present several lemmas before stating our main theorem. Throughout this section $\delta$ will always denote a closed derivation in $C(I)$.

Lemma 1. Let $f$ be a function in $\mathcal{D}(\delta)$ with $f(x_0) = \delta(f)(x_0) = 0$ for some $x_0$ in $(0, 1)$ and define $\tilde{f}$ and $\widehat{\delta(f)}$ by the following:

$$
\tilde{f} = \begin{cases} f & \text{on } [0, x_0] \\ 0 & \text{on } [x_0, 1] \end{cases} \quad \text{and} \quad \widehat{\delta(f)} = \begin{cases} \delta(f) & \text{on } [0, x_0] \\ 0 & \text{on } [x_0, 1] \end{cases}.
$$

Then $\tilde{f}$ belongs to $\mathcal{D}(\delta)$ and satisfies $\delta(\tilde{f}) = \widehat{\delta(f)}$. 

PROOF. If \( f = 0 \) on \([0, x_0]\), by [3, Lemma 1.1.5], the lemma is clear. Thus we assume \( \max_{0 \leq x \leq x_0} |f(x)| \neq 0 \). Let \( \varepsilon \) be a positive number with \( \varepsilon \leq \max_{0 \leq x \leq x_0} |f(x)|/2 \) and set \( \alpha = \max\{0 \leq x \leq x_0; |f(x)| = 2\varepsilon\} \), \( \beta = \min\{\alpha \leq x \leq x_0; |f(x)| = \varepsilon\} \), and \( \gamma = \min\{1 \cup \{x \geq x_0; |f(x)| = \varepsilon/3\}\} \). Taking \(-f\) instead of \( f \) if necessary, we may assume \( f(\alpha) = 2\varepsilon \) and \( f(\beta) = \varepsilon \). Let \( \eta \) be an arbitrary positive number. Then, by [3, Lemma 1.1.5] and the continuity of \( f \) and \( \delta(f) \), \( \alpha \leq x \leq x_0 \) implies \( |\delta(f)(x)| \leq \eta \) for \( \varepsilon \) sufficiently small. Since \( \mathcal{D}(\delta) \) is a Silov algebra, we can find \( g_1 \) and \( g_2 \) in \( \mathcal{D}(\delta) \) in such a way that \( 0 \leq g_1 \leq 1 \), \(-1 \leq g_2 \leq 0\), and 
\[
\begin{align*}
g_1 &= \begin{cases} 
0 & \text{on } [\beta, 1] \\
1 & \text{on } [0, \alpha],
\end{cases} \quad \text{and} \quad 
g_2 &= \begin{cases} 
0 & \text{on } [0, x_0] \\
-1 & \text{on } [\gamma, 1].
\end{cases}
\end{align*}
\]
Then \( h = f + 2\|f\|(g_1 + g_2) \) belongs to \( \mathcal{D}(\delta) \) and we have \( h(x) \geq \varepsilon \) for \( x \in [0, \beta] \) and \( h(x) \leq \varepsilon/3 \) for \( x \in [x_0, 1] \). Let \( p \) be a \( C^1 \)-function with \( 0 \leq p' \leq 2 \) and
\[
p(x) = \begin{cases} 
x & \text{if } x \geq \varepsilon \\
0 & \text{if } x \leq \varepsilon/3,
\end{cases}
\]
where \( p' \) is the usual derivative of \( p \). Then we have \( (p(h) - 2\|f\|g_2 - \tilde{f})(x) = 0 \) for \( x \in [0, \beta] \cup [x_0, 1] \) and \( |(p(h) - 2\|f\|g_1 - \tilde{f})(x)| = |p(h)(\beta) + |f(x)| | \leq 4\varepsilon \) for \( x \in [\beta, x_0] \). By [6, Theorem 3.8] and [3, Lemma 1.1.5], we also have \( p(h) \in \mathcal{D}(\delta) \),
\[
(\delta(p(h) - 2\|f\|g_2 - \tilde{f}))(x) = 0 \quad \text{for } x \in [0, \beta] \cup [x_0, 1],
\]
and
\[
|\delta(p(h) - 2\|f\|g_1 - \tilde{f}))(x)| 
\leq 2|\delta(h)(x)| + 2\|f\| |\delta(g_1)(x)| + |\delta(f)(x)| 
= 3\eta \quad \text{for } x \in [\beta, x_0].
\]
It follows that \( \|p(h) - 2\|f\|g_1 - \tilde{f}\| \leq 4\varepsilon \) and \( \|\delta(p(h) - 2\|f\|g_2 - \tilde{g}(f))\| \leq 3\eta \). Since we can take \( \varepsilon \) and \( \eta \) arbitrarily small, the closedness of \( \delta \) implies that \( \tilde{f} \in \mathcal{D}(\delta) \) and \( \delta(\tilde{f}) = \tilde{\delta}(\tilde{f}) \). This completes the proof.

LEMMA 2. Let \( f_1 \) and \( f_2 \) be functions in \( \mathcal{D}(\delta) \) such that \( f_1(x_0) = f_2(x_0) \) and \( \delta(f_1)(x_0) = \delta(f_2)(x_0) \) for some \( x_0 \) in \((0, 1)\). We define functions \( f \) and \( F \) in \( C(I) \) by the following:
\[
\begin{align*}
f &= \begin{cases} 
f_1 & \text{on } [0, x_0] \\
f_2 & \text{on } [x_0, 1]
\end{cases} \quad \text{and} \quad 
F &= \begin{cases} 
\delta(f_1) & \text{on } [0, x_0] \\
\delta(f_2) & \text{on } [x_0, 1].
\end{cases}
\end{align*}
\]
Then $f$ belongs to $\mathcal{D}(\delta)$ and satisfies $\delta(f) = F$.

PROOF. By assumption we have $(f_1 - f_2)(x_0) = \delta(f_1 - f_2)(x_0) = 0$. Put
\[ g = \begin{cases} f_1 - f_2 & \text{on } [0, x_0], \\ 0 & \text{on } [x_0, 1] \end{cases} \quad \text{and} \quad G = \begin{cases} \delta(f_1 - f_2) & \text{on } [0, x_0], \\ 0 & \text{on } [x_0, 1]. \end{cases} \]
Then Lemma 1 shows that $g \in \mathcal{D}(\delta)$ and $\delta(g) = G$, so that we have $f = g + f_2 \in \mathcal{D}(\delta)$ and $\delta(f) = G + \delta(f_2) = F$. This completes the proof.

We put $A_f = \{ x \in I; \delta(f)(x) \neq 0 \text{ for some } f \in \mathcal{D}(\delta) \}$. Note that $A_f$ is an open set in $I$.

LEMMA 3. Let $x_0$ be in $A_f$ and $U$ an arbitrary neighborhood of $x_0$. Then there exists $f$ in $\mathcal{D}(\delta)$ satisfying $\delta(f)(x_0) = 1$, $0 \leq \delta(f) \leq 2$, and $\text{supp} \delta(f) \subset U$, where $\text{supp} \delta(f)$ is the support of $\delta(f)$.

PROOF. $x_0 \in A_f$ implies that there exists a function $g$ in $\mathcal{D}(\delta)$ with $\delta(g)(x_0) = 1$. We assume $x_0 \in (0, 1)$. If $x_0$ is zero or one, we can also prove this lemma in a similar way. Take $\alpha$ and $\beta$ in $I$ such that $[\alpha, \beta] \subset U$, $x_0 \in (\alpha, \beta)$, and $0 < \delta(g) \leq 2$ on $[\alpha, \beta]$. By [3, Lemma 1.1.5], $\delta(g) \neq 0$ on $[\alpha, \beta]$ implies that there exist $x_1 \in (\alpha, x_0)$ and $x_2 \in (x_0, \beta)$ with $g(x_1) \neq g(x_0)$ and $g(x_2) \neq g(x_0)$. We shall consider only the case where $g(x_1) < g(x_0) < g(x_2)$. In the other case the proof is the same. Take a number $k$ with $0 < k < \min \{ g(x_0) - g(x_1), g(x_2) - g(x_0) \}$ and put
\[ \alpha' = \max \{ \alpha \} \cup \{ x \in [\alpha, x_1]; g(x) = g(x_0) - k \} \quad \text{and} \quad \beta' = \min \{ \beta \} \cup \{ x \in [x_0, \beta]; g(x) = g(x_0) + k \}. \]
Since $\mathcal{D}(\delta)$ is a Silov algebra, there exists a function $h$ in $\mathcal{D}(\delta)$ satisfying $-1 \leq h \leq 0$ on $[\alpha', x_1]$, $0 \leq h \leq 1$ on $[x_2, \beta']$, and
\[ h = \begin{cases} -1 & \text{on } [0, \alpha'], \\ 0 & \text{on } [x_1, x_2], \\ 1 & \text{on } [\beta', 1]. \end{cases} \]
Then $e = g + 2\|g\|h$ is an element in $\mathcal{D}(\delta)$ such that $e(x) \in [e(x_0) - k, e(x_0) + k]$ for $x \in [0, x_1] \cup [x_2, 1]$ and, by [3, Lemma 1.1.5], $\delta(e) = \delta(g)$ on $[x_0, x_2]$. Take a function $p$ in $C^1(R)$ such that $0 \leq p' \leq 1$, $p' = 0$ on $R \setminus [e(x_0) - k, e(x_0) + k]$, and $p'(e(x_0)) = 1$. Then, by [6, Theorem 3.8], $p(e)$ is a function in $\mathcal{D}(\delta)$ with $\delta(p(e)) = p'(e)\delta(e)$, so that we have $\delta(p(e))(x_0) = 1$, $0 \leq \delta(p(e)) \leq 2$, and $\text{supp} \delta(p(e)) \subset [x_0, x_2] \subset U$. Setting $f = p(e)$, this completes the proof.

Let $E$ be an arbitrary closed subinterval of $I$ and denote the restriction of a function $g$ in $C(I)$ to $E$ by $g|_E$. We define the restriction $\delta_E$ of $\delta$ to $E$ by $\delta_E(f|_E) = \delta(f)|_E$ for $f$ in $\mathcal{D}(\delta)$. Then, by [3, Lemma 1.1.5], $\delta_E$ is well defined and becomes a derivation in $C(E)$ whose domain $\mathcal{D}(\delta_E)$ is $\{ f|_E; f \in \mathcal{D}(\delta) \}$. 

PROPOSITION 4. Let $E$ be a closed subinterval of $I$. Then $\delta_E$ is a closed derivation in $C(E)$.

PROOF. Set $E = [x_0, x_1]$ ($x_0 < x_1$) and let $f_n$ be a sequence in $\mathcal{D}(\delta_E)$ such that $f_n \to f$ and $\delta_E(f_n) \to F$ as $n \to \infty$ in $C(E)$. If $x_i \in A_i$ ($i = 0, 1$), by Lemma 3, there exists $h_i$ in $\mathcal{D}(\delta)$ such that $\delta(h_i)(x_i) = 1$ and $\delta(h_i)(x_{i-1}) = 0$. If $x_i \notin A_i$, we put $h_i = 0$. Setting $g_n = f_n - \sum_{i=0}^{1} \delta_E(f_n)(x_i)h_i|_E$, we have $g_n \in \mathcal{D}(\delta_E)$, $\delta_E(g_n)(x_i) = 0$, $\lim_{n \to \infty} g_n = f - \sum_{i=0}^{1} F(x_i)h_i|_E$, and $\lim_{n \to \infty} \delta_E(g_n) = F - \sum_{i=0}^{1} F(x_i)\delta(h_i)|_E$ in $C(E)$. We put

$$
\bar{g}_n(x) = \begin{cases} g_n(x_i) & \text{if } x \geq x_1 \\ g_n(x) & \text{if } x_1 \leq x \leq x_0 \\ g_n(x) & \text{if } x \leq x_0.
\end{cases}
$$

Since $\delta_E(g_n)(x_i) = 0$ for all $n$ and $i = 0, 1$, by Lemma 2, $\bar{g}_n$ belongs to $\mathcal{D}(\delta)$ and satisfies $\delta(\bar{g}_n)|_E = \delta_E(g_n)$. Furthermore $\bar{g}_n$ and $\delta(\bar{g}_n)$ are Cauchy sequences in $C(I)$. From the closedness of $\delta$, we have $\lim_{n \to \infty} \bar{g}_n \in \mathcal{D}(\delta)$ and $\delta(\lim_{n \to \infty} \bar{g}_n) = \lim_{n \to \infty} \delta(\bar{g}_n)$, and it follows that $f - \sum_{i=0}^{1} F(x_i)h_i|_E \in \mathcal{D}(\delta_E)$ and $\delta_E(f - \sum_{i=0}^{1} F(x_i)h_i|_E) = F - \sum_{i=0}^{1} F(x_i)\delta(h_i)|_E$. Thus we have $f \in \mathcal{D}(\delta_E)$ and $\delta_E(f) = F$, so that $\delta_E$ is closed, this completes the proof.

We set $\mathcal{D}_\varepsilon = \{f \in \mathcal{D}(\delta); f(x) = 0\}$ for $x$ in $I$ and $B_\varepsilon = \{x \in I; \text{there exists a positive number } K \text{ and an open interval } U \text{ which contains } x \text{ such that } \|f\|_U \leq K\|\delta(f)\|_U \text{ for all } f \in \mathcal{D}_\varepsilon\}$, where $\|\ |_U$ is the uniform norm on $U$. Note that $B_\varepsilon$ is an open subset of $I$.

LEMMA 5. Let $x_0$ be in $I \setminus B_\varepsilon$, $\varepsilon$ an arbitrary positive number, and $J = (\alpha, \beta)$ an arbitrary open subinterval of $I$ which contains $x_0$. Then there exists an element $f$ in $\mathcal{D}(\delta)$ such that $0 \leq f \leq 1$, $f = 1$ on $[\beta, 1]$, $f = 0$ on $[0, \alpha]$, and $\|\delta(f)\| \leq \varepsilon$.

PROOF. By the definition of $B_\varepsilon$, $x_0 \in I \setminus B_\varepsilon$ implies that there exists $g$ in $\mathcal{D}_\varepsilon$ with $\|g\|_J = 4$, $\|\delta(g)\|_J \leq \varepsilon$. Let $x_1$ be an element in $J$ with $|g(x_1)| = 4$. We may assume that $g(x_1) = 4$ and $x_0 < x_1$. Otherwise, the proof is the same. Put $\gamma = \min\{x > x_0; g(x) = 1\}$ and $\sigma = \max\{x < x_1; g(x) = 3\}$. Then we can find $h$ in $\mathcal{D}(\delta)$ such that $-1 \leq h \leq 0$ on $[x_0, \gamma]$, $0 \leq h \leq 1$ on $[\sigma, x_1]$, and

$$
h = \begin{cases} 1 & \text{on } [x_1, 1] \\ 0 & \text{on } [\gamma, \sigma] \\ -1 & \text{on } [0, x_0].
\end{cases}
$$

Let $p$ be a $C^1$-function satisfying $0 \leq p \leq 1$, $0 \leq p' \leq 1$, $p(x) = 0$ if $x \leq 1$, and $p(x) = 1$ if $x \geq 3$. Putting $f = p(g + 2\|g\|_h)$, by [6, Theorem 3.8],
we have \( f \in \mathcal{D}(\delta), \| \delta(f) \| = \| p'(g + 2) \| g \| h \| \delta(g + 2 \| g \| h) \| \leq \epsilon, f = 0 \) on [0, \( \gamma \)], and \( f = 1 \) on [\( \sigma, 1 \)]. This completes the proof.

We recall the closed derivations induced by non-atomic signed measures (cf. [4]). Let \( E = [x_0, x_1] \) be a closed interval and \( \mu \) a non-atomic measures on \( E \) with the support \( E \). We define a linear mapping \( \delta_\mu \) in \( C(E) \) by the following:

\[
\delta_\mu \left( \lambda 1_E + \int_{x_0}^x f d\mu \right) = f \quad \text{for} \quad f \in C(E) \quad \text{and} \quad \lambda \in \mathbb{R},
\]

where \( 1_E \) is the unit element of \( C(E) \). [4, Theorem 2.2] has shown that \( \delta_\mu \) is well defined and becomes a closed derivation in \( C(E) \) whose domain is

\[
\left\{ \lambda 1_E + \int_{x_0}^x f d\mu ; f \in C(E) \quad \text{and} \quad \lambda \in \mathbb{R} \right\}.
\]

Now we state our main theorem.

**Theorem 6.** Let \( \delta \) be a closed derivation in \( C(I) \). Then the following conditions are satisfied:

(i) \( A_\delta \cap B_\delta \) is a dense open subset in \( A_\delta \).

(ii) There exists a continuous real-valued function \( \mu \) on \( A_\delta \cap B_\delta \) such that, for any closed interval \( E \) contained in \( A_\delta \cap B_\delta \), the restriction \( \mu_E \) of \( \mu \) to \( E \) is a non-atomic signed measure on \( E \) and satisfies \( \delta_E = \delta_{\mu_E} \).

**Proof.** We suppose that \( A_\delta \cap B_\delta \) is not dense in \( A_\delta \). Then we can take a closed interval \( J = [\alpha, \beta] \) with \( \alpha < \beta \) and \( J \subseteq A_\delta \cap B_\delta \), where \( B_\delta \) is the complement of \( B_\delta \). By Proposition 4, the restriction \( \delta_J \) of \( \delta \) to \( J \) is a closed derivation in \( C(J) \).

For an element \( g \) in \( C(J) \) and a positive number \( \varepsilon \), there exists a \( C'(J) \)-function \( h \) with \( \| h - g \|_J \leq \varepsilon/2 \). Furthermore we can find an integer \( n \) such that \( n^{-1}(\beta - \alpha) \| h' \|_J \leq 1 \) and \( |g(x) - g(y)| \leq \varepsilon/2 \) for every \( x \) and \( y \) in \( J \) with \( |x - y| \leq n^{-1}(\beta - \alpha) \). Put \( x_k = \alpha + kn^{-1}(\beta - \alpha) \) for \( k = 0, 1, \ldots, n \). Since \( J \subseteq B_\delta \), Lemma 5 shows that, for \( k = 1, 2, \ldots, n \), there exists an element \( f_k \) in \( \mathcal{D}(\delta_J) \) satisfying \( 0 \leq f_k \leq 1 \), \( \| \delta_J(f_k) \|_J \leq \epsilon, f_k = 0 \) on \( [\alpha, x_{k-1}] \), and \( f_k = 1 \) on \( [x_k, \beta] \). Note that \( \delta_J(f_k) = 0 \) on \( [\alpha, x_{k-1}] \cup [x_k, \beta] \). If we set \( f = \alpha 1_J + n^{-1}(\beta - \alpha) \sum_{k=1}^n f_k \), where \( 1_J \) is the unit element in \( C(J) \), we have \( f \in \mathcal{D}(\delta_J) \) and \( \| \delta_J(f) \|_J \leq n^{-1}(\beta - \alpha) \), \( f(x_k) = x_k(k = 0, 1, \ldots, n) \), and \( x_{k-1} \leq f(x) \leq x_k \) for \( x_{k-1} \leq x \leq x_k \) \( (k = 1, 2, \ldots, n) \). By [6, Theorem 3.8], we have \( h(f) \in \mathcal{D}(\delta_J) \) and \( \| \delta_J(h(f)) \|_J = \| h'(f) \delta_J(f) \|_J \leq n^{-1}(\beta - \alpha) \| h' \|_J \leq \epsilon \).

On the other hand, if \( x_{k-1} \leq x \leq x_k \), we also have \( |(h(f) - g)(x)| \leq |h(f(x)) - g(x)| \leq \varepsilon, \) so that \( \| h(f) - g \|_J \leq \varepsilon \). Since \( \varepsilon \) is arbitrary and \( \delta_J \) is closed, it follows that \( g \in \mathcal{D}(\delta_J) \) and \( \delta_J(g) = 0 \). This is a contradiction, so that \( A_\delta \cap B_\delta \) is dense in \( A_\delta \).
Now we prove the second part of the theorem. Let $E$ be a closed interval in $A \cap B$. By $E \subseteq B$, and the compactness of $E$, there exist points $x_i$ in $E$ and open subintervals $V_i$ of $I$ and positive numbers $K_i (i = 1, 2, \ldots, n)$ such that $x_i \in V_i, E \subseteq \bigcup_{i=1}^{n} V_i$, and $\| f \|_{V_i} \leq K_i \| \delta(f) \|_{V_i}$ for $f$ in $D$, then we have $\| f \|_{V_i} \leq 2n(\max_i K_i) \| \delta(f) \|_{V_i}$ for $f$ in $D$, where we put $E = [\gamma, \sigma]$ and $V = \bigcup_{i=1}^{n} V_i$. Let $g$ be an arbitrary element in $D(\delta)$ with $g(\gamma) = 0$. Lemma 3 shows that there exists $h_1$ in $D(\delta)$ such that $\delta(h_1)(\gamma) = \delta_E(g)(\gamma), \delta(h_1)(\sigma) = \delta_E(g)(\sigma)$, and $\| \delta(h_1) \| \leq 2 \| \delta_E(g) \|_E$. Since $D(\delta)$ is a Silov algebra, we can find $h_2$ in $D(\delta)$ with $h_2 = g(\gamma) - h_1(\gamma) = -h_1(\gamma)$ on $[0, \gamma]$ and $h_2 = g(\sigma) - h_1(\sigma)$ on $[\sigma, 1]$. Note that $\delta(h_2) = 0$ on $[0, \gamma] \cup [\sigma, 1]$. We put $\bar{g} = \begin{cases} g & \text{on } [\gamma, \sigma], \\ h_1 + h_2 & \text{on } [0, \gamma] \cup [\sigma, 1]. \end{cases}$

Lemma 2 implies that $\bar{g} \in D(\delta)$ and $\| \delta(\bar{g}) \| \leq 2 \| \delta_E(g) \|_E$, so that we have

$$\| g \|_E \leq \| \bar{g} \|_E \leq 2n(\max_i K_i) \| \delta(\bar{g}) \|_E \leq 2n(\max_i K_i) \| \delta_E(g) \|_E \leq 4n(\max_i K_i) \| \delta_E(g) \|_E.$$ 

It follows that the kernel $K(\delta_E)$ of $\delta_E$ is $\{ \lambda 1_E, \lambda \in R \}$ and the range $R(\delta_E)$ of $\delta_E$ is a closed linear subspace in $C(E)$.

Now we show that $R(\delta_E) = C(E)$. By Lemma 3 and the compactness of $E$, there is an element $\nu$ in $D(\delta)$ with $\delta(\nu)(x) \neq 0$ on $E$. Taking $(\delta(\nu)|_E)^{-1} \delta_E$ instead of $\delta_E$ if necessary, we may assume that $R(\delta_E)$ contains $1_E$, where $1_E$ is the unit element of $C(E)$. Let $K$ be an arbitrary subinterval of $E$ and $\chi(K)$ a characteristic function of $K$. Then, by Lemmas 2 and 3, the same argument as above implies that there exists a sequence $g_n$ in $D(\delta_E)$ such that $\delta(\bar{g}_n)$ pointwise converges to $\chi(K)$ and $\| \delta_E(g_n) \|_E \leq 2 \| 1_E \|_E = 2$. Suppose that $\phi$ is a continuous linear functional on $C(E)$ such that $\phi(R(\delta_E)) = 0$. Then we have $\lim_{n \to \infty} \chi(K) d\phi = \lim_{n \to \infty} \phi(\delta_E(g_n)) = 0$ so that $\phi = 0$. Since $R(\delta_E)$ is a closed subspace of $C(E)$, by the Hahn-Banach theorem, we have $R(\delta_E) = C(E)$. It follows from [4, Theorem 2.3] that there exists a unique non-atomic signed measure $\mu_E$ on $E$ such that $\delta_E = \delta_{\mu_E}$.

Let $G$ be a connected component of $A \cap B$ and $G_n$ a sequence of closed subintervals of $G$ such that $G_n \subseteq G_{n+1}$ and $\bigcup_{n} G_n = G$. By the above argument, for each $G_n$, there is a non-atomic signed measure $\mu_{G_n}$ with $\delta_{G_n} = \delta_{\mu_{G_n}}$. The uniqueness of $\mu_{G_n}$ implies that $\mu_{G_n}$ is the restriction of $\mu_{G_{n+1}}$ to $G_n$. Considering $\mu_{G_n}$ as a function of bounded variation on
which is the restriction of \( \mu_{\sigma_{n+1}} \) to \( G_n \), we put \( \mu(x) = \lim_{n \to \infty} \mu_{\sigma_n}(x) \) for \( x \) in \( G \). Since \( \mu_{\sigma_n} \) is non-atomic, \( \mu \) is continuous. Thus we get a continuous function \( \mu \) on \( A_i \cap B_i \) and our assertion follows from the uniqueness of measure.

**Remark 7.** Let \( \delta_0 \) be the closed extension of the usual derivative \( d/dx \) whose kernel is the closed subalgebra of \( C(I) \) generated by the Cantor function and the unit element of \( C(I) \) (cf. [3]). Then we have \( A_{\delta_0} = I \) and \( B_{\delta_0} \) is the complement of the Cantor set.

**Remark 8.** It follows from the proof of [4, Theorem 2.2] that there exists a dense subset \( U \) of \( A_i \cap B_i \) such that

\[
\delta(f)(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{\mu(x + h) - \mu(x)} \quad \text{for } f \text{ in } \mathcal{D}(\delta) \text{ and } x \text{ in } U.
\]

We set \( M_\mu = \{ x \in A_i \cap B_i ; \text{ there exists a neighborhood of } x \text{ on which } \mu \text{ is monotone} \} \). The following corollary is clearly verified by [4, Theorem 3.1].

**Corollary 9.** Let the notation be the same as in Theorem 6. Then \( \delta \) is quasi well-behaved if and only if \( M_\mu \) is dense in \( A_i \cap B_i \).

**References**


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