RATIONAL FUNCTIONS OF $C^*$-TYPE ON
THE TWO-DIMENSIONAL COMPLEX
PROJECTIVE SPACE

TAKASHI KIZUKA

(Received February 14, 1985)

Introduction. A non-constant rational function $f$ on a smooth algebraic surface $X$ defines a holomorphic mapping of $X \setminus I_f$ onto the Riemann sphere $P$, where $I_f$ denotes the set of all points of indetermination of $f$. The set $L_c = \{ p \in X \setminus I_f | f(p) = c \}$ is called the level curve of $f$ with value $c \in P$. Following Nishino, we call an irreducible component of $L_c$ a prime curve of $f$ (with value $c$). If a smooth prime curve $S$ is analytically isomorphic to the punctured Gaussian plane $C^*$, we say that $S$ is of $C^*$-type. If all the prime curves of $f$, except for a finite number of them, are of $C^*$-type, we say that $f$ is of $C^*$-type. The terms “$C$-type” and “$P$-type” for prime curves, and for rational functions, are defined similarly, where $C$ is the Gaussian plane. If a rational function $f$ is of $C$-type, or if $f$ is of $C^*$-type, we say that $f$ is of special type. In the previous paper [3], we have shown the following fact.

**Theorem 0.** Let $C$ be an algebraic curve in the complex projective plane $P^2$. If the complement $P^2 \setminus C$ has an analytic transcendental automorphism, then $C$ is a smooth cubic curve or there exists a rational function $f$ of special type on $P^2$ whose restriction to $P^2 \setminus C$ is still of special type. In the latter case, $C$ contains at least one prime curve of $f$. 
This theorem poses the problem to determine all the rational functions of special type on $P^2$. If a rational function $f$ on $P^2$ is of special type, then, for each non-constant rational function $\psi$ on $P$ and for each analytic automorphism $S$ of $P^2$, $\psi \circ f \circ S$ is of special type. So the problem is reduced to that of determining a canonical form of a rational function of special type. If a rational function $f$ of special type on $P^2$ has a prime curve $S$ of degree one (a complex line), we say that $f$ belongs to the family $\mathcal{F}_I$. In this case, regarding the closure $\overline{S}$ of $S$ as the line at infinity, we may regard $f$ as a rational function of special type on $C^2 = P^2 \setminus \overline{S}$. The rational functions of $C$-type on $C^2$ were determined by Jung [1] and the rational functions of $C^*$-type on $C^2$ were determined by Kashiwara (née Saito). If a rational function $f$ of special type on $P^2$ has no prime curve of degree one, we say that $f$ belongs to the family $\mathcal{F}_{II}$. The rational functions belonging to $\mathcal{F}_I$ are simpler than those belonging to $\mathcal{F}_{II}$. Recently, Kashiwara [2] has determined all the rational functions of $C$-type belonging to $\mathcal{F}_{II}$ by her systematic study. In this paper, we resolve the remaining problem of determining all the rational functions of $C^*$-type belonging to $\mathcal{F}_{II}$.

The author would like to express his gratitude to Professor T. Kuroda, Dr. M. Suzuki and Dr. T. Ueda for their constant encouragement and important advice. This work was completed while he was a research fellow at the Research Institute for Mathematical Sciences, Kyoto University. Thanks are also due to Professor S. Nakano for his hospitality.

Chapter 0. Summary.

§ 1. Reciprocity.

1. A prime curve $S$ of a rational function $f$ with value $c$ ($\neq \infty$) is said to be of order $\nu$ if the function $f - c$ takes the value zero of order $\nu$ on $S$. A prime curve $S$ with value $\infty$ is said to be of order $\nu$ if the function $1/f$ takes the value zero of order $\nu$ on $S$. If the order $\nu$ of $S$ is greater than one, we say that $S$ is multiple. If each level curve of $f$ is irreducible except for a finite number of level curves, $f$ is called primitive. Proposition 1 in Chapter I implies that we have only to determine all the primitive rational functions of $C^*$-type on $P^2$.

Let $f$ be a rational function of special type on $P^2$. Denote by

$$M = M_m \xrightarrow{\sigma_m} M_{m-1} \xrightarrow{\sigma_{m-1}} \cdots \xrightarrow{\sigma_2} M_1 \xrightarrow{\sigma_1} M_0 = P^2$$

the minimal sequence of $\sigma$-processes which resolves the indetermination points of $f$. Set $\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_m$. The pull-back $\sigma^* f$ of $f$ by $\sigma$ is a
rational function on $M$ with no indetermination point. We denote by $\Sigma(f)$ the complete inverse image $\sigma^{-1}(I_f)$ of the set of indetermination points $I_f$ of $f$ under $\sigma$ in this paper.

Let $g$ be a non-constant primitive rational function on a smooth rational surface $V$. Consider an algebraic compactification $(M, \iota)$ of $V$, which means that there exist an algebraic curve $E$ on a compact smooth algebraic surface $M$ and a birational biregular isomorphism $\iota$ of the complement $M \setminus E$ of $E$ onto $V$. Suppose that $\iota^*g$ has no indetermination point on $M$. An irreducible component $C$ of the curve $E$ is called a basic section of $\iota^*g$ if the restriction $\iota^*g|_C$ of $\iota^*g$ to $C$ is not constant. Denote by $B$ the union of basic sections of $\iota^*g$. If the function $g$ on $V$ is of $C^*$-type, then $B$ consists of one or two irreducible components (cf. Chapter I §1.1). If $B$ is reducible, we say that $g$ is of direct $C^*$-type. If $B$ is irreducible, we say that $g$ is of torsional $C^*$-type. We say also that $g$ is of proper $C^*$-type if $g$ satisfies the following three conditions; (i) $g$ has no indetermination point on $V$, (ii) $g$ does not take the values 0, $\infty$ and the regular mapping $g: V \to C^*$ is surjective and (iii) each level curve of $g$ is irreducible, of $C^*$-type and of order one. The following theorem proved in Chapter III, §1.3 is used several times in this paper.

**Theorem 1.** Let $C$ be an algebraic curve in $P^2$. Suppose that there exists a rational function of proper (resp. proper direct) $C^*$-type on $V = P^2 \setminus C$. If a non-constant primitive rational function $g$ on $V$ does not take the value 0, $\infty$, then $g$ is also of proper (resp. proper direct) $C^*$-type on $V$.

2. Rational functions of special type on $P^2$ are intimately related to each other as is seen by Theorem 1. The rational functions belonging to the class $(D_0)$, defined in the following, play a pivotal role in this relation of functions. We say that a primitive rational function $f$ of direct $C^*$-type on $P^2$ belongs to the class $(D_0)$ if $f$ satisfies the following five conditions; (i) $I_f$ consists of only one point, (ii) the level curve $L_0$ consists of two prime curves both of which are of $C$-type and of order one, and those two prime curves intersect each other transversally, (iii) the level curve $L_\infty$ is irreducible and of $C$-type, (iv) the level curve $L_1$ is irreducible, of $C^*$-type and multiple and (v) the other level curves are irreducible and of $C^*$-type.

In Chapter II, we determine the graph of $\Sigma(f)$ of a rational function $f$ belonging to the class $(D_0)$, using the following two properties; (i) $\Sigma(f)$ is a (reducible) exceptional curve of the first kind and (ii) $\sigma^*f$ is a rational function of $P$-type on $M$. Therefore we can construct inductively
all the rational functions belonging to the class \((D_0)\). (cf. Example [B] and Example [C] in Chapter II, §1.2.) In this process, we obtain all the rational functions of \(C\)-type belonging to \(\mathscr{S}_{II}\). By the same method, we can determine all the rational functions of \(C^*\)-type on \(P^2\). But it is very complicated. In this paper, we use Theorem 1 to determine those functions as is seen in the following paragraph.

Suppose that \(f\) is of direct \(C^*\)-type on \(P^2\). By a topological lemma in Chapter III, we know that \(f\) has prime curves \(C_1, C_2, C_3\) such that the restriction \(f_0|_V\) of a rational function \(f_0\) to \(V = P^2 \setminus (\overline{C_1} \cup \overline{C_2} \cup \overline{C_3})\) is of proper \(C^*\)-type, where \(\overline{C_i}\) is the closure of \(C_i\) in \(P^2\). Let \(t_i\) be a homogeneous polynomial defining \(\overline{C_i}\). Theorem 1 implies that, for each triple \((\alpha_1, \alpha_2, \alpha_3)\) \((\neq (0, 0, 0))\) of integers satisfying \(\alpha_1 \deg(t_1) + \alpha_2 \deg(t_2) + \alpha_3 \deg(t_3) = 0\), the rational function \(g = t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3}\) is of special type on \(P^2\). By this fact, if \(f\) belongs to \(\mathscr{S}_{II}\), \(f\) is related to a rational function belonging to the class \((D_0)\). Hence \(f\) can be determined concretely. On the other hand, there exists no rational function of torsional \(C^*\)-type on \(P^2\) as is seen in Chapter III, §3, which completes our study.

§2. Statement of results.

1. Here, we give a summary of Chapter II to explain our recurrence formulas. Suppose that a primitive rational function \(f\) belongs to the class \((D_0)\). Let \(\sigma: M \to P^2\) be the minimal resolution of the points of indetermination of \(f\) by a finite sequence of \(\sigma\)-processes. We denote by \(F_c\) the level curve of \(\sigma^*f\) with value \(c\). If the level curve \(F_1\) of \(\sigma^*f\) with value 1 contains \(n-1\) irreducible components with the self-intersection numbers smaller than \(-2\), we say that \(f\) belongs to \(D_n\). The class \((D_0)\) divides into subclasses \(D_n (n=1, 2, 3, \ldots)\). We have the following in Chapter II.

**Proposition 0.** If \(f\) belongs to \(D_n\), then \(F_1\) is represented by the diagrams in Figure 1.

\[
\begin{align*}
2 & 1 & 2 & 5 & 1 & 4 & 2 & 1 & 2 & r & 12 & r & 5 & r & 12 & r & 3 \\
\odot \square & \odot \square & \odot \square \odot & \odot \square \odot \square \odot \square \odot & \odot \square \odot \square \odot \square \odot \square \odot \\
(n=1) & (n=2) & (n=2r+1) & (n=2r+2)
\end{align*}
\]

**Figure 1.**

In this diagram, a circle or a square represents a non-singular rational curve. A circle represents an irreducible component of \(\Sigma(f)\). A square represents the proper image of \(\overline{L_i}\) (the closure of the level curve \(L_i\) of \(f\) with value 1 in \(P^i\)) under the mapping \(\sigma^{-1}\). The numbers attached to
circles and squares are those obtained by the multiplication of $-1$ to the self-intersection numbers of the corresponding curves. A short segment connecting circles or squares represents a transversal intersection of the corresponding curves.

A long line with a non-negative integer $p$ attached in Figure 2(i) is the abbreviation for the diagram in Figure 2(ii). A sign with a non-negative integer $r$ attached in Figure 2(iii) is the abbreviation for the diagram in Figure 2(iv). A sign with a non-negative integer $r$ attached in Figure 2(v) is the abbreviation for the diagram in Figure 2(vi).

We shall see in Chapter II, §2.1, that, in the case where the graph of $\Sigma(f)$ is linear, $\Sigma(f) \cup F_0 \cup F_0 \cup F_1$ is represented by the diagrams in Figure 3.

In each diagram, a square represents the proper image of the closure of one of the prime curves of $f$ with values 0, 1, $\infty$ under the mapping $\sigma^{-1}$. The connected square in the center of each diagram represents the level curve $F_0$ of $\sigma^*f$ with value 0. The components intersecting $F_0$ are the basic sections of $\sigma^*f$. The right-hand side of each diagram is the level curve $F_1$ with value 1 and the left-hand side is the level curve $F_0$ with the value $\infty$.

If $f$ belongs to $D_n$ and if the graph of $\Sigma(f)$ is linear, then we say that $f$ belongs to $D_n\sigma$. There exists only one rational function belonging
to \( D_n^0 \) up to the projective transformations of \( P^2 \). In Chapter II, using the diagrams in Figure 3, we see that a function belonging to \( D_n^0 \) has a connection with a function belonging to \( D_{l-1}^0 \). Therefore, a function \( f \) belonging to \( D_n^0 \) can be written as \( f = v_{n+1}v_{n-1}/v_n^2 \) for a homogeneous coordinate \((X: Y: Z)\) of \( P^2 \), where \( v_i \) is a homogeneous polynomial of \((X, Y, Z)\) defined by the following recurrence formula;

\[
v_{l+1} = (v_l^2 + w(m_l))/v_{l-1} \quad (m_l = \deg(v_l); \ l = 0, 1, \ldots, n),
\]

\[
v_{l-1} = X, \ v_0 = Y, \ u = XYZ - X^3 - Y^3.
\]

Let \( \{b_l\} \) be the so-called Fibonacci sequence satisfying \( b_{-1} = 0, \ b_0 = 1 \) and \( b_{l+1} = b_l + b_{l-1} \). We know that \( m_n = b_{2n} \).

We divide the complement \( D_n^0 / D_0 \) into \( D_{k+n} \) and \( D_{k-n} \) according to the shape of \( \Sigma(f) \) where the integer \( k \) is the number of the diverging components of \( \Sigma(f) \) (for the definition of a diverging component, see Chapter I, §2.1). A function belonging to \( D_{l-1}^0 \) has a connection with a function belonging to \( D_{l-2}^0 \). By induction, a function \( f \) belonging to \( D_{k+n} \) has a connection with a function belonging to \( D_{k+n}^0 \) so that \( f \) can be represented as \( f = w_kw_{k-1}/v_k^2 \) for a homogeneous coordinate \((X: Y: Z)\) of \( P^2 \), where \( u_l \) and \( w_l \) are homogeneous polynomials of \((X, Y, Z)\) defined by the following recurrence formulas;

\[
\mu_{l+1} = \mu_{l-1}v_l + \alpha_ldeg(w_{l-1}) - P_{\alpha}(w_l, w_{l-1}),
\]

where \( P_{\alpha}(z_1, z_2) \) is a homogeneous polynomial in \((z_1, z_2)\) of degree \( \alpha \) with \( P_{\alpha}(1, 0) \neq 0 \) \((l = 1, 2, \ldots, k)\), and

\[
w_l = (v_l^2 + w_{l-1}^2)/w_{l-1}.
\]

When \( f \) belongs to \( D_{k+n}^0 \), we define \( s_l(n) \) and \( w_0 \) as follows; \( s_l(n) = [b_{2n} + 3(-1)^l]/2 \) and \( w_0 = v_{n+1} \). When \( f \) belongs to \( D_{k-n}^0 \), we define \( s_l(n) \) and \( w_0 \) as follows; \( s_l(n) = [b_{2n} + 3(-1)^{l-1}]/2 \) and \( w_0 = v_{n-1} \). In both cases, \( \mu_l(n) = \mu_l(n)deg(w_{l-1}) - \deg(u_{l-1})]/m_n \) \((l = 1, 2, \ldots, k)\). When \( f \) does not belong to \( D_{k+n}^0 \), we define \( u_0 = 0 \). When \( f \) belongs to \( D_{k-n}^0 \), we define \( u_0 \) as \( u_0 = X \). (In the case where \( n = 1 \), we may suppose that \( \alpha_l > 0, \ l = 1, 2, \ldots, k \).)

2. Set \( R_{n,k} = w_{k-1}^n/w_{n+1}^2 \), where \( w_{k-1} \) and \( v_n \) are those defined above. The rational function \( R_{n,k} \) is of C-type on \( P^2 \). If the rational function \( f \) does not belong to \( D_1^0 \), then \( R_{n,k} \) belongs to \( \mathcal{S}_{11} \). Otherwise, \( R_{1,1} \) belongs to \( \mathcal{S}_{11} \). Conversely, a rational function \( R \) of C-type belonging to \( \mathcal{S}_{11} \) can be expressed in the form \( R = \Lambda(R_{n,k}) \) for a homogeneous coordinate \((X: Y: Z)\) of \( P^2 \), where \( \Lambda(z) \) is a non-constant rational function in one variable \( z \). Set \( \nu_{n,k} = u_{k-1}v_{n+1}/w_{k-1}^2 \). If \( f \) does not belong to \( D_1^0 \), then the
rational function $\psi_{n,k}$ is of $C^*$-type on $P^2$ and belongs to $F_{II}$. Otherwise, $\psi_{1,1}$ is of $C$-type on $P^2$ and belongs to $F_I$. The mapping $\tau$ defined by $\tau(p) = (R_{n,k}(p), \psi_{n,k}(p))$, $p \in P^2 \setminus \{v_n w_{k-1} = 0\}$, is a biregular birational mapping of $P^2 \setminus \{v_n w_{k-1} = 0\}$ onto $C^* \times C$.

Set $\varphi(z) = P(z)/z^l$, where $l \in Z^+ \cup \{0\}$ and $P(z)$ is an arbitrary polynomial in $z$. Set $\varphi_0 = (R_{n,k})^p (\psi_{n,k} - P(R_{n,k}))^q$, where $p \in Z$, $q \in Z^+$ and $(p, q) = 1$. Except when $f$ belongs to $D_1^-$, the rational function $\varphi_0$ is of $C^*$-type on $P^2$ and belongs to $F_{II}$. Conversely, we get the following in Chapter III, §2.

**Theorem 2.** A rational function $\varphi$ of $C^*$-type belonging to $F_{II}$ can be expressed as $\varphi = \varphi_0 (\psi_{n,k})$ for a homogeneous coordinate $(X: Y: Z)$ of $P^2$. Here $\varphi(z)$ is a non-constant rational function in one variable $z$.

3. Suppose that a rational function $f$ belongs to the class $(D_0)$. If $f$ belongs to $D_k^{\pm}$, we call the graph of $\Sigma(f) \cup F_\infty \cup F_o \cup F_1$ simply the graph of $D_k^{\pm}$. The following is the list of the graph of $D_k^{\pm}$.

(a) The graph of $D_1^{\pm}$.

In the above, the circle labeled $H_i(a_k)$ represents the following diagrams:

![Diagram](image1.png)

![Diagram](image2.png)
In the graph of $D_{1}^{-}$, the case $a_{1} = 0$ does not occur.

(b) The graph of $D_{n}^{k+}$ $(n \geq 2)$.

\[ \begin{array}{c}
T_{n}^{k}
\end{array} \]

$k$: odd

\[ \begin{array}{c}
K_{n}(a_{1}) \rightarrow K_{n}^{*}(a_{2}) \rightarrow K_{n}(a_{3}) \rightarrow K_{n}^{*}(a_{4}) \rightarrow \cdots \rightarrow K_{n}^{*}(a_{k-1}) \rightarrow H_{n}(a_{k})
\end{array} \]

\[ \begin{array}{c}
(k: \text{odd})
\end{array} \]

$k$: even

\[ \begin{array}{c}
K_{n}(a_{1}) \rightarrow K_{n}^{*}(a_{2}) \rightarrow K_{n}(a_{3}) \rightarrow K_{n}^{*}(a_{4}) \rightarrow \cdots \rightarrow K_{n}(a_{k-1}) \rightarrow H_{n}^{*}(a_{k})
\end{array} \]

\[ \begin{array}{c}
(k: \text{even})
\end{array} \]

\textbf{FIGURE 6.}

(c) The graph of $D_{n}^{k-}$ $(n \geq 2)$: Figure 7.

\[ \begin{array}{c}
T_{n}^{k}
\end{array} \]

$k$: odd

\[ \begin{array}{c}
K_{n}^{*}(a_{1}) \rightarrow K_{n}(a_{2}) \rightarrow K_{n}^{*}(a_{3}) \rightarrow K_{n}(a_{4}) \rightarrow \cdots \rightarrow K_{n}(a_{k-1}) \rightarrow H_{n}^{*}(a_{k})
\end{array} \]

\[ \begin{array}{c}
(k: \text{odd})
\end{array} \]

$k$: even

\[ \begin{array}{c}
K_{n}^{*}(a_{1}) \rightarrow K_{n}(a_{2}) \rightarrow K_{n}^{*}(a_{3}) \rightarrow K_{n}(a_{4}) \rightarrow \cdots \rightarrow K_{n}^{*}(a_{k-1}) \rightarrow H_{n}(a_{k})
\end{array} \]

\[ \begin{array}{c}
(k: \text{even})
\end{array} \]

\textbf{FIGURE 7.}

\[ \begin{array}{c}
K_{n}(a)
\end{array} \]

\[ \begin{array}{c}
K_{n}^{*}(a)
\end{array} \]

\textbf{FIGURE 8.}
In Figures 6 and 7, the marks labeled $K_n(a_1), K^*_n(a_1), H_n(a_2), H^*_n(a_2), T_n, T^*_n$, represent the diagrams in Figures 8 through 13.

(i) Case $n = 2r + 1$ ($r = 1, 2, \cdots$): Figure 8 through 10.

(ii) Case $n = 2r + 2$ ($r = 0, 1, \cdots$): Figure 11 through 13.

When $r = 0$, $T_n$ is empty. The square in the graph of $T^*_n$ is the proper image of $L_\infty$ under the mapping $\sigma^{-1}$. In the case where $f$ belongs to $D_n^{+}$, the proper image of $L_\infty$ under the mapping $\sigma^{-1}$ intersects a component in the part labeled $K^*_n(a_1)$, as is illustrated in Figure 14.

Chapter I. Curves with the property $(P)$. In this chapter, we state several elementary facts on level curves of rational functions of $P$-type.

§ 1. Definition.

1. Let $f$ be a non-constant rational function on a compact smooth algebraic surface $X$.

Proposition 1. If the set $I_f$ is not empty, then there exists a pair of a primitive rational function $f_1$ on $X$ and a rational function $\varphi$ on $P$ such that $f = \varphi \circ f_1$. If another pair of a primitive rational function $f_1$ on $X$ and a rational function $\lambda$ on $P$ satisfies the condition $f = \lambda \circ f_1$, then...
then there exists an analytic automorphism (a linear fractional transformation) $T$ of $P$ such that $f_1 = T \circ f_0$ and $\varphi = \lambda \circ T$.

**Proof.** We recall the “Stein factorization”. Let $\sigma: M \to X$ be a resolution of the indetermination points of $f$ by a finite sequence of $\sigma$-processes. The mapping $\sigma$ of the smooth surface $M$ onto $X$ is holomorphic and the pull-back $\sigma^*f$ of $f$ by $\sigma$ has no indetermination point. If two points $p_i, p_j$ on $M$ are contained in the same connected component of $(\sigma^*f)^{-1}(\sigma^*f(p_i))$, then we write the fact as $p_i \sim p_j$. The relation $\sim$ is an equivalence relation. Let $\pi$ be the canonical projection of $M$ onto the quotient space
RATIONAL FUNCTIONS OF C*-TYPE

$H_n(a)$

$H_2^*(a)$

$R = M/\sim$ provided with the quotient topology. The space $R$ is connected and there exists a unique mapping $\varphi$ of $R$ onto $P$ such that $\sigma^*f = \varphi \circ \pi$. By Stein’s theorem on a proper holomorphic mapping, $R$ has an analytic structure such that $\varphi$ and $\pi$ are holomorphic. (cf. Nishino [4])

Set $\Sigma(f) = \sigma^{-1}(I_f)$. The algebraic curve $\Sigma(f)$ consists of a finite number of non-singular rational curves. Let $B_i$ ($l = 1, 2, \cdots, m$) be the basic sections of $\sigma^*f$. The mapping $\pi|_{B_i}$ of $B_i$ into $R$ is non-constant and holomorphic. Hence $R$ is the Riemann sphere. Setting $f_0 = (\sigma^{-1})^*\pi$, we see that $f_0$ is a primitive rational function on $X$ and that $f = \varphi \circ f_0$. We
easily get the remainder of the statement.

Suppose that \( f \) is primitive. If there exists a neighbourhood \( U_a \) of a point \( a \) on \( P \) such that the triple \( \langle N, f|_N, U_a \rangle \) defines a trivial topological fibre bundle where \( N = f^{-1}(U_a) \), then \( a \) is called a regular value of \( f \). The fibre \( L_a = f^{-1}(a) \), that is, the level curve of \( f \) with regular value \( a \), is called a regular level curve of \( f \). A regular level curve is irreducible, non-singular and of order one. A point \( c \) on \( P \) which is not a regular value of \( f \) is called a critical value of \( f \). A level curve of \( f \) with critical value is called critical.

Let \( E \) be the set of critical points of the mapping \( \sigma^*f: M \to P \). The image \( \sigma^*f(E) \) of \( E \) consists of a finite number of points on \( P \). Set \( E' = (\sigma^*f)^{-1}\langle \sigma^*f(E) \rangle \) and \( R' = \sigma^*f(M \setminus E') \). For each point \( p \) on \( R' \) except for a finite number of points, the number of intersections of the fibre \( F = \pi^{-1}(p) \) with the sum \( B_1 \cup B_2 \cup \cdots \cup B_m \) of the basic sections of \( \sigma^*f \) equals a constant number independent of \( p \). Hence the set of critical values of \( f \) is finite and prime curves of a rational function on \( X \) are homeomorphic to each other except for a finite number of them.

Let \( n \) be the number of boundary components of general prime curves of \( f \). Denote by \( \#(I_f) \) the number of indetermination points of \( f \). We see easily that \( \#(I_f) \leq m \leq n \). If \( f \) is of \( C^* \)-type, then \( n = 2 \). If \( f \) is of direct \( C^* \)-type, that is, if \( m = 2 \), then \( \#(I_f) \) is 1 or 2. If \( f \) is of torsional \( C^* \)-type, that is, if \( m = 1 \), then \( \#(I_f) = 1 \).
2. The following lemma will be applied again and again in this paper.

**Noether's Lemma.** Let $C$ be a smooth irreducible rational compact curve on a compact smooth rational surface $M$. Suppose that the self-intersection number $(C^2)$ of $C$ is zero. Then there exists a rational function $h$ of $P$-type on $M$ such that $C$ is a regular level curve of $h$.

The rational function $h$ of $P$-type in the above lemma is primitive because the level curve $C$ is of order one.

If a compact algebraic curve $C$ on a smooth rational surface $M$ is a level curve of a primitive rational function $f$ of $P$-type on $M$, then we say that $C$ has the property $(P)$. Suppose that $C$ is also a level curve of a primitive rational function $g$ on $M$. For each compact prime curve $S$ of $g$ not intersecting $C$, the restriction $f|_{S}$ does not take the value $f(C)$ so that $f|_{S}$ must be constant. Hence there exists an analytic automorphism $T$ of $P$ such that $g = T \circ f$. This means that the order of an irreducible component $C_i$ of $C$ is independent of the choice of the function $f$. We call this order of $C_i$ as a component of a curve $C$ with the property $(P)$ the component order of $C_i$ with respect to $C$. Suppose that $C$ is irreducible. The curve $C$ must be non-singular and rational. The self-intersection number $(C^2)$ of $C$ must be zero. By Noether's lemma, there exists a primitive rational function $g$ of $P$-type on $M$ such that $C$ is a level curve of order one of $g$. Hence the (component) order of $C$ as a curve with the property $(P)$ is one.

Suppose that $C$ is reducible. Denote by $[C]$ the divisor defined by the equation $f - f(C) = 0$. (When $f(C) = \infty$, $[C]$ denotes the divisor defined by the equation $1/f = 0$.) The self-intersection number $(\lvert C \rvert^2)$ of the divisor $[C]$ is zero. The virtual genus of $[C]$ is zero because $f$ is of $P$-type. As is well-known, at least one component $D$ of $C$ is an exceptional curve of the first kind. Let $\sigma: M \to \sigma(M)$ be the $\sigma$-process which contracts $D$. The image curve $\sigma(C)$ is the level curve of $(\sigma^{-1})^*f$ with the value $f(C)$. Hence the curve $\sigma(C)$ has the property $(P)$. Let $D'$ be any component of $C$ different from $D$. Clearly, the component order of $\sigma(D')$ with respect to $\sigma(C)$ equals the component order of $D'$ with respect to $C$. Contracting exceptional components of the first kind of $C$ successively, we get an irreducible curve with the property $(P)$. Hence we obtain the following lemma.

**Lemma 1.** An algebraic curve with the property $(P)$ on a smooth rational algebraic surface contains at least one irreducible component of order one.
The following proposition is fundamental in our later discussion.

**Proposition 2.** Suppose there exists an algebraic curve $C$ on a compact smooth rational surface $M$ such that the complement $M \setminus C$ contains no compact algebraic curve. Then there exists a pair of rational functions $(h_1, h_2)$ such that the mapping $\theta: M \rightarrow \mathbb{P} \times \mathbb{P}$, defined by $\theta(p) = (h_1(p), h_2(p))$ ($p \in M$), is a biregular isomorphism of $M$ onto $\mathbb{P} \times \mathbb{P}$ if $C$ satisfies one of the following three conditions (i), (ii), (iii): (i) $C$ consists of two irreducible components $C_1, C_2$ with the property (P) such that $(C_1 \cup C_2) = 1$, (ii) $C$ consists of three irreducible components $C_1, C_2, C_3$ with the property (P) such that $(C_1 \cup C_2) = (C_2 \cup C_3) = 1$ and $(C_1 \cup C_3) = 0$, and (iii) $C$ consists of four irreducible components $C_1, C_2, C_3, C_4$ with the property (P) such that $(C_1 \cup C_2) = (C_2 \cup C_3) = (C_3 \cup C_4) = (C_4 \cup C_1) = 1$ and $(C_1 \cup C_3) = (C_2 \cup C_4) = 0$. (See Figure 15)

**Proof.** Suppose first that (i) holds. There exist primitive rational functions $h_1, h_2$ of $\mathbb{P}$-type on $M$ such that $C_l$ ($l = 1, 2$) is a regular level curve of $h_l$. We prove that each level curve of $h_l$ is irreducible. Assume that a level curve $F_1$ of $h_1$ is reducible. The restriction $h_1|_{C_2}$ of $h_1$ to $C_2$ is a rational function of degree one on $C_2$. Hence $F_1$ intersects $C_2$ at a regular point of $F_1$ transversally. Let $D$ be an irreducible component of $F_1$ which does not intersect $C_2$. We get $D \cap C_1 = D \cap C_2 = \emptyset$. Hence $D$ is a compact curve in $M \setminus C$, a contradiction to the assumption. Therefore each level curve of $h_1$ is irreducible. Similarly, each level curve of $h_2$ is irreducible. For each level curve $F'_l$ of $h_l$, we have $(F'_l \cdot C_l) = 1$. Hence the restriction $h_2|_{F'_1}$ of $h_2$ to $F'_1$ is a rational function of degree one on $F'_1$. This means that $\theta$ is a biregular isomorphism of $M$ onto $\mathbb{P} \times \mathbb{P}$.

Next suppose that (ii) holds. Since $C_1$ and $C_2$ satisfy (i), it is sufficient to prove that there exists no compact curve in $M \setminus (C_1 \cup C_2)$. Let $h_1, h_2$ be the same functions as in the first case. There exists a primitive rational function $h_3$ on $M$ such that $C_3$ is a level curve of $h_3$. Since $C_1 \cap C_3 = \emptyset$, there exists an analytic automorphism $T_1$ of $\mathbb{P}$ such that $h_3 = T_1 \circ h_1$. Suppose that a compact algebraic curve $D$ is contained in

![Figure 15](image-url)
By assumption, $D$ must intersect $C_6$. Hence the restriction $h_3|_D$ is not constant. On the other hand, since $C_1 \cap D = \emptyset$, the restriction $h_1|_D$ must be constant, a contradiction to the fact $h_3 = T_1 \circ h_1$. Hence there is no compact algebraic curve in $M \setminus (C_1 \cup C_2)$.

Finally suppose (iii) holds. Since $C_1$, $C_2$ and $C_3$ satisfy (ii), it is sufficient to prove that there exists no compact algebraic curve in $M \setminus (C_1 \cup C_3 \cup C_5)$. There exists a primitive rational function $h_4$ on $M$ such that $C_4$ is a level curve of $h_4$. Since $C_2 \cap C_4 = \emptyset$, there exists an analytic automorphism $T_2$ of $P$ such that $h_4 = T_2 \circ h_4$. Suppose that a compact algebraic curve $D$ is contained in $M \setminus (C_1 \cup C_3 \cup C_5)$. By assumption, $D$ must intersect $C_4$. Hence $h_4|_D$ is not constant, a contradiction to the fact $h_4 = T_1 \circ h_4$. Thus we have our proposition.

§ 2. Reducible curves with the property $(P)$.

1. A compact connected algebraic curve $E$ on a smooth algebraic surface $V$ is called an (reducible) exceptional curve of the first kind if there exists a sequence of regular mappings of smooth surfaces

$$V = V_n \xrightarrow{\tau_n} V_{n-1} \xrightarrow{\tau_{n-1}} \cdots \xrightarrow{\tau_2} V_1 \xrightarrow{\tau_1} V_0 = V'$$

such that each $\tau_i$ is a $\sigma$-process which contracts an irreducible component of the curve $(\tau_{i+1} \circ \tau_{i+2} \circ \cdots \circ \tau_n)(E)$ on $V_i$ and that $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_n(E)$ is a one point set. Hence $E$ is a tree of rational curves on $V$ and a singular point of $E$ is an ordinary double point where two components of $E$ intersect each other.

An irreducible curve of the first kind is non-singular, rational and with the self-intersection number $-1$. Conversely, by Castelnuovo’s theorem, we know that a compact non-singular irreducible rational curve $E$ with the self-intersection number $-1$ on a smooth algebraic surface $V$ is an exceptional curve of the first kind. If an algebraic curve $C$ on $V$ intersects $E$, then $(\tau(C)) = (C') + k^2$ where $k$ is the multiplicity of the point $\tau(E)$ on the curve $\tau(C)$.

**Lemma 2.** Suppose that $E$ is reducible exceptional curve of the first kind on $V$ which contains only one irreducible component with the self-intersection number $-1$. If $E$ is a linear tree, then the graph of $E$ is given by Figure 16.

Here the numbers $a_i$ and $b_i$ are non-negative integers.

**Proof.** We prove this lemma by induction on the number of the irreducible components of $E$. Let $\tau_i: V \to \tau_i(V)$ be the $\sigma$-process which contracts the component $D_i$ of $E$ whose self-intersection number is $-1$. 
The image $\tau_1(E)$ of $E$ is an exceptional curve of the first kind on $\tau_1(V)$. Since each singular point of $\tau_1(E)$ is an ordinary double point, $D_0$ intersects at most two other components of $E$. The self-intersection number of a component $D$ of $E$ differs from the self-intersection number of $\tau_1(D)$ if and only if $D_0$ intersects $D$. Suppose that $D_0$ intersects two other components $D_1, D_2$ of $E$ and that $(\tau_1(D_1)^2) = (\tau_1(D_2)^2) = -1$. Let $\tau_2: \tau_1(V) \to \tau_2(\tau_1(V))$ be the $\sigma$-process which contracts $\tau_1(D_2)$. The self-intersection number of $\tau_2(\tau_1(D_1))$ is 0. Noether's lemma leads us to a contradiction because $\tau_2(\tau_1(E))$ must be an exceptional curve of the first kind on $\tau_2(\tau_1(V))$ or one point. Hence we see that $\tau_1(E)$ contains only one component with the self-intersection number $-1$. Therefore we have our lemma.

Now, we can study the case where $E$ is non-linear. Let $E$ be an exceptional curve of the first kind which contains only one component $D_0$ with the self-intersection number $-1$. A component $D$ of $E$ is called a diverging component of $E$ if $D$ intersects at least three other components of $E$. Let $A_1'$ be the maximal connected linear tree of components of $E$ containing $D_0$ which does not contain a diverging component of $E$. Then $A_1'$ must intersect a diverging component $D_i$ of $E$ at one of the edge of $A_1'$. Set $A_1 = A_1' \cup D_i$. The curve $A_1$ is an exceptional curve of the first kind. Denote by $\sigma_1$ the composite of $\sigma$-processes which contracts all the components of $A_1'$. The self-intersection number $(\sigma_1(D_i)^2)$ of the image $\sigma_1(D_i)$ must be $-1$. Hence we obtain the following lemma.

**Lemma 2'.** If the number of diverging components of $E$ is $k - 1$, then the outline of the graph of $E$ is as in Figure 17.
intersection number of the diverging component common to $A_{i-1}$ and $A_i$, then the part $A_i$ of the graph becomes the graph of an exceptional curve of the first kind satisfying the condition in Lemma 2. The part of $E$ represented by $A_i$ is called the $i$-th branch of $E$. By the sequence $\tau_0, \tau_{n-1}, \ldots, \tau_2, \tau_1$, the branches of $E$ are contracted successively.

2. An algebraic curve $C$ with the property $(P)$ on a rational smooth surface consists of non-singular rational curves. Each singular point of $C$ is an ordinary double point. If $C$ is reducible, then the self-intersection number $(C_i^2)$ of each component $C_i$ of $C$ is negative. This fact is easily proved by Noether’s lemma. The first Betti number of $C$ is zero.

**Lemma 3.** Suppose that $C$ is a reducible algebraic curve with the property $(P)$ on a smooth rational surface $M$.

(i) If the component order of a component $C_0$ of $C$ (with respect to $C$) is one, then each connected component of the closure of $C \setminus C_0$ is an exceptional curve of the first kind.

(ii) If two irreducible components $C_1, C_2$ of $C$ satisfy $(C_1 \cdot C_2) = 1$ and $(C_1^2) = (C_2^2) = -1$, then the component orders of the curves $C_1, C_2$ with respect to $C$ are both one and $C$ consists of only these curves, that is, $C = C_1 \cup C_2$.

(iii) Suppose that $C_1$ is the unique exceptional component of the first kind of $C$ and suppose that the closure of $C \setminus C_1$ consists of two connected components, both of which contain components of component order one with respect to $C$. Then the graph of $C$ is given by Figure 18.

Here, the numbers $p_i, q_i$ are non-negative integers. When $C$ has $n-1$ components with the self-intersection number smaller than $-2$, the number $r$ in the graph of $C$ means the integer $\lfloor n/2 \rfloor$.

**Proof.** (i) Let $C_l$ ($l = 0, 1, 2, \ldots, n$) be the irreducible components
of $C$ and $v_i$ be the component order of $C_i$ with respect to $C$. By assumption, the divisor $[C] = \sum_{i=0}^{n} v_i[C_i]$ satisfies $([C]^g) = \pi([C]) = 0$, where $\pi([C])$ is the virtual genus of $[C]$. Let $K$ be the canonical bundle of $M$. Since $\pi([C]) = ([C]^g) + (K \cdot [C])]/2 + 1$, we have $(K \cdot [C]) = (K \cdot [C_0]) + \sum_{i=1}^{n} v_i(K \cdot [C_i]) = -2$. By $\pi([C_i]) = 0$, we get $(K \cdot [C_i]) = -2 - (C_i^2) \ (l = 0, 1, \ldots, n)$. The fact $(C_0^2) \leq -1$ shows $(K \cdot [C_0]) \leq -1$. Hence $\sum_{i=1}^{n} v_i(K \cdot [C_i]) = \sum_{i=1}^{n} v_i(-2 - (C_i^2)) \leq -1$. Therefore, there exists a component $C_m$ such that $(C_m^2) \leq -1$. By $(C_m^2) \leq -1$, we get $(C_m^2) = -1$. By Castelnuovo’s theorem, we see that $C_m$ is an exceptional curve of the first kind. Let $\tau: M \rightarrow \tau(M)$ be the $\sigma$-process which contracts $C_m$. The image $\tau(C)$ of $C$ is a curve with the property (P) on $\tau(M)$. The component order of $\tau(C_0)$ with respect to $\tau(C)$ is also one. By induction, we thus have (i).

(ii) Let $\tau: M \rightarrow \tau(M)$ be the $\sigma$-process which contracts $C_i$. The self-intersection number $(\tau(C_i)^g)$ of the image $\tau(C_i)$ is zero. By Noether’s lemma, $\tau(C_i)$ has the property (P). On the other hand, the image $\tau(C)$ has the property (P). Hence $\tau(C) = \tau(C_i)$, $C = C_i \cup C_2$ and the curves $C_1, C_2$ are of order one.

(iii) Let $C_2, C_3$ be the components of $C$ intersecting $C_i$ and $\tau: M \rightarrow \tau(M)$ be the $\sigma$-process which contracts $C_i$. We may suppose that the self-intersection number $(\tau(C_2)^g)$ of $\tau(C_2)$ is $-1$. Assume that the self-intersection number $(\tau(C_3)^g)$ is $-1$. By (ii) of this lemma, we get $C = C_i \cup C_2 \cup C_3$. This is the case where $r = p_i = q_i = 0$ in Figure 18. Assume $(\tau(C_3)^g) \leq -1$. The curve $\tau(C_2)$ is the unique exceptional component of the first kind of $\tau(C)$. By (i) of this lemma, the component order of $\tau(C_0)$ with respect to $\tau(C)$ is not one. Since $\tau(C_i)$ intersects at most two other components of $\tau(C)$, the closure of $\tau(C) \setminus \tau(C_0)$ consists of two connected components, each of which contains a component of order one. By induction, we thus have (iii).

By the following lemma, we can calculate orders of components of $C$ in Lemma 3. The proof is easy and may be omitted.

**Lemma 4.** Let $\Gamma$ be the bicylinder $\{(x, y) \in C^2 \mid |x| < 1, |y| < 1\}$. Denote by $\sigma: \sigma^{-1}(\Gamma) \rightarrow \Gamma$ the $\sigma$-process which gives the blowing-up at the origin $(0, 0) \in \Gamma$. Set $f(x, y) = x^p y^q$ for a pair of non-negative integers $p, q$. Then the curve $\sigma^{-1}((0, 0))$ is a prime curve of order $p + q$ of $\sigma^* f$.

**Chapter II. Functions belonging to the class $(D_0)$.**

§ 1. Outlines.

1. **Sketch of the graph of $\Sigma(f)$**. We say that a primitive rational function $f$ of $C^*$-type on $P^*$ belongs to the class $(D_0)$ if $f$ satisfies the
following conditions: (i) \( f \) has only one indetermination point \( p_0 \), (ii) the level curve of \( f \) with value 0 consists of two prime curves \( S_{01}, S_{02} \) of \( C \)-type and of order one, and the curve \( S_{01} \) intersects \( S_{02} \) transversally at a point in \( P^1 \setminus \{ p_0 \} \), (iii) the level curve of \( f \) with value \( \infty \) consists of only one prime curve \( S_{\infty} \) of \( C \)-type, (iv) the level curve of \( f \) with value 1 consists of only one multiple prime curve \( S_1 \) of \( C^* \)-type, and (v) the other level curves are irreducible and of \( C^* \)-type. By the classification in Chapter III, §1.4, \( f \) is of direct \( C^* \)-type. In this section, we suppose that \( f \) belongs to the class \( (D_0) \) and give a sketch of the graph of \( \sigma^{-1}(p_0) \) and level curves of \( \sigma f \) where \( \sigma : M \to P^2 \) is the minimal resolution of the indetermination point of \( f \).

There exist two basic sections \( B_1, B_2 \) of \( \sigma f \). Since \( \sigma \) is minimal, we may suppose \( (B_i) = -1 \). Each restriction \( \sigma f|_{B_i} \) \((i = 1, 2) \) is a rational function of degree one on \( B_i \). Hence each level curve \( F_c \) of \( \sigma f \) with value \( c \) intersects \( B_i \) transversally at a regular point of \( F_c \). The prime curve of \( \sigma f \) intersecting \( B_i \) must be of order one. Let \( S \) be a prime curve of \( f \) with value \( c \). We denote by \( \bar{S} \) the proper transform by \( \sigma^{-1} \) of the closure of \( S \) in \( P^1 \). If a component \( C \) of \( F_c \) is not a proper transform of the closure of a prime curve of \( f \), \( C \) is a component of \( \Sigma(f) \), and, since \( \sigma \) is minimal, \( (C^*) < -1 \). Suppose that \( F_c \) is reducible. Since \( F_c \) has the property \( (P) \), at least one component of \( F_c \) is an exceptional curve of the first kind. Hence at least one proper transform of the closure of a prime curve with value \( c \) is an exceptional curve of the first kind. Suppose \( c \neq 0 \). The level curve of \( f \) with value \( c \) consists of only one prime curve \( S_c \). Hence the proper transform \( \bar{S}_c \) of the closure of \( S_c \) must be an exceptional curve of the first kind.

Suppose that \( c \neq 0, \infty \). If \( S_c \) is of order one, Lemma 3(i) shows that \( F_c \) is irreducible, that is, \( F_c \) is the proper transform of the closure of \( S_c \). Since \( S_c \) is of \( C^* \)-type and since \( F_c \) is simply connected, \( F_c \) intersects each \( B_j \) \((j = 1, 2) \) at one point and \( (F_c \cap B_j) \neq (F_c \cap B_j) \). If \( S_c \) is multiple, then \( F_c \) must be reducible. Hence \( \bar{S}_c \) is an exceptional curve of the first kind. Since \( \bar{S}_c \cap (B_1 \cup B_2) = \varnothing \) and \( S_c \) is of \( C^* \)-type, \( F_c \) and \( \bar{S}_c \) satisfy the condition of Lemma 3(iii). Especially \( F_c \) must be a linear tree of rational curves. Since a component of \( F_c \) intersecting \( B_1 \cup B_2 \) must be of order one and since \( \Sigma(f) \) must be connected, two components represented at the edge of the graph in Figure 18 intersect \( B_1 \cup B_2 \). We denote by \( K_i \) \((i = 1, 2) \) the component of the level curve \( F_1 \) with value 1 which intersects \( B_i \). Let \( \tau : M \to \tau(M) \) be the composite of \( \sigma \)-processes which contracts \( F_1 \setminus K_i \). Then \( K_j \) \((i \neq j) \) is the last component contracted by those \( \sigma \)-processes.
By assumption, $S_{o1}$ and $S_{o2}$ are of order one. By Lemma 3(i), both of $\tilde{S}_{o1}$ and $\tilde{S}_{o2}$ are exceptional curves of the first kind. By Lemma 3(ii), $F_o = \tilde{S}_{o1} \cup \tilde{S}_{o2}$. We may suppose that $\tilde{S}_{o1}$ intersects $B_1$ and that $\tilde{S}_{o2}$ intersects $B_2$. Since $S_o$ is of $C$-type, $\tilde{S}_o$ intersects $\Sigma(f)$ at only one point.

Here we prove that $B_1$ and $B_2$ have no common point. Suppose that $B_1 \cap B_2 \neq \emptyset$. Since $\Sigma(f)$ is simply connected, $B_i$ intersects $B_i$ at only one point $q$. The level curve of $\sigma^*f$ with value $\sigma^*f(q)$ intersects $B_i \cup B_i$ only at $q$. Hence $\sigma^*f(q) = \infty$. Since a singular point of $\Sigma(f)$ is necessarily an ordinary double point, $F_o = \tilde{S}_o$. Since $(B_2^2) = -1$ and since $\Sigma(f)$ is an exceptional curve of the first kind, $B_i$ intersects at most two other components of $\Sigma(f)$. Hence $S_i$ is the unique multiple prime curve of $C^*$-type. The shape of $\Sigma(f)$ is given by solid lines in Figure 19, while interrupted lines in the figure represent proper transforms of the closure of prime curves of $f$.

Let $\tau:M \to \tau(M)$ be the composite of $\sigma$-processes which contracts the closure of $F_1 \setminus K_1$ and $\tilde{S}_{o1}$. The curve $\tau(B_2) \cup \tau(K_2)$ satisfies the condition of Proposition 2(i). Hence $\tau(M)$ is biregularly isomorphic to $P \times P$. On the other hand, since $\Sigma(f)$ is an exceptional curve of the first kind, $(B_i^2) \leq -2$ (see Lemma 2). Hence $(\tau(B_i^2)) \leq -1$, a contradiction to the fact that there is no algebraic curve with the negative self-intersection number in $P \times P$. Hence $B_1 \cap B_2 = \emptyset$.

Since $\Sigma(f)$ is connected, $\tilde{S}_o \cap (B_i \cup B_i) = \emptyset$ and the closure of $F_o \setminus \tilde{S}_o$ intersects both $B_i$ and $B_i$. Since $B_i$ intersects at most two other components of $\Sigma(f)$, $S_i$ must be the unique multiple prime curve of $C^*$-type. Since $\tilde{S}_o$ is the unique exceptional component of the first kind of $F_o$, we have by Lemma 3(i) that a component of $F_o$ intersecting $B_i \cup B_i$.

![Figure 19.](image_url)
RATIONAL FUNCTIONS OF $C^*$-TYPE

intersects only one more component of $F_\infty$. One or two components of $F_\infty$ intersect $B_1 \cup B_2$. Denote the component of $F_\infty$ intersecting $B_i$ by $T$. By Lemma 3(i), the closure of $F_\infty \setminus T$ is an exceptional curve of the first kind.

Suppose that $T$ also intersects $B_3$. In this case, the shape of $\Sigma(f)$ is as in Figure 20. Let $\tau: M \to \tau(M)$ be the composite of $\sigma$-processes.
which contracts $\bar{S}_{01}$, the closure of $F_1 \setminus K_1$ and the closure of $F_\infty \setminus T$. The curves $\tau(T)$, $\tau(K_1)$, $\tau(B_1)$ satisfy the condition of Proposition 2(ii). Hence $\tau(M)$ is biregularly isomorphic to $\mathbb{P} \times \mathbb{P}$. Hence $(\tau(B_1^2)) = 0$, and $(B_1^2) = -1$.

Suppose that $T$ does not intersect $B_2$. In this case, the shape of $\Sigma(f)$ is as in Figure 21. Let $\rho: M \to \rho(M)$ be the composite of $\sigma$-processes which contracts the branches of the closure of $F_\infty \setminus T$ except the last branch. Then $\rho(F_\infty)$ satisfies the condition of Lemma 3(iii). Hence the component of $F_\infty$ intersecting $B_2$ is the last component of $F_\infty \setminus T$ contracted by this sequence of $\sigma$-processes.

Let $\tau: M \to \tau(M)$ be the composite of $\sigma$-processes which contracts $\bar{S}_{01}$, the closure of $F_1 \setminus K_1$ and the closure of $F_\infty \setminus T$. The curves $\tau(T)$, $\tau(K_1)$, $\tau(B_1)$ satisfy the condition of Proposition 2(ii). Hence $\tau(M)$ is biregularly isomorphic to $\mathbb{P} \times \mathbb{P}$. Thus $(\tau(B_2^2)) = 0$ and $(B_2^2) = -2$.

2. Basic examples.

[A] In the first place, we introduce the simplest rational function $f_0$ which satisfies the conditions (i), (ii), (iii) and (v) in the former subsection. Suppose that the graph of $\Sigma(f_0)$ is as given by solid lines in Figure 22. The mapping $\sigma: M \to \mathbb{P}^2$ is the composite of $\sigma$-processes which contracts exceptional components of $\Sigma(f_0)$ successively. Hence we can calculate the self-intersection numbers of the curves $S_{01}$, $S_{02}$, $S_\infty$ on $\mathbb{P}^2$ by this graph. It shows that the degrees of the curves $S_{01}$, $S_{02}$, $S_\infty$ are 2, 1, 1, respectively. There exists a homogeneous coordinate $(X: Y: Z)$ of $\mathbb{P}^2$ such that $S_{01} = \{YZ - X^2 = 0\}$, $S_{02} = \{X = 0\}$ and $S_\infty = \{Y = 0\}$. By Lemma 4, $S_\infty$ is a prime curve of order 3 of $\sigma^*f_0$ with value $\infty$. Hence, by using a suitable constant $a$, we can write $f_0$ as $(*) f_0 = aX(YZ - X^2)/Y^3$.

Conversely, for a homogeneous coordinate $(X: Y: Z)$, a rational func-

\[\begin{align*}
|F_0| & \quad B_1 \quad |\bar{S}_{01}| \\
|S_0| & \quad -2 \quad |\bar{S}_{02}| \\
|S_\infty| & \quad -1 \quad |\bar{S}_\infty| \\
B_2 & \quad -1
\end{align*}\]

Figure 22.
tion $f_0$ given by $(*)$ has only one indetermination point $(0:0:1)$ whose level curve $L_c$ with value $c$ ($c \neq 0, \infty$) is given by \(aX(YZ - X^2) - cY^3 = 0\). Hence $L_c$ is irreducible, of $C^*$-type and of order one. The graph of $\Sigma(f_0)$ is the same as that in Figure 22.

Consider the level curve $L_1$ of $f_0$ with value 1. By the projective transformation $\Phi: X' = \alpha X, Y' = \alpha^2 Y, Z' = Z$ where $\alpha^3 = a^{-1}$, $f_0$ can be written as $f_0 = X'(Y'Z' - X'^2)/Y'^3$. Hence $L_1 = \{X'Y'Z' - X'^2 - Y'^3 = 0\}$. This fact is used in the next example.

[B] Here, we introduce the simplest rational function $f_{1,0}$ belonging to the class $(D_0)$. The function $f_{1,0}$ belongs to $D_0^*$. Suppose that the graph of $\Sigma(f_{1,0})$ is as in Figure 23. (See also Figure 3.) By Lemma 4, the prime curve $S_1$ of $f_{1,0}$ is of order 2 and the prime curve $S_\infty$ of $f_{1,0}$ is
of order 3. Denote by $T_i$ the component of $F_\sigma$ intersecting $B_i$, and by $	au: M \to \tau(M)$ the composite of $\sigma$-processes with contracts $T_i \cup B_i \cup K_i$ and gives the blowing-up at the intersection of $\tilde{S}_i$ and $K_i$. The graph of the total image of $\Sigma(f_{1,0})$ under $\tau$ is as in Figure 24, where the image $\tau(\tilde{S}_{01})$ is omitted. Denote by $\tau(\tilde{S}_i)$ the proper image of $\tilde{S}_i$ under $\tau$. Removing $\tau(\tilde{S}_i)$ from this graph, we get the same graph as that of $\Sigma(f_0)$, with $\tilde{S}_{02}$ removed, of $f_0$ in Example [A]. Denote by $\omega: \tau(M) \to \omega(\tau(M))$ the composite of $\sigma$-processes which contracts $\tau(\tilde{S}_0)$ and $\tau(T_i) \cup \tau(B_i) \cup \tau(K_i) \cup \tau(\tilde{S}_0)$. The graph of the total image $\Sigma^*$ of $\Sigma(f_{1,0})$ under $\omega \circ \tau$ is as in Figure 25, where the cross represents the point $\omega(\tau(\tilde{S}_0))$. Since

\[ \omega \circ \tau(M) \setminus (\Sigma^* \cup \omega \circ \tau(\tilde{S}_i)) \]

is analytically isomorphic to the complement of an algebraic curve on $P^2$, it contains no compact curve. By Proposition 2(iii), $\omega(\tau(M))$ is biregularly isomorphic to $P \times P$. There exists a rational function $h$ on $\omega(\tau(M))$ such that $\omega(\tau(\tilde{S}_i))$ is a level curve of order one of $h$. Set $f = (\omega \circ \tau \circ \sigma^{-1})^* h$. We may suppose that $h(\omega(\tau(\tilde{S}_{01}))) = \infty$, $h(\omega(\tau(\tilde{S}_{02}))) = 0$ and $h(\omega(\tau(\tilde{S}_{02}))) = 1$. Then $f(S_{01}) = \infty$, $f(S_{02}) = 0$, $f(S_1) = 1$. Denote by $S'$ the level curve of $h$ with value 0 and denote by $S$ the proper transform of $S'$ under $(\omega \circ \tau \circ \sigma^{-1})^{-1}$. Then $f(S \setminus \{p_0\}) = 0$ and $\sigma \circ \tau^{-1}: \tau(M) \to P^2$ is the minimal resolution of the indeterminacy point of $f$. By the graph of $\Sigma(f)$, we see that $f$ if the function $f_0$ in Example [A]. Hence, in a homogeneous coordinate $(X: Y: Z)$ of $P^3$, $\tilde{S}_1 = (XYZ - X^2 - Y^3 = 0)$, $\tilde{S}_{01} = (YZ - X^2 = 0)$, $\tilde{S}_{02} = (Y = 0)$ and $S = \{X = 0\}$. Set $v_{1,-1} = X$, $v_0 = Y$, $v_1 = YZ - X^2$, $u = XYZ - X^3 - Y^3$. Consider the rational function $g = v_1^2/u^2$ on $P^3$. Since $S_1$ is a level curve of order 2 of $f_{1,0}$ and since $\tilde{S}_{02} = \{v_1 = 0\}$, there must exist an analytic automorphism $\phi$ of $P$ such that $f_{1,0} = \phi \circ g$. The level curve of $g$ with value $-1$ is $\{v_1^2 + u^2 = 0\} \setminus \{p_0\}$. Since $v_1^2 + u^2 = v_1^2 + (v_{1,-1}v_1 - v_0)^2 \equiv v_1^2 v_{1,-1} + u^2 (\mod. v_0)$, $v_1 = (v_1^2 + u^2)/v$ is a homogeneous polynomial of $(X, Y, Z)$. Hence the level curve of $g$ with value $-1$ is $S_{01} \cup S_{02}$. Thus $\tilde{S}_{01} = \{v_2 = 0\}$ and $f_{1,0} = (g + 1)/g = v_0 v_1/v_1$. 

\[ \text{Figure 25.} \]
Conversely, starting from the graph of $\Sigma(f_0)$, we can construct $f_{1,0}$. Consider the rational function $f_0 = X(YZ - X^3)/Y^3$ for a homogeneous coordinate $(X: Y: Z)$ of $P^2$. Removing $\tilde{S}_{02}$ from the graph of $\Sigma(f_0)$ and applying to the graph the operation inverse to $\tau$, we get Figure 26(a). Contracting the encircled components, we get Figure 26(b). By Proposition 2(iii), there exists a rational function $\tilde{h}$ such that the curves corresponding to the vertical solid lines are level curves of $\tilde{h}$. We may suppose that $\tilde{h}$ takes the values 0, 1, and $\infty$ on the proper images of the curves $\tilde{S}_\infty$, $\tilde{S}_1$, and $\tilde{S}_{01}$ of $f_0$. The graph of $\Sigma(f)$ of the transform $f$ of $\tilde{h}$ on $P^2$ is as in Figure 23, which assures the existence of the function belonging to $D^q_0$.

Another proof of this fact is as follows. A smooth rational surface $M$ and a rational function $f$ on $M$ for which the graph of $\Sigma(f)$ is as in Figure 23 are constructed by a finite sequence of blowing-ups on $P \times P$. Let $\sigma': M \to N$ be the composite of $\sigma$-processes which contracts $\Sigma(f)$. By the formula on the Euler characteristic in Chapter III, §1.4, the Euler characteristic of $N$ is 3. Hence $N \cong P^2$.

[C] In the last place, we introduce a rational function $f_{1,1}$ belonging to $D^q_1$. Suppose that the graph of $\Sigma(f_{1,1})$ is as in Figure 27. (See Figures 4 and 5.) The order of $S_{1}$ is 2. Denote by $\rho_1: M \to \rho_1(M)$ the composite of $\sigma$-processes which contracts the encircled components of $\Sigma(f_{1,1})$ in Figure 27. The graph of $\rho_1(\Sigma(f_{1,1}))$ is given in Figure 28. Denote by $C$ the component of $\Sigma(f_{1,1})$ with the self-intersection number $-(a+1)$. The self-intersection number $(\rho_1(C)^\circ)$ of the image $\rho_1(C)$ is $-1$. Denote by $\omega: \rho_1(M) \to \omega(\rho_1(M))$ the composite of $\sigma$-processes which contracts the components of $\rho_1(\Sigma(f_{1,1}) \cup \tilde{S}_\infty \cup \tilde{S}_{\infty})$ encircled by fine interrupted lines. The
The graph of $\omega(\rho_i(S_{\ast}))$ is as in Figure 29. The interrupted curved line represents the image $\omega(\rho_i(\tilde{S}))$. The curve $\omega(\rho_i(\tilde{S}))$ is tangent to $\omega(\rho_i(C))$ with order $a - 1$, that is, $(\omega(\rho_i(\tilde{S})) \cdot \omega(\rho_i(C))) = a$. By Proposition 2(ii), there exist rational functions $h_1, h_2$ on $\omega(\rho_i(M))$ such that the curves corresponding to the vertical lines in the graph of $\omega(\rho_i(\Sigma(f_{1,1}))))$ are level curves of $h_1$ and such that $\omega(\rho_i(C))$ is a level curve of $h_2$. The mapping $\theta$ defined by $\theta(p) = (h_1(p), h_2(p))$ for $p \in \omega(\rho_i(M))$ is a birational isomorphism of $\omega(\rho_i(M))$ onto $P \times P$. Set $R = (\omega \circ \rho_i \circ \sigma^{-1})^* h_1$. The rational function $R$ is of $C$-type on $P^1$. The graph of $\rho_i(\Sigma(f_{1,i}))$ is that of the minimal resolution of indetermination points of $R$. In the graph of $\omega(\rho_i(\Sigma(f_{1,i}))))$, the crosses represent the images $\omega(\rho_i(\tilde{S}_{\ast}))), \omega(\rho_i(\tilde{S}_{\ast})))$. Hence the curves $S_{\ast}, S_{\ast}$ are level curve of $R$. Set $\varphi = (\omega \circ \rho_i \circ \sigma^{-1})^* h_2$. The rational function $\varphi$ is of $C^*$-type on $P^1$. Denote by $S'$ the level curve of $h_2$ which contains the image $\omega(\rho_i(\tilde{S}_{\ast})))$. Denote by $\tilde{S}$ the proper transform of $S'$ under the mapping $(\omega \circ \rho_i)^{-1}$. The curve $(\rho_i(\Sigma(S_{\ast})) \setminus \{p_0\}) \cup S_{\ast}$ is a level curve of $\varphi$ and the curve $S_{\ast}$ is another level curve of $\varphi$. Denote by $\mu: \rho_i(M) \rightarrow \mu(\rho_i(M))$ the composite of $\sigma$-processes which contracts the component of $\rho_i(\Sigma(f_{1,i}))))$ encircled by a fine curved line in the graph of $\rho_i(\Sigma(f_{1,i}))$ in Figure 28. The graph of $\mu(\rho_i(\Sigma(f_{1,i}))))$ is as in Figure 30. In this graph, $S''$ is the proper transform of $S = \sigma(\tilde{S})$ under the mapping $\mu \circ \rho_i \circ \sigma^{-1}$. This graph is the same as that of $\Sigma(f_{1,i,0})$, with $S_{\ast}$ removed, of $f_{1,0}$ in
Example [B]. By Proposition 2(iii), we can show that there exists a rational function $k$ on $\mu(\rho(M))$ having the same family of level curves as that of $\sigma f_{1,0}$ in Example [B]. Especially, the level curve of $k$ with value $k(\mu \circ \rho(\tilde{S}_{\infty}))$ consists of two prime curves one of which is $\mu \circ \rho(\tilde{S}_{\infty})$ and another of which is an algebraic curve relevant to $\tilde{S}_{\infty}$ of $f_{1,0}$ in Example [B]. Hence, in a homogeneous coordinate $(X:Y:Z)$ of $P^2$, $S_{\infty} = \{v_1 = 0\}$, $\tilde{S}_{\infty} = \{v_2 = 0\}$ and $S = \{u = 0\}$. Since $R$ is primitive and since $\deg v_1$ and $\deg v_2$ are coprime, we may suppose that $R = R_{1,1} = v_2^{\deg(v_1)/\deg(v_2)} = v_2^{\delta}/v_2^{\delta}$. From the graph of $\rho(\Sigma(f_{1,0}))$, we see that the curve $\rho(\tilde{S}_{\infty})$ is a prime curve of order one of $\omega^*h_2$ and that $\rho(\tilde{S})$ is a prime curve of order one of $\omega^*h_2$. Hence we may suppose that $\psi = \psi_{1,1} = v_{1,1}/v_{1,1}$. Then $\omega(\rho(\tilde{S}))$ is defined by the equation $h_2 = P(h_1)$ on $\omega(\rho(M))$, where $P(z)$ is a polynomial in $z$ of degree $\alpha$. Hence $S_1$ is the level curve of order one of the rational function $\psi_{1,1} - P(R_{1,1})$ on $P^2$ with value 0. Hence, for a homogeneous polynomial $P_a(z_1, z_2)$ in $(z_1, z_2)$ of degree $\alpha$ with $P_a(1, 0) \neq 0$, we have $S_1 = \{uv_2^{\alpha+1} - P_a(v_2^{\alpha}, v_2^{\alpha})v_2 = 0\}$.

![Figure 30](image-url)

Set $u_1 = uv_2^{\alpha+1} - P_a(v_2^{\alpha}, v_2^{\alpha})v_2$. Then $\tilde{S}_1 = \{u_1 = 0\}$. Since $\psi_{1,1} - P(R_{1,1})$ is a rational function of degree $\deg v_2 + \alpha \deg v_1^\delta = 10\alpha + 5$ on $P^2$, $\tilde{S}_1$ is of degree $10\alpha + 5$. Consider the rational function $g = v_2^{\deg(u_1)/u_2^\delta}$ on $P^2$. Since $S_1$ is a level curve of order 2 of $f_{1,1}$ and since $\tilde{S}_\infty = \{v_1 = 0\}$, there must exist an analytic automorphism $\tilde{f}_{1,1}$ of $P$ such that $f_{1,1} = \phi(g)$. The level curve of $g$ with the value $-1$ is $(v_1^{\deg(u_1)} + u_2^\delta = 0) \setminus \{p_0\}$. Since $v_1^{\deg(u_1)} + u_2^\delta \equiv v_1^{\deg(u_1)} + u_2^\delta \equiv v_1^{\delta(\alpha+1)} = v_1^{\delta(\alpha+1)}(v_1^{\delta} + u_2^\delta) = v_1^{\delta(\alpha+1)}v_2 = 0 \pmod{v_2}$, we see that $w_1 = (v_1^{\deg(u_1)} + u_2^\delta)/v_2$ is a homogeneous polynomial of $(X, Y, Z)$. Hence the level curve of $g$ with the value $-1$ is $S_{\infty} \cup S_{\infty}$. We obtain $\tilde{S}_{\infty} = \{w_1 = 0\}$ and $f_{1,1} = (g + 1)/g = v_2w_2/v_2^{\delta+1}$. Conversely, starting from the graph of $\Sigma(f_{1,0})$, we can construct the
graph of $\Sigma(f_{1,1})$, using Proposition 2(ii). It assures the existence of the functions $R_{1,1}$, $\varphi_{1,1}$, $f_{1,1}$, for a homogeneous polynomial $P_n(z_1, z_2)$ in $(z_1, z_2)$ satisfying $P_n(1, 0) \neq 0$.

§ 2. Determination.

1.1. Now we prove that, if $f$ belongs to the class $(D_0)$ and if the graph of $\Sigma(f)$ is linear, the graph of $\Sigma(f)$ is given by Figure 3 in Chapter 0. Suppose that $T$ intersects $B_1$. Then, as was seen in §1.1, the shape of $\Sigma(f)$ must be as in Figure 31. Let $\rho: M \to \rho(M)$ be the $\sigma$-process which contracts $\tilde{S}_\omega$. Since $\sigma$ is minimal, $(T') \leq -2$. So $(\rho(T')) \leq -1$, a contradiction to the fact that $\rho(T)$ must have the property $(P)$. We thus obtain $T \cap B_1 = \emptyset$ and $(B_1^2) = -2$.

![Figure 31](image)

For simplicity, we suppose that $F_1$ has at least one component with the self-intersection number smaller than $-2$. Let $K_l$ ($l = 1$ or 2) be the component of $F_1$ corresponding to the left edge of the graph in Lemma 3(iii) and $K_m$ ($m = 2$ or 1) be the component corresponding to the right edge of the graph.

(i) The case $(l, m) = (1, 2)$. By Lemma 2, the graph of $\Sigma(f)$ must be as in Figure 32. In Figure 32, the portion in the parenthesis may not exist. First we prove $p_1 = 0$. Suppose that $p_1 > 0$. Then the number of components of $F_1$ with the self-intersection number smaller than $-2$

![Figure 32](image)
is $2r$. Hence the portion in the parenthesis must exist. By Lemma 3(iii), we have $k = 0$ and $p_1 + 3 = p_1 + 2$, a contradiction. So we obtain $p_1 = 0$.

If the portion in the parenthesis does not exist, we have $q_1 + 3 = q_1 + 2$ by Lemma 3(iii), a contradiction. Hence the portion in the parenthesis must exist. Let $\rho: M \to \rho(M)$ be the $\sigma$-process which contracts $\hat{S}_\omega$. As was seen in §1.1, $\rho(F_{\omega})$ satisfies the condition of Lemma 3(iii). We thus see that the graph of $F_{\omega} \cap \Sigma(f)$ must be as in Figure 33.

\[\begin{array}{c}
| & q_r & | \\
| & \downarrow & | \\
pr + 3 & 2 & | \\
\end{array}\]

\[\text{(pr+3=q_r+1)}\]

**Figure 33.**

Therefore, the graph of $F_1$ must be as in Figure 34.

\[\begin{array}{c}
B_1 & \leftarrow & k & 3 & q_1 & \cdots & q_{r-1} & 1 \\
& & pr & pr-1 & q_{r-1}+3 & q_{r-1}+3 & q_{r-1}+3 & K_1 \\
& & qr+3 & qr+3 & qr+3 & pr+3 & p_1+3 & B_2 \\
\end{array}\]

\[\text{(p_1=0)}\]

**Figure 34.**

By Lemma 3(iii), we see $k + 2 = q_1 + 3$, $q_1 + 3 = q_2 + 3$, $q_r + 3 = q_{r-1} + 3$, $(q_{r-1} - 2) + 3 = q_r + 3$; $3 = p_2 + 3$, $p_2 + 3 = p_3 + 3$, $p_3 + 3 = p_r + 3$. Hence $p_1 = p_2 = \cdots = p_r = 0$; $q_r = 2$, $q_{r-1} = q_{r-2} = \cdots = q_1 = 4$ and $k = 5$. Thus $f$ belongs to $D_0^r$ ($r = 1, 2, \cdots$).

(ii) The case $(l, m) = (2, 1)$. By Lemma 2, the graph of $\Sigma(f)$ must be as in Figure 35.

\[\begin{array}{c}
\tilde{S}_1 & \leftarrow & k & 3 & p_{r-1} & p_r & pr & q_r & q_{r-1} & q_{r-1} & q_{r-2} & \cdots & q_1 & 12 \\
& & pr & pr-1 & pr & pr & pr & q_{r-1}+3 & q_r+3 & T & T & T & T \\
& & q_{r-1}+3 & q_r+3 & q_r+3 & q_r+3 & q_r+3 & pr+3 & p_1+3 & p_2+3 & p_2+3 & p_1+3 & p_1+3 & p_1+2 \\
\end{array}\]

\[\text{(p_1>0)}\]

\[\begin{array}{c}
\tilde{S}_1 & \leftarrow & k & p_r & pr & q_r & q_{r-1} & q_{r-2} & 12 & q_{r-1}+3 & q_r+3 & T & T & T & T \\
& & pr & pr & pr & pr & pr & q_{r-1}+3 & q_r+3 & T & T & T & T \\
& & q_{r-1}+3 & q_r+3 & q_r+3 & q_r+3 & q_r+3 & pr+3 & p_1+3 & p_2+3 & p_2+3 & p_1+3 & p_1+3 & p_1+2 \\
\end{array}\]

\[\text{(p_1=0)}\]

**Figure 35.**
The portion in the parenthesis may not exist. If \( p_1 = 0 \), the number of components of \( F_1 \) with the self-intersection number smaller than \(-2\) is \( 2r - 1 \) and a component of \( F_\infty \) cannot exist in the graph, a contradiction. Hence \( p_1 \) must be positive.

By Lemma 3(iii), the graph of \( F_\infty \cap \Sigma(f) \) must be as in Figure 36.

\[
\begin{array}{c}
\begin{array}{c}
B_i \\
\end{array}
\end{array}
\xrightarrow{q_r+3}
\begin{array}{c}
\begin{array}{c}
B_i \\
\end{array}
\end{array}
\]
\[
\xrightarrow{p_r-1}
\]

\[
(q_r+3=p_r-1)
\]

FIGURE 36.

Therefore, the graph of \( F_1 \) must be as in Figure 37.

\[
\begin{array}{c}
\begin{array}{c}
B_i \\
\end{array}
\end{array}
\xrightarrow{q_r+3}
\begin{array}{c}
\begin{array}{c}
K_i \\
\end{array}
\end{array}
\]
\[
\xrightarrow{p_r+3}
\begin{array}{c}
\begin{array}{c}
S_i \\
\end{array}
\end{array}
\]

FIGURE 37.

By Lemma 3(iii), we see \( p_1 = k \), \( p_2 = p_1 - 1 \), \( p_3 = p_2 \), \( \cdots \), \( p_r = p_{r-1} ; \)
\( q_1 = 0, q_2 = q_1 \), \( \cdots \), \( q_r = q_{r-1} \). Hence \( q_1 = q_2 = \cdots = q_r = 0, p_2 = p_3 = \cdots = p_r = 4, p_1 = 5, k = 5 \). So we see that \( f \) belongs to \( D_{2r+1} (r = 0, 1, \cdots) \).

By (i) and (ii), we obtain the graph of \( \Sigma(f) \) in Figure 3 in Chapter 0, § 2.

1.2. Let \( f_{n,0} \) be the rational function belonging to \( D_n^0 \). Let \( \sigma; M_n \to P^2 \) be the minimal resolution of indetermination points of \( f_{n,0} \). We define a \( \tau \)-transformation \( \tau_n; M_n \to \tau_n(M_n) \) of \( f_{n,0} \) as follows. If \( n \) is odd, \( \tau_n \) is the composite of \( \sigma \)-processes which contracts the components of \( \Sigma(f_{n,0}) \) represented by the diagram in Figure 38(i). If \( n \) is even, \( \tau_n \) is the composite of \( \sigma \)-processes which contracts the components of \( \Sigma(f_{n,0}) \) represented by the diagram in Figure 38(ii).

\[
\begin{array}{c}
\begin{array}{c}
(i) \\
\end{array}
\end{array}
\xrightarrow{312}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
(ii) \\
\end{array}
\end{array}
\xrightarrow{221}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]

FIGURE 38.

The graph of \( \tau_n(\Sigma(f_{n,0}) \cup \tilde{S}_0 \cup \tilde{S}_1 \cup \tilde{S}_\infty) \) is the graph of \( D_{n-1}^0 \) with \( \tilde{S}_0 \) removed. By Proposition 2(iii), there exists a curve relevant to \( \tilde{S}_0 \) in the graph of \( D_{n-1}^0 \). Hence there must exist a unique rational function \( f_{n-1,0} \) on \( P^2 \) belonging to \( D_{n-1}^0 \) such that \( \Sigma(f_{n-1,0}) = \tau_n(\Sigma(f_{n,0})) \), whose
minimal resolution of indetermination points is \( \sigma \circ \tau_n^{-1} : \tau_n(M_n) \to P^2 \). Set \( M_{n-1} = \tau_n(M_n) \). Let \( \tau_{n-1} : M_{n-1} \to \tau_{n-1}(M_{n-1}) \) be the \( \tau \)-transformation on \( M_{n-1} \) of \( f_{n-1,0} \).

As stated before, there exists a curve relevant to \( S_0 \) in the graph of \( D_{n-2} \) and a unique rational function \( f_{n-2,0} \) on \( P^2 \) belonging to \( D_{n-2} \) such that \( \sigma \circ \tau_n^{-1} \circ \tau_{n-1} \circ \tau_{n-2} : \tau_{n-2}(M_{n-2}) \to P^2 \) is the minimal resolution of the indetermination points of \( f_{n-2,0} \). Set \( M_{n-2} = \tau_{n-1}(M_{n-1}) \).

Repeating these processes, we get a sequence of \( \tau \)-transformations

\[
M_n \xrightarrow{\tau_n} M_{n-1} \xrightarrow{\tau_{n-1}} M_{n-2} \xrightarrow{\tau_{n-2}} \cdots \xrightarrow{\tau_2} M_2 \xrightarrow{\tau_1} M_1
\]

and rational functions \( f_{j,0} \) (\( j = 1, 2, \ldots, n \)) on \( P^2 \) belonging to \( D_j \) such that \( \Sigma(f_{j,0}) = \check{\tau}_j(\Sigma(f_{n,0})) \), where \( \check{\tau}_j = \tau_{j+1} \circ \tau_{j+2} \circ \cdots \circ \tau_n \).

By Example [B], \( f_{1,0} \) is written as \( f_{1,0} = v_0/v_3^2 \), where \( v_0 = Y, v_1 = YZ - X^3, u = XYZ - X^3 - Y^3 \) and \( v_2 = (v_3^2 + u^3)/v_0 \) in a homogeneous coordinate \((X: Y: Z)\) of \( P^2 \). Denote by \( S_{01}, S_{02}, S_1, S_\infty \) the prime curves \( S_{01}, S_{02}, S_1, S_\infty \) of \( f_{j,0} \), respectively. We get the recurrence relation \( S_{01} = S_{01}^{j+1}, S_1 = S_1^{j+1} \) (\( j = 1, 2, \ldots, n - 1 \)). Suppose that \( v_i \) (\( i = j - 2, j - 1, j, j + 1 \)) are irreducible homogeneous polynomials defining \( S_i \), respectively. Suppose furthermore \( v_{j+1} = (v_3^j + u^{m_j})/v_{j-1} \) and \( v_j = (v_{j-1}^3 + u^{m_{j-1}})/v_{j-2} \), where \( m_i = \deg(v_i) \). Since \( S_{1j} = S_{1j}^1 = \{ u = 0 \} \), the level curve of a rational function \( v_{j+1}v_j/v_j^3 \) with value 1 is \( S_{1j}^1 \). Hence \( f_{j,0} \) is written as \( v_{j+1}v_j/v_j^3 \) and \( 3m_j = m_{j+1} + m_{j-1} \). So we have

\[
v_{j-1}(v_{j-1}^3 + u^{m_{j-1}+1}) = (v_j^3 + u^{m_j})^3 + v_{j-1}^3u^{m_{j+1}} + v_j^3u^{m_{j+1}} (\text{mod. } v_j)
= u^{m_{j+1}}(v_j^3 + v_{j-1}^3) + v_j^3u^{m_{j+1}}v_{j-2} = 0 \quad (\text{mod. } v_j).
\]

By our assumption, \( v_j \) and \( v_{j-1} \) are coprime. Hence \( v_{j+2} = (v_{j+1}^3 + u^{m_{j+1}})/v_j \) is a homogeneous polynomial.

Consider the rational function \( g = v_{j+1}^3/u^{m_{j+1}} \). Since \( \{ v_{j+1} = 0 \} \setminus \{v_0\} \) is the level curve of \( f_{j+1,0} \) of order 3 and since \( S{\check{\tau}_1}^{j+1} = \{ u = 0 \} \), there must exist an analytic automorphism \( \Phi \) of \( P \) such that \( f_{j+1,0} = \Phi \circ g \). Since \( \{ g = -1 \} = \{ v_{j+1} + u^{m_{j+1}} = 0 \} = \{ v_jv_{j+2} = 0 \} \), we have \( \check{\tau}_1^{j+1} = \{ v_{j+2} = 0 \} \), so that \( f_{j+1,0} = (g + 1)/g = v_{j+1}v_j/v_j^3 \). By induction, we get the recurrence formula in Chapter 0, \( \S \ 2.1 \).

Conversely, starting from the graph of \( \Sigma(f_{1,0}) \), we can construct the graph of \( \Sigma(f_{n,0}) \) by a method similar to that in Example [B]. It assures that the function gotten by the recurrence formula for a homogeneous coordinate \((X: Y: Z)\) belongs to \( D_{n,0} \).

The restriction \( f_{n,0}|_V \) of \( f_{n,0} \) to \( V = P^2 \setminus (\check{S}_0 \cup S_1 \cup S_\infty) \) is of proper \( C^* \)-type. Since \( m_n \) and \( m_{n+1} \) are coprime, the rational function \( R = v_{m_n}^{m_n}/v_{m_{n+1}}^{m_{n+1}} \) is primitive. By Theorem 1, the restriction \( R|_V \) is of proper direct \( C^* \)-type. By the classification in Chapter III, \( \S \ 1.4 \), \( R \) must be of \( C \)-type.
Let $\xi : N \to P^1$ be the minimal resolution of the indetermination points of $R$. The graph of $\Sigma(R) = \xi^{-1}(p_i)$ is easily determined, which appears in 2.2 in this section.

2.1. In this subsection, we suppose that the graph of $\Sigma(f)$ is not linear and $\Sigma(f)$ has $k (> 0)$ diverging components. For simplicity, suppose furthermore that $F_1$ has at least three components with the self-intersection number smaller than $-2$. Suppose that $T$ intersects $B_2$. Then, using Lemma 3(iii) for $F_1$, we see that $\Sigma(f)$ cannot be an exceptional curve of the first kind, a contradiction. We thus obtain $T \cap B_2 = \emptyset$.

Define $K_1$ and $K_2$ as in § 1 and define $K_i$ and $K_m$ as in § 2.1.1. Denote by $A_j$ ($j = 1, 2, \ldots, k + 1$) the $j$-th branch of $\Sigma(f)$ (see Chapter I, § 2.1). By Lemma 3(i), the curve $F_\infty \setminus T$ is an exceptional curve of the first kind which contains only one irreducible exceptional curve $S_{\infty}$ of the first kind. Denote by $A'_j$ the $j$-th branch of $F_\infty \setminus T$. By assumption, $F_\infty \setminus T$ has $k$ diverging components. There occur four cases, that is, (i) $(l, m) = (1, 2)$ and $p_i > 0$, (ii) $(l, m) = (2, 1)$ and $p_i > 0$, (iii) $(l, m) = (1, 2)$ and $p_i = 0$, (iv) $(l, m) = (2, 1)$ and $p_i = 0$. In the following, we determine $\Sigma(f)$ in the case (i). In the remaining cases, $\Sigma(f)$ is determined similarly.

By Lemma 3(iii), we know that the graph of $F_1$ is as in Figure 39.

\[
\begin{array}{c}
\text{FIGURE 39.} \\
\end{array}
\]

By Lemma 2, the graph of $A_1$ is as in Figure 40.

\[
\begin{array}{c}
\text{FIGURE 40.} \\
\end{array}
\]

The portion in the parenthesis may not exist. In such a case, we put $a_k = 0$. We denote by $C_j$ the diverging component of $F_\infty \setminus T$ common to $A'_j$ and $A'_{j+1}$. By Lemma 3, the graph of $A'_{k+1}$ is as in Figure 41.

Hence, by Lemma 2, the graph of $A_z$ is as in Figure 42.

In Figure 42, the symbol in Figure 43(i) is the abbreviation for the diagram in Figure 43(ii) and the symbol in Figure 43(iii) is the abbrevi-
By Lemma 3, the graph of $A_k'$ is as in Figure 44. When $a_k = 0$, the portions in the brakets must vanish. By Lemma 2, the graph of $A_3$ is as in Figure 45.

Repeating these processes, we obtain the graph of $A_k$. If $k$ is odd, the graph of $A_k$ is the same as the graph of $A_3$. If $k$ is even, the graph of $A_k$ is the same as the graph of $A_2$. So we see that the graph of $A'_0 \setminus \overline{S}_\infty$ must be as in Figure 46.
The portions in the braket must vanish if $a_1 = 0$. Since $A'_0$ is an exceptional curve of the first kind, we can determine $p_j$, $q_j$ ($j = 1, 2, \cdots, r$) by Lemma 2. If $k$ is odd, the portion in the parenthesis does not exist. We see $q_1 + 3 = 3$, $q_2 + 3 = q_1 + 3$, $\cdots$, $q_r + 3 = q_{r-1} + 3$, $q_r + 5 = p_r + 1$, $p_r + 3 = p_{r-1} + 3$, $\cdots$, $p_3 + 3 = p_2 + 3$ and $p_3 + 3 = (p_2 - 1) + 3$. Hence $q_1 = q_2 = \cdots = q_r = 0$, $p_2 = p_3 = \cdots = p_r = 4$ and $p_1 = 5$. So, if $k$ is odd,
we see that \( f \) belongs to \( D_{k+1}^{\pm} \) \((r = 2, 3, \ldots)\). If \( k \) is even, the portion in the parenthesis must exist. We see \( l + 3 = p_i + 3, (p_i - 1) + 3 = p_i + 3, p_i + 3 = q_i + 3, \ldots, p_{r-1} + 3 = q_r + 3, \ldots, q_1 + 3 = q_1 + 3 \) and \( 3 = q_1 + 3 \). Hence \( q_1 = q_2 = \cdots = q_r = 0, p_2 = p_3 = \cdots, p_r = 4 \) and \( p_i = l = 5 \). Thus, if \( k \) is even, we see that \( f \) belongs to \( D_{k-1}^{\pm} \) \((r = 2, 3, \ldots)\). In the remaining cases, we can determine the graph of \( \Sigma(f) \) similarly. We get the graph of \( \Sigma(f) \) in Chapter 0, \$2.3. Thus we obtain Proposition 0.

2.2. Let \( f = f_{n,k} \) be a rational function belonging to \( D_n^{k\pm} \). Let

\[
M_k = M'_m \xrightarrow{\sigma_m} M'_{m-1} \xrightarrow{\sigma_{m-1}} \cdots \xrightarrow{\sigma_2} M'_1 \xrightarrow{\sigma_1} M'_0 = P^2
\]

be the minimal resolution of the indetermination points of \( f_{n,k} \) by \( \sigma \)-processes. We define the birational mapping \( \rho_k: M_k \to N_k \) as follows. Let \( v, w \) be homogeneous polynomials which define \( S_\infty, S_0 \). The restriction \( f|_V \) of \( f \) to \( V = P^2 \setminus (S_\infty \cup S_1 \cup S_\infty) \) is of proper \( C^* \)-type on \( V \). Consider the rational function \( R = w^{\alpha_1/v^{\alpha_2}} \) on \( P^2 \) for coprime positive integers \( \alpha_1 \) and \( \alpha_2 \) satisfying \( \alpha_1 \deg(w) = \alpha_2 \deg(v) \). Since \( \alpha_1 \) and \( \alpha_2 \) are coprime, \( R \) is primitive. Hence, by Theorem 1, the restriction \( R|_V \) is of proper direct \( C^* \)-type on \( V \). Hence \( R \) is of special type on \( P^2 \) and is not of torsional \( C^* \)-type. By the classification in Chapter III, \$1.4, \( R \) must be of \( C \)-type. Let \( \xi_k: N_k \to P^2 \) be the minimal resolution of the indetermination point \( p_0 \) of \( R \) by \( \sigma \)-processes. The graph of \( \Sigma(R) = \xi_k^{-1}(p_0) \) is easily obtained from the graph of \( \Sigma(f_{n,k}) = \xi^{-1}(p_0) \) where \( \xi_k = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_m \). Set \( \rho_k = (\xi_k^{-1}) \circ \xi_k \). The definition of \( \rho_k \) is independent of the choice of \( v, w \). Another definition of \( \rho_k \) is as follows. Suppose \( a_k > 0 \). If \( f \in D_n^{k+} \) and \( k \) is odd, or, if \( f \in D_n^{k-} \) and \( k \) is even, then \( \rho_k \) is the composite of \( \sigma \)-processes which contracts the components of \( \Sigma(f_{n,k}) \) represented by the graph in Figure 47.

If \( f \in D_n^{k+} \) and \( k \) is even, or, if \( f \in D_n^{k-} \) and \( k \) is odd, then \( \rho_k \) is the composite of \( \sigma \)-processes which contracts the components of \( \Sigma(f_{n,k}) \) repre-
sented by the graph in Figure 48.

If $a_k = 0$, the birational mapping $\rho_k$ is determined similarly. The graph of $\Sigma(R)$ when $k = 1$ and $f \in D_1^{1+}$ is as in Figure 49.

In the following, we suppose $f \in D_1^{-}$. Let $H_0, H_\infty$ be the level curves of $\xi_k^* R$ with values 0, $\infty$, respectively. The graph of $H_0$ is linear. Let $E$ be the unique exceptional irreducible component of the first kind of $\Sigma(R)$. The restriction $\xi_k^* R|_E$ of $\xi_k^* R$ to $E$ is non-constant. Let $C_\infty$ be the irreducible component of $H_\infty$ intersecting $E$ and let $C_0$ be the irreducible component of $H_0$ located at the edge of the graph of $H_0$ which does not intersect $E$. Let the mapping $\omega: N_k \to \omega(N_k)$ be the composite of $\sigma$-processes which contracts $(H_0 \setminus C_0) \cup (H_\infty \setminus C_\infty)$. The image $\omega(\Sigma(R))$ consists of three smooth rational curves $\omega(H_0)$, $\omega(E)$, $\omega(H_\infty)$ with the property (P) such that $(\omega(H_0) \cdot \omega(E)) = (\omega(E) \cdot \omega(H_\infty)) = 1$, $(\omega(H_0) \cdot \omega(H_\infty)) = 0$. By Proposition 2(ii), there exist rational functions $h_1, h_\infty$ of $P$-type on $\omega(N_k)$ with no critical value such that $\omega(H_0)$ and $\omega(H_\infty)$ are level curves of $h_1$ and such that $\omega(E)$ is a level curve of $h_\infty$. Since $R$ is primitive, we may
suppose that $\xi^t R = \omega^* h_i$. Let $S'$ be the level curve of $h_2$ passing through the point $\omega(H_\infty \setminus C_\infty)$. The curve $S'$ satisfies $(S' \cdot \omega(H_\infty)) = (S' \cdot \omega(H_0)) = 1$ and $(S' \cdot \omega(E)) = 0$.

For $k \geq 2$, we define the birational mapping $\tau_k: M_k \to \tau_k(M_k)$ in the following way. If $n = 1$ and $a_k = 0$, then $\tau_k$ is the composite of $\sigma$-processes which contracts one of the parts of $\Sigma(f_{n,k})$ represented by the diagram in Figure 50(i). If $n = 2$ and $a_k = 0$ and if the right hand side of the graph of $\Sigma(f_{n,k})$ is $H_n(a_k)$, then $\tau_k$ is the composite of $\sigma$-processes which contracts the components of $\Sigma(f_{n,k})$ represented by the diagram in Figure 50(ii). If $n = 2$ and $a_{k-1} = 0$ and if the right-hand side of the graph of $\Sigma(f_{n,k})$ is $H_n^*(a_k)$, then $\tau_k$ is the composite of $\sigma$-processes which contracts the encircled components of $\Sigma(f_{n,k})$ in the graphs in Figure 51 and blows-up a point marked by a cross in these graphs. In the other cases, we define $\tau_k$ by $\tau_k = \sigma_{i+1} \circ \sigma_{i+2} \circ \cdots \circ \sigma_m$, where $i$ is the integer such that $\sigma_i: M'_{i-1} \to M'_i$ is the $\sigma$-process which contracts the image of $\sigma_{i+1} \circ \cdots \circ \sigma_m(B_k)$ on $M'_i$. We call the mapping $\tau_k$ the $\tau$-transformation with respect to $f_{n,k}$ and set $M_{k-1} = \tau_k(M_k)$.

Let $\tilde{S}'$ be the proper image of $S'$ under the mapping $\rho_{k-1} \circ \omega^{-1}$. The graph of $\tau_k(\Sigma' \cup \tilde{S}_{m-1} \cup \tilde{S} \cup \tilde{S}')$ is the graph of $D^{k-1}_{n-1}$ with $\tilde{S}_{m-1}$ removed and $\tau_k(\tilde{S}')$ corresponds to $\tilde{S}'$ in this graph. By Proposition 2(iii), there exists a curve relevant to $\tilde{S}_{m-1}$ in the graph of $D^{k-1}_{n-1}$ (see Example [C]). Hence there exists a unique rational function $f_{n,k-1}$ on $P^1$ belonging to $D^{k-1}_{n-1}$.
whose minimal resolution of the indetermination point is $\sigma_k \circ \tau_k^{-1} : M_{k-1} \to \mathbb{P}^2$ and such that $\Sigma(f_{n,k-1}) = \tau_k(\Sigma(f_{n,k}))$. If $f \in D_n^{k+}$, then $f_{n,k-1} \in D_{n-1}^{k+}$. If $f \in D_n^{k-}$, then $f_{n,k-1} \in D_{n-1}^{k-}$. Let $\tau_{k-1} : M_{k-1} \to \tau_{k-1}(M_{k-1})$ be the $\tau$-transformation with respect to $f_{n,k-1}$ and set $\tau_{k-1}(M_{k-1}) = M_{k-2}$. As stated before, we know that there exist curves relevant to $\bar{S}_{02}$ and $\bar{S}_1$ in the graph of $D_{n,2-k}$. There exists a unique rational function $f_{n,k-2}$ on $\mathbb{P}^2$ belonging to $D_{n,2-k}$ such that $\Sigma(f_{n,k-2}) = \tau_{k-1}(\Sigma(f_{n,k-1}))$. Repeating these processes, we get a sequence of $\tau$-transformations $\tau_j : M_j \to M_{j-1}$ and rational functions $f_{n,j}$ on $\mathbb{P}^2$ belonging to $D_{n,2-k}^j$ $(j = 1, 2, \ldots, k)$, such that $\Sigma(f_{n,j}) = \tau_j(\Sigma(f_{n,k}))$ where $\tau_j = \tau_{j+1} \circ \tau_{j+2} \circ \cdots \circ \tau_k$. The level curve of $f_{n,k}$ with value $\infty$ is also the level curve of $f_{n,j}$ with value $\infty$ $(j = 1, 2, \ldots, k-1)$. Hence we may suppose that $v = v_n$. Denote by $S_{01,j}, S_{02,j}, S_{1,j}$ the prime curves $S_{01}, S_{02}, S_1$ of $f_{n,j}$, respectively. We get the recurrence relation $S_{01,j} = S_{02,j+1}$.

Let $u_{n,k}$ be a homogeneous polynomial defining $\bar{S}_{1,k-1} = \bar{s}_k(\bar{S}')$. The proper image of $S_{01,k}$ under the mapping $\omega \circ \xi_k^{-1}$ is the point $\omega(H_0 \cap C_0)$. The proper image of $S_{02,k}$ under $\omega \circ \xi_k^{-1}$ is the point $\omega(H_0 \cap C_0)$. Hence we may suppose $(\omega \circ \xi_k^{-1})^* h_z = u_{n-1}^{-1} \nu_{n-1} / \nu^k$, where $\mu_k$ and $s_k$ are positive integers determined as follows. Let $\eta_k : N_k \to \mathbb{P}^2$ be a minimal resolution of the indetermination point $p_0$ of $\eta = (\omega \circ \xi_k^{-1})^* h_z$. The graph of $\Sigma(\eta) = \eta_k^{-1}(p_0)$ is easily obtained from the graph of $\Sigma(R)$. The graph of the level curve of $\eta_k^* \eta$ with value $\infty$ is given in Figure 52(a) if $n = 1$, given in Figure 52(b) if $f \in D_n^{k+}$ and $k$ is odd, or, if $f \in D_n^{k-}$ and $k$ is even, and given in Figure 52(c) if $f \in D_n^{k+}$ and $k$ is even, or, if $f \in D_n^{k-}$ and $k$ is odd.

![Figure 52](image)

By Lemma 4, the order $s_k$ of the level curve $S_{02,k}$ of $\eta$ with the value $\infty$ is determined. If $f \in D_n^{k+}$, then $s_k = (b_{2n} + 3(-1)^k b_{2n-3})/2$. If $f \in D_n^{k-}$, then $s_k = (b_{2n} + 3(-1)^k b_{2n-3})/2$. The integer $\mu_k$ satisfies $\mu_k = (s_k \deg w - \deg u_{k-1})/m_n$ for $m_n = \deg v_n$. 

### Lemma 4

The order of the level curve $S_{02,k}$ of $\eta$ with the value $\infty$ is determined by $s_k$.
The proper image $S$ of $\tilde{S}_{1,k}$ under the mapping $\omega \circ h^{-1}$ is a smooth rational curve satisfying $(S \cdot \omega(E)) = a_k$ and $(S \cdot \omega(H)) = (S \cdot \omega(H_0)) = 1$. If $a_k > 0$, then $S$ is tangent to $\omega(E)$ with order $a_k - 1$ at the point $\omega(E) \cap \omega(H_0)$. Since $\omega(H_0)$ is the level curve of $h_0$ with value $\infty$ and $\omega(E)$ is the level curve of $h_0$ with value $\infty$, the equation $h_0 = P_{a_k}(h_0)$ on $\omega(N_k)$ defines $S$, where $P_{a_k}(x)$ is a polynomial of degree $a_k$. Hence $S_{1,k}$ is the level curve of the rational function $\psi - P_{a_k}(R)$ of order one with the value $0$. Set $w_{k-1} = w$. The polynomial $u_k = u_{k-1} - v_n^{a_k + a_k \deg(w_{k-1})} - P_{a_k}(w_{k-1})$, $v_n^{a_k \deg(w_{k-1})}w_{k-1}$ defines $S_{1,k}$ where $P_{a_k}(z)$ is a homogeneous polynomial in $(z_1, z_2)$ of degree $a_k$. Hence $S_{1,k}$ is the level curve of the rational function $\psi - P_{a_k}(R)$ of order one with the value $0$. Set $w_{k-1} = w$. The polynomial $u_k = u_{k-1} - v_n^{a_k + a_k \deg(w_{k-1})} - P_{a_k}(w_{k-1})$, $v_n^{a_k \deg(w_{k-1})}w_{k-1}$ defines $S_{1,k}$ where $P_{a_k}(z)$ is a homogeneous polynomial in $(z_1, z_2)$ of degree $a_k$ and $P_{a_k}(1, 0) \neq 0$. The degree of $u_k$ is $a_k m_a \deg(w_{k-1}) + s_a \deg(w_{k-1}) = m_a (\mu_k + a_k \deg(w_{k-1})) + \deg(u_{k-1})$. Suppose furthermore $v_n^{a_k \deg(w_{k-1})} + u_{k-1}^{m_a} = w_{k-1}w_{k-2}$, where $w_{k-2}$ is a homogeneous polynomial defining $S_{02,k-1}$. Hence $f_{n,k} = (g + 1)/g = w_kw_{k-1}/v_n^{a_k \deg(w_{k-1})}$. We obtain the recurrence formula in Chapter 0, §2.1 by induction.

Conversely, starting from the graph of $\Sigma(f_{n,0})$, we can construct the graph of $\Sigma(f_{n,k})$ using Proposition 2 (ii). So the function obtained by the recurrence formula belongs $D_{n,k}$. Since $\deg(v_j)$ and $\deg(v_{j+1})$ are coprime and since $\deg(w_k) = m_a \deg(u_k) - \deg(w_{k-1})$, we can prove inductively that $\deg(w_k)$ is primitive. Kashiwara has proved that a rational function $R$ of $C$-type belonging to $\mathcal{T}_1$ on $P^2$ is written as $R = \Lambda(R_{n,k})$ in a homogeneous coordinate $(X : Y : Z)$, where $\Lambda(z)$ is a rational function of $z$. By a method similar to those in Chapter II and Chapter III, §1, we can prove the result of Kashiwara. Set $\psi_{n,k} = u_{n-1}v_n^{a_k}/w_{n-1}^{a_k}$. As was seen in this section, the mapping $\theta$ defined by $\theta(p) = (R_{n,k}(p), \psi_{n,k}(p))$ is a birational biregular isomorphism of $P^2 \setminus (\tilde{S}_n \cup \tilde{S}_{02,k})$ onto $C^* \times C$.

When $f$ belongs to $D_1$, we can also prove the recurrence formula in Chapter 0, §2.1 similarly. The rational function $R_{1,e}$ of $C$-type belongs
Chapter III. Rational functions of $C^*$-type on $P^2$.

§ 1. Critical level curves.

1. Rational functions of proper $C^*$-type. Here we prove the following.

**Proposition 3.** Let $f$ be a primitive rational function of direct $C^*$-type on $P^2$. There exist a triple $S_1, S_2, S_3$ of prime curves of $f$ and an analytic automorphism $T$ of $P$ such that the restriction $T \circ f|_V$ of $T \circ f$ to $V = P^2 \setminus (\overline{S}_1 \cup \overline{S}_2 \cup \overline{S}_3)$ is of proper direct $C^*$-type, where $\overline{S}_i$ is the closure of $S_i$ in $P^2$.

We use the following lemma to prove this. (See M. Oka [5, p. 233].)

**Lemma 5.** Let $C$ be an algebraic curve on $P^2$. If $C$ has $l$ irreducible components, then the first Betti number $b_1(P^2/C)$ of $P^2/C$ equals $l - 1$.

**Proof of Proposition 3.** Let $\sigma: M \to P^2$ be the minimal resolution of the indetermination points of $f$ by $\sigma$-processes. Let $B_1$ and $B_2$ be basic sections of $\sigma^*f$. We may suppose $(B_2) = -1$. The curve $B_1$ intersects the other components of $\Sigma(f) = \sigma^{-1}(I_f)$ at most two points. Hence, for a suitable analytic automorphism $T$ of $P$, each point $p$ of $B_1$ satisfying $T \circ f(p) \neq 0$ is a regular point of $\Sigma(f)$. Clearly, $\sigma$ is the minimal resolution of the indetermination points of $g = T \circ f$. Since $\sigma^*g|_{B_1}$ is a rational function degree one on $B_1$, each level curve $F_c$ of $\sigma^*g$ with value $c$ intersects $B_1$ at one ordinary point of $F_c$ transversally for each $i$. Since a component of $F_c$ intersecting $B_1$ is of order one, Lemma 3 (i) shows that the union $E_c$ of all prime curves of the level curve $F_c$ which do not intersect $B_1$ is an exceptional curve of the first kind. Let the mapping $\tau: M \to \tau(M)$ be the composite of $\sigma$-processes which contracts the curve $\cup E_c$, where $c$ varies over $C^*$. The restriction $h|_{V'}$ of $h = (\sigma \circ \tau^{-1})^*g$ to $V' = \tau(M) \setminus (\{h = 0\} \cup \{h = \infty\} \cup \tau(B_1) \cup \tau(B_2))$ is of proper direct $C^*$-type. The restriction $\sigma \circ \tau^{-1}|_{V'}$ of the birational mapping $\sigma \circ \tau^{-1}$ is a birational mapping of $V'$ onto $V = \sigma \circ \tau^{-1}(V')$. Since the first Betti number of $V'$ equals 2, Lemma 5 implies that $P^2/V$ is an algebraic curve with three irreducible components $C_1, C_2, C_3$ which are the closures of prime curves $S_1, S_2, S_3$ of $f$, respectively. Since $h|_{V'}$ is of proper direct $C^*$-type, $T \circ f|_{V'}$ is of proper direct $C^*$-type.

2. The first Betti numbers and the Euler characteristics of level curves.

**Lemma (Nishino).** Let $f$ be a primitive rational function on $P^2$. 

Denote by $L$ the topological model of regular level curves of $f$ and let $L_0$ be a critical level curve of $f$. Then $b_1(L_0) \leq b_1(L)$ and $\chi(L_0) \geq \chi(L)$, where $b_1(*)$ and $\chi(*)$ are the first Betti-number and the Euler characteristic of $*$, respectively.

**Proof.** Let $(X:Y:Z)$ be a homogeneous coordinate of $P^2$. Suppose that $f$ is a rational function of degree $n$ on $P^2$. The function $f$ is represented as $f = P/Q$, where $P, Q$ are homogeneous polynomials in $(X:Y:Z)$ of degree $n$. Set $N = n + 3$. We denote homogeneous coordinates in $P^N$ be $W_{k_0k_1k_2}$ where $k_0, k_1, k_2$ are arbitrary non-negative integers such that $k_0 + k_1 + k_2 = n$. The Veronese mapping $v: P^2 \rightarrow P^N$ defined by $W_{k_0k_1k_2} = X^{k_0}Y^{k_1}Z^{k_2}$ is an analytic imbedding of $P^2$ into $P^N$.

The so-called Veronese variety $v(P^2)$ is smooth in $P^N$. Suppose $P = \sum A_{k_0k_1k_2}X^{k_0}Y^{k_1}Z^{k_2}$ and $Q = \sum B_{k_0k_1k_2}X^{k_0}Y^{k_1}Z^{k_2}$. Set $\tilde{P} = \sum A_{k_0k_1k_2}W_{k_0k_1k_2}$ and $\tilde{Q} = \sum B_{k_0k_1k_2}W_{k_0k_1k_2}$. Then, the rational function $\tilde{P}/\tilde{Q}$ of degree one on $P^N$ satisfies $f = v^*(\tilde{P}/\tilde{Q})$.

Let $L_0$ be a critical level curve of $f$ with value $c_0$ and let $c_1 (\neq c_0)$ be a complex number. We regard $H = \{ \tilde{P} - c_1\tilde{Q} = 0 \}$ as the hyperplane at infinity and denote by $(w_1, w_2, \ldots, w_N)$ an inhomogeneous coordinate of $C^N = P^N \setminus H$. The manifold $V = \{ p \in P^2 \setminus I_f | f(p) \neq c_1 \}$. Set $\tilde{Q}^\alpha = \{ \sum_{k=1}^N |w_k|^\alpha < \alpha^2 \} \cap \tilde{V}$ for each real positive number $\alpha$. Denote by $Q^\alpha$ the inverse image $v^{-1}(\tilde{Q}^\alpha)$. Then $\Omega^\alpha \subset \subset V$. Set $\bar{\varphi} = \sum_{k=1}^N |w_k|^\alpha$ and $\varphi = v^*\bar{\varphi}$. The function $\varphi$ is strongly pluri-subharmonic on $V$.

Let $\alpha_\beta$ be a number such that $L_0 \cap Q^\alpha_\beta \neq \emptyset$. Then there is an open neighbourhood $U$ of $c_\beta$ such that $L_\beta = L_{c_\beta} \cap Q^\alpha_\beta \neq \emptyset$ for any $c_\beta \in U$, where $L_\beta$ is the level curve of $f$ with value $c_\beta$. For each real number $\alpha > \alpha_\beta$, we denote by $L_{\alpha_\beta}$ the analytic continuation of $L_\beta$ in $\Omega^\alpha$.

If $\alpha_\beta < \alpha < \beta$, then $b_1(L_{\alpha_\beta}) \leq b_1(L_\beta)$ and $\chi(L_{\alpha_\beta}) \geq \chi(L_\beta)$. To see this, suppose that $l = b_1(L_{\alpha_\beta}) > b_1(L_\beta)$. Let $c_1, c_2, \ldots, c_l$ be $l$ cycles on $L_{\alpha_\beta}$ whose homology classes $[c_1]_\alpha, [c_2]_\alpha, \ldots, [c_l]_\alpha$ generate $H_1(L_{\alpha_\beta}, Z)$. There must exist a set of integers $(m_1, m_2, \ldots, m_l)$ such that at least one $m_j$ is not zero and such that the homology class $[c_\beta]$ of $c = m_1c_1 + m_2c_2 + \cdots + m_lc_l$ is the zero element of $H_1(L_\beta, Z)$. Hence there exists a subdomain $S$ of $L_\beta$ such that $\partial S \subset \text{supp} c$ and such that $S \not\subset L_\beta$. Since $\partial S \subset \Omega^\alpha$, $\varphi|_S$ takes a maximal value at an interior point of $S$, which contradicts the fact that $\varphi$ is strongly pluri-subharmonic. Hence $b_1(L_{\alpha_\beta}) \leq b_1(L_\beta)$. Suppose that $\chi(L_{\alpha_\beta}) < \chi(L_\beta)$. By assumption, there exists a simply connected component of $L_{\alpha_\beta} \setminus L_\beta$ whose boundary is contained in $\partial \Omega^\alpha$. It leads us to a contradiction. Hence $\chi(L_{\alpha_\beta}) \geq \chi(L_\beta)$.

From the above fact, we see that $b_1(L_{\alpha_\beta}) \leq b_1(L_\beta)$ and $\chi(L_{\alpha_\beta}) \geq \chi(L_\beta)$.
Suppose that \( \alpha \) is so large that \( L_{\alpha}^{0} \) intersects all the irreducible components of \( L_{\alpha}^{0} \). Let \( p_{1}, p_{2}, \ldots, p_{k} \) be the singular points of \( L_{\alpha}^{0} \). For a sufficiently large \( \alpha \), we have \( b_{1}(L_{\alpha}^{0}) = b_{1}(L_{\alpha}) \), \( \chi(L_{\alpha}^{0}) = \chi(L_{\alpha}) \) and \( \{p_{j}\} \subset L_{\alpha}^{0} \).

Suppose that \( L_{\alpha} \) is a regular level curve of \( f \). There exist bicylinders \( \Gamma_{j} \) with the center \( p_{j} \) (\( j = 1, 2, \ldots, k \)) in coordinate neighbourhoods such that \( L_{\alpha}^{0} \cap \gamma_{j} \) is simply connected and for \( c \) sufficiently near to \( c_{0} \), \( L_{\alpha}^{0} \cup \gamma_{j} \) is a topological covering surface of \( L_{\alpha}^{0} \cup \gamma_{j} \) (with no branch point and with no relative boundary). Hence \( b_{1}(L_{\alpha}^{0}) \geq b_{1}(L_{\alpha}) \), \( \chi(L_{\alpha}^{0}) \leq \chi(L_{\alpha}) \). Therefore \( b_{1}(L_{\alpha}) \geq b_{1}(L_{\alpha}^{0}) \) and \( \chi(L_{\alpha}) \leq \chi(L_{\alpha}^{0}) \).

**Corollary.** Each prime curve of a rational function of special type on \( P^{2} \) is of C-type or of \( C^{*} \)-type.

**Remark.** Let \( S \) be a prime curve of \( f \) with value \( c \) and \( \{p_{j}\} \) be the set of intersections of \( S \) with the other prime curves of \( f \) with value \( c \). Then \( S' = S \setminus \{p_{j}\} \) satisfies \( b_{1}(S') \leq b_{1}(L) \) and \( \chi(S') \geq \chi(L) \). The proof is almost the same as that of Lemma 6.

### 3. Proof of Theorem 1.

Let \( C \) be an algebraic curve on \( P^{2} \) such that the restriction \( f|_{V} \) of a rational function \( f \) on \( P^{2} \) to \( V = P^{2} \setminus C \) is of proper \( C^{*} \)-type. Let \( \sigma: M \to P^{2} \) be a resolution of the indetermination points of \( f \) by a finite sequence of \( \sigma \)-processes. Set \( \tilde{V} = \sigma^{-1}(V) \). The mapping \( \sigma|_{\tilde{V}}: \tilde{V} \to V \) is a biholomorphic mapping of \( \tilde{V} \) onto \( V \). Set \( \tilde{h} = \sigma^{*}f|_{\tilde{V}} \) and \( M' = M \setminus (\{\sigma^{*}f = 0\} \cup \{\sigma^{*}f = \infty\}) \). Let \( L_{\alpha} \) denote the level curve of \( \tilde{h} \) with value \( c \) and \( F_{\alpha} \) be the level curve of \( \tilde{h}_{\alpha} = \sigma^{*}f|_{M'} \) with value \( c \).

Assume that \( F_{\alpha} \) is reducible. Since \( L_{\alpha} \) is irreducible and of order one, we see by Lemma 3(i) that the closure of \( F_{\alpha} \setminus \tilde{L} \) is an exceptional curve of the first kind. Let \( \tau: M' \to \tau(M') \) be the composite of \( \sigma \)-processes which contracts \( F_{\alpha} \setminus L_{\alpha} \). Set \( h = (\tau^{-1})^{*}\tilde{h} \) and set \( h_{\alpha} = (\tau^{-1})^{*}\tilde{h}_{\alpha} \). The function \( h_{\alpha} \) is of \( P \)-type and each level curve of \( h_{\alpha} \) is irreducible. The image of the union of basic sections of \( \sigma^{*}f \) under \( \tau \) is \( H = \tau(M') \setminus \tau(V) \). So \( H \) intersects each \( \tau(F_{\alpha}) \) at two points transversally. Hence \( H \) is a smooth algebraic curve in \( \tau(M') \).

Let \( g \) be a primitive rational function on \( P^{2} \) whose restriction \( g|_{V} \) to \( V \) does not take the values 0, \( \infty \) on \( V \). Set \( k = (\sigma \circ \tau^{-1})^{*}g|_{\tau(V)} \) and \( k_{\alpha} = (\sigma \circ \tau^{-1})^{*}g|_{\tau(M')} \). Since \( k \) does not take the values 0, \( \infty \) on \( \tau(V) \), \( k \) has no point of indetermination on \( H \). Therefore, if the restriction \( k|_{\tau(L_{\alpha})} \) of \( k \) to some level curve \( \tau(L_{\alpha}) \) of \( h \) is constant, then the restriction of \( k \) to any \( \tau(L_{\alpha}) \) must be constant. Hence \( g \) is of proper \( C^{*} \)-type.

Suppose that the restriction \( k|_{\tau(L_{\alpha})} \) is non-constant for each \( c \in C^{*} \). For a fixed number \( c \), let \( \mu_{c}: \tau(L_{\alpha}) \to C^{*} \) be an analytic isomorphism of \( \tau(L_{\alpha}) \) onto \( C^{*} \). The variable \( \zeta = \mu_{c}(p) \) (\( p \in \tau(L_{\alpha}) \)) is a global coordinate of
τ(L). The mapping \( k \circ \mu_{c}^{-1} \) is a non-constant regular mapping of \( C^{*} \) onto \( C^{*} \). Hence we have \( k \circ \mu_{c}^{-1} = \alpha \zeta^{m} \) for a non-zero constant \( \alpha \) and a non-zero integer \( m \). So each prime curve of \( k \) is non-singular and of order one and the mapping \( k: \tau(V) \rightarrow C^{*} \) is surjective. An irreducible component of \( H \) is a prime curve of \( k_{i} \) with value 0 or \( \infty \). Hence the closure of a prime curve of \( k \) does not intersect \( H \). A prime curve \( S \) of \( k \) is a covering surface over \( C^{*} \) with the projection \( h|_{S}: S \rightarrow C^{*} \) which has no branch point and no relative boundary. Hence each prime curve of \( k \) is of \( C^{*}\)-type. By Lemma 6, each level curve of \( k \) is irreducible. Therefore \( k \) is of proper \( C^{*}\)-type on \( \tau(V) \). This means that the restriction \( g|_{r} \) of \( g \) to \( V \) is of proper \( C^{*}\)-type. If \( f \) is of direct \( C^{*}\)-type, the first Betti number of \( \tau(V) \) is 2. Hence \( g \) must be of direct \( C^{*}\)-type, if \( f \) is of direct \( C^{*}\)-type. This Theorem 1 is established.

4. Classification. Let \( f \) be a primitive rational function of \( C^{*}\)-type on \( P^{2} \) and \( B \) be the union of basic sections of \( \sigma^{*}f \). For a suitable set \( e^{*} = \{ a_{1}, a_{2}, \ldots, a_{m} \} \) of values of \( \sigma^{*}f \) the triple \( F = \langle M^{*}, \sigma^{*}f|_{M^{*}}, P \backslash e^{*} \rangle \) is a locally trivial analytic family of curves with the fibre \( C^{*} \), where \( M^{*} = M \backslash (B \cup (\cup_{k} (\sigma^{*}f)^{-1}(a_{k}))) \). If \( f \) is of direct \( C^{*}\)-type, then \( b_{1}(M^{*}) = m \). If \( f \) is of torsional \( C^{*}\)-type, then \( b_{1}(M^{*}) = m - 1 \).

Set \( C = \bigcup_{k} L_{a_{k}} \) where \( L_{a_{k}} \) is the level curve of \( f \) with value \( a_{k} \). Since \( P^{2} \backslash C \) is homeomorphic to \( M^{*} \), we get the following proposition from Lemma 5.

**Proposition 4.** If \( f \) is of direct \( C^{*}\)-type on \( P^{2} \), then \( f \) has one level curve with two irreducible components and the other level curves of \( f \) are irreducible. If \( f \) is of torsional \( C^{*}\)-type on \( P^{2} \), then each level curve of \( f \) is irreducible.

Since each level curve of \( \sigma^{*}f \) is simply connected, a singular point of each level curve of \( f \) is an ordinary double point. Since each level curve of \( \sigma^{*}f \) intersects \( B \) at most two points, a connected component of each level curve of \( f \) has at most two boundary points. Each prime curve of \( f \) is smooth and non-compact. If a level curve of \( f \) is irreducible, of \( C^{*}\)-type and of order one, then it is regular. By Lemma 6 and Remark after it, we see the following facts for a critical level curve \( L_{0} \) of \( f \). (a) If \( \chi(L_{0}) = 0 \), then \( L_{0} \) is of \( C^{*}\)-type, irreducible and multiple. (b) If \( \chi(L_{0}) = 1 \), then one of the following three cases occurs. (i) \( L_{0} \) consists of two prime curves both of which are of \( C\)-type. They intersect each other at one point in \( P^{2} \backslash I_{f} \) transversally. (ii) \( L_{0} \) consists of two prime curves disjoint in \( P^{2} \backslash I_{f} \) and one of them is of \( C\)-type and the other is of \( C^{*}\)-type. (iii) \( L_{0} \) is irreducible and of \( C\)-type. (c) If \( \chi(L_{0}) = 2 \), then
L_0 consists of two prime curves disjoint in \( P^2 \setminus I_f \), both of which are of \( C \)-type. Let \( e = \{ c_1, c_2, \ldots, c_m \} \) be the set of critical values of \( f \) and \( L_i \) be the level curve of \( f \) with value \( c_i \). Set \( X = P^2 \setminus (I_f \cup (\cup_{i=1}^m L_i)) \). The triple \( \langle X, f|_X, P \setminus e \rangle \) is a locally trivial analytic family of curves with the fibre \( C^* \). Hence \( \chi(X) = \chi(C^*) \chi(P \setminus e) \). Therefore, \( \chi(P^2 \setminus I_f) = \chi(C^*) \chi(P) + \sum_{i=1}^m (\chi(L_i) - \chi(C^*)) \). Since \( \chi(C^*) = 0 \), we see \( \chi(P^2 \setminus I_f) = \sum_{i=1}^m \chi(L_i) \). Suppose that \( f \) has two points of indetermination. Since \( \chi(P^2 \setminus I_f) = 1 \), we have \( \sum_{i=1}^m \chi(L_i) = 1 \). By Lemma 6, \( \chi(L_i) \geq 0 \) for each \( i \). Hence one critical level curve \( L_{i_0} \) satisfies \( \chi(L_{i_0}) = 1 \). If \( i \neq i_0 \), then \( \chi(L_i) = 0 \) and all the level curves of \( f \), except \( L_{i_0} \), are irreducible. Since \( f \) must be of direct \( C^* \)-type, Proposition 4 shows that \( L_{i_0} \) must be reducible. Therefore, \( f \) belongs to one of the following two classes.

Class (A): A level curve \( L_1 \) satisfies the condition in (b), (i) and the other level curves are irreducible and of \( C^* \)-type.

Class (B): A level curve \( L_1 \) satisfies the condition in (b), (ii) and the other level curves are irreducible and of \( C^* \)-type.

Suppose that \( I_f \) consist of only one point. Since \( \chi(P^2 \setminus I_f) = 2 \), we have \( \sum_{i=1}^m \chi(L_i) = 2 \). We may suppose that \( \chi(L_1) = \chi(L_2) = 1 \) and \( \chi(L_i) = 0 \) for \( i \neq 1, 2 \), or that \( \chi(L_1) = 2 \) and \( \chi(L_i) = 0 \) for \( i \neq 1 \). Assume that \( \chi(L_1) = \chi(L_2) = 1 \). If \( f \) is of direct \( C^* \)-type, then, by Proposition 4, we may suppose that \( L_1 \) satisfies the condition in (b), (iii) and \( L_2 \) satisfies the condition in (b), (i) or (b), (ii). Therefore, \( f \) belongs to one of the following two classes.

Class (C): A level curve \( L_1 \) satisfies the condition in (b), (iii) and another level curve \( L_2 \) satisfies the condition (b), (ii) and, furthermore, the other level curves are irreducible and of \( C^* \)-type.

Class (D): A level curve \( L_1 \) satisfies the condition in (b), (iii) and another level curve \( L_2 \) satisfies the condition (b), (i) and, furthermore, the other level curves are irreducible and of \( C^* \)-type.

If \( f \) is of torsional \( C^* \)-type, then, by Proposition 4, each level curve of \( f \) is irreducible. Hence we have the following class.

Class (T): Two level curves \( L_1, L_2 \) satisfy the condition (b), (iii) and the other level curves are irreducible and of \( C^* \)-type.

Assume that \( \chi(L_i) = 2 \). Then we have the following class.

Class (E): A level curve \( L_1 \) satisfies the condition in (c) and the other level curves are irreducible and of \( C^* \)-type.

By Proposition 4, \( f \) is of torsional \( C^* \)-type if and only if \( f \) belongs to Class (T). This fact is used in \( \S \) 3. Suppose that \( R \) is a primitive rational function of \( C \)-type on \( P^2 \). By the same method as in the proof of
Proposition 4, we can prove that each level curve of \( R \) is irreducible. By Lemma 6, each level curve of \( R \) is of \( C \)-type.

§ 2. Functions of direct \( C^* \)-type.

1. Functions of \( C^* \)-type with one point of indetermination. In this subsection, we determine the rational functions of direct \( C^* \)-type on \( P^2 \) with one point of indetermination. By Proposition 3, there is an analytic automorphism \( T \) on \( P \) such that \( T \circ f|_V \) is a rational function of proper \( C^* \)-type on \( V = P^2 \setminus (C_1 \cup C_2 \cup C_3) \), where each \( C_i \) (\( i = 1, 2, 3 \)) is the closure of the prime curve \( S_i \) of \( f \). Let \( S_a \) be a prime curve of \( C \)-type of \( f \) with value \( a \) different from \( S_1, S_2, S_3 \). Since \( S_a \setminus V \) is a prime curve of \( C^* \)-type of \( f|_V \), the level curve of \( f \) with the value \( a \) must satisfy the condition (b) (i) in §1.4 of this chapter. By the classification of rational functions of \( C^* \)-type in that subsection, at least two of the prime curves \( S_1, S_2, S_3 \) are of \( C \)-type. We suppose that \( S_1 \) and \( S_2 \) are of \( C \)-type.

Let \( t_i \) be an irreducible homogeneous polynomial defining \( C_i \) for each \( i \). The function \( T \circ f \) is written as \( T \circ f = a_1 t_1^{a_1} t_2^{a_2} t_3^{a_3} \) for a non-zero constant \( a_1 \) and three integers \( a_1, a_2, a_3 \) satisfying \( \sum_{i=1}^{3} a_i \deg t_i = 0 \). We may suppose that \( a_1 \) is positive. Let \( \beta_1 \) and \( \beta_2 \) be coprime positive integers such that \( \beta_1 \deg t_1 = \beta_2 \deg t_2 \). The rational function \( R = t_1^{\beta_1} / t_2^{\beta_2} \) is primitive. By Theorem 1, the restriction \( R|_V \) of \( R \) to \( V \) is of proper direct \( C^* \)-type. Since \( R \) is not of torsional \( C^* \)-type, the classification in §1.4 shows that \( R \) must be of \( C \)-type. Each level curve of \( R \) with the value different from 0, \( \infty \) is irreducible and intersects \( C_3 \) at a point in \( P^2 \setminus I_f \). Since the restriction \( R|_{C_3} \) is a rational function of degree one on \( C_3 \), a level curve \( L \) of \( R \) is of order one and each \( L \) intersects \( C_3 \) transversally.

Suppose that \( f \) belongs to \( \mathcal{F}_{11} \). Then \( R \) belongs to \( \mathcal{F}_{11} \). Since \( R \) is primitive, \( R \) is written as \( R = a_2 R_{n,k} \) or as \( R = a_2 (R_{n,k})^{-1} \) in a homogeneous coordinate \((X:Y:Z)\) of \( P^2 \), where \( a_2 \) is a non-zero constant. So \( \deg t_1 \) and \( \deg t_2 \) are coprime and \( \beta_1 = \deg t_1 \), \( \beta_2 = \deg t_2 \). Hence there exists a rational function \( \psi_{n,k} \) of \( C^* \)-type on \( P^2 \) such that the mapping \( \theta \) defined by \( \theta(p) = \langle R_{n,k}(p), \psi_{n,k}(p) \rangle, p \in P^2 \setminus (C_1 \cup C_2) \) is a birational isomorphism of \( P^2 \setminus (C_1 \cup C_2) \) onto \( C^* \times C \). Then the equation \( \psi_{n,k} = F(R_{n,k}) \) on \( P^2 \setminus (C_1 \cup C_2) \) defines \( C_3 \) for some rational function \( F(z) = P(z)/z^l \), where \( P(z) \) is a polynomial in \( z \) and \( l \) is a non-negative integer. Hence \( C_3 \) is a prime curve of order one of the rational function \( \psi_{n,k} - R_{n,k} \). The locus of poles of \( \psi_{n,k} - R_{n,k} \) is contained in \( C_1 \cup C_2 \). Since \( \deg t_1 \) and \( \deg t_2 \) are coprime, there exists an integer \( p \) such that \( S_i \) is a prime
curve of order $\alpha_i$ with the value 0 of $f_0' = (R_{n,k})^{\alpha_i}[\varphi_{n,k} - \Upsilon(R_{n,k})]^{\alpha_i}$. Then the function $f$ is written as $f = a_3f_0$ for a non-zero constant $a_3$. Since $f$ is of $C^*$-type, $a_3 \neq 0$. Since $f$ is primitive, the integers $p$ and $\alpha_3$ are coprime.

Conversely, in a homogeneous coordinate $(X: Y: Z)$ of $P^2$, the equation $\varphi_{n,k} = \Upsilon(R_{n,k})$ on $P^2 \setminus \{(w_{k-1} = 0) \cup \{v_n = 0\}\}$ defines a prime curve $S$ of order one of the rational function $\varphi_{n,k} - \Upsilon(R_{n,k})$ on $P^2$ where $\Upsilon(z) = P(z)/z^l$ and a polynomial $P(z)$ and a non-negative integer $l$ are arbitrary. Since the level curves of $R_{n,k}$ with value 0, $\Upsilon$ are multiple, the closure $\overline{S}$ does not intersect them. Hence $S$ is of $C^*$-type. The restriction $R_{n,k}|_V$ of $R_{n,k}$ to $V = P^2 \setminus \{(w_{k-1} = 0) \cup \{v_n = 0\} \cup S\}$ is of proper direct $C^*$-type. Consider the rational function $f_0 = (R_{n,k})^{\alpha_i}[\varphi_{n,k} - \Upsilon(R_{n,k})]^{\alpha_i}$ for coprime integers $p$ and $q (\neq 0)$. The curve $S$ is a prime curve of $C^*$-type of $f_0$ with the value 0. Hence, by Theorem 1, $f_0$ is direct $C^*$-type on $P^2$. The curves $\{w_{k-1} = 0\}$ and $\{v_n = 0\}$ must be the closures of prime curves of $f_0$. Hence $f_0$ belongs to $\mathcal{F}_{11}$. The restriction $\varphi|_L$ of $\varphi = \varphi_{n,k} - \Upsilon(R_{n,k})$ to each level curve $L$ of $R_{n,k}$ is a rational function of degree one on $L$. Hence the mapping $\theta_1$ defined by $\theta_1(p) = (R_{n,k}(p), \varphi(p))$, $p \in P^2 \setminus \{(w_{k-1} = 0) \cup \{v_n = 0\}\}$ is a birational biregular isomorphism of $P^2 \setminus \{(w_{k-1} = 0) \cup \{v_n = 0\}\}$ onto $C^* \times C$. Hence $f_0$ is primitive. As is seen in Proposition 5, a rational function of $C^*$-type with two points of indetermination belongs to $\mathcal{F}_1$. Hence we obtain the following theorem announced in Chapter 0, §2.2.

**Theorem 2.** A primitive rational function $f$ on $P^2$ is of direct $C^*$-type and belongs to $\mathcal{F}_{11}$ if and only if $f$ is represented as $f = T \circ f_0$ where $T$ is an analytic automorphism of $P$ and $f_0 = R_{n,k}[\varphi_{n,k} - \Upsilon(R_{n,k})]^{\alpha_i}$. Here $(R_{n,k}, \varphi_{n,k})$ is a pair of rational functions of special type belonging to $\mathcal{F}_{11}$ given in Chapter 0, §2, $p$ and $q (\neq 0)$ are coprime integers and $\Upsilon$ is a rational function $P(z)/z^l$ in one variable $z$ for a polynomial $P$ in $z$ and a non-negative integer $l$.

Suppose that $f$ belongs to $\mathcal{F}_1$. Then $R$ belongs to $\mathcal{F}_1$. Let $C_2$ be a prime curve of degree one. The rational function $R$ defines a regular function $T_1$ on $C^2 = P^2 \setminus C_2$. Let $(x, y)$ be an inhomogeneous coordinate of $P^2$ with $C_2$ regarded as the complex line at infinity. Then $T_1$ is a polynomial function of $(x, y)$. By Jung [1], we know that there exists a polynomial $T_2(x, y)$ such that the transformation $(x', y') = (T_1(x, y), T_2(x, y))$ is an algebraic automorphism of $C^2$. Hence $f$ is written as $f = a(T_1^{\alpha_1}(T_2 - \Upsilon(T_1))^{\alpha_3})$ for coprime $\alpha_1 \in Z^+$ and $\alpha_3 \in Z \setminus \{0\}$, where $\Upsilon(z) = P(z)/z^l$, $P(z)$ is a polynomial in $z$ and $l$ is a non-negative integer. Since $f$ is primitive, $\alpha_1$ and $\alpha_3$ are coprime. The converse is not true. There is a case where $T_1^{\alpha_1}(T_2 - \Upsilon(T_1))^{\alpha_3}$ is of $C$-type.
2. Functions of \( C^* \)-type with two points of indetermination. In this subsection, we determine the rational functions of \( C^* \)-type with two points of indetermination. We use the following lemma.

**Lemma 7.** Let \( C \) be an algebraic curve on \( P^2 \) whose complement \( P^2 \setminus C \) is simply connected. Then \( C \) is an irreducible curve of degree one, that is, a complex line.

**Proof.** Suppose that the degree \( \nu \) of the curve \( C \) is not 1. Let \( L \) be a curve of degree one on \( P^2 \). Let \( (X: Y: Z) \) be a homogeneous coordinate of \( P^2 \) such that \( L = \{ Z = 0 \} \). Let \( P(X, Y, Z) \) be an irreducible homogeneous polynomial of \( (X, Y, Z) \) defining \( C \). Consider an analytic function \( \xi = (Z/|P(X, Y, Z)|)^{1/\nu} \). The Riemann domain of \( \xi \) over \( P^2 \setminus C \) is \( \nu \)-sheeted and unramified with no relative boundary, which contradicts the assumption that \( P^2 \setminus C \) is simply connected. Thus we have our lemma.

Let \( f \) be a primitive rational function of \( C^* \)-type with two points of indetermination on \( P^2 \). Let \( \sigma: M \to P^2 \) be the minimal resolution of indeterminacy points of \( f \) by \( \sigma \)-processes. Let \( B_1, B_2 \) be two basic sections of \( \sigma^* f \). Since \( \sigma \) is minimal, we have \( (B_1) = (B_2) = -1 \).

First, we prove that there is at most one irreducible multiple level curve of \( C^* \)-type of \( f \). Suppose that \( f \) has two irreducible multiple level curves with the values \( c_1, c_2 \), respectively. The graphs of level curves \( \sigma^* f \) with the value \( c_1, c_2 \) are determined by Lemma 3(iii) in a way similar to that of Chapter II, §1.1. Each \( B_i \) (\( i = 1, 2 \)) intersects at most two other components of \( \Sigma(f) = \sigma^{-1}(I_f) \). Hence a level curve of \( \sigma^* f \) with the value different from \( c_1, c_2 \) consists only of proper transforms of prime curves of \( f \) under the mapping \( \sigma^{-1} \). Since the restriction \( \sigma^* f|_{B_i} \) of \( \sigma^* f \) to \( B_i \) is a rational function of degree one on \( B_i \), \( f \) satisfies the condition of Class (A) in Chapter III, §1.4. Only one level curve \( L \) of \( f \) consists of two prime curves \( S_i, S_j \) of \( C \)-type which intersect at a point in \( P^2 \setminus I_f \). Denote by \( \tilde{S}_i, \tilde{S}_j \) the proper transforms of \( S_i, S_j \) under the mapping \( \sigma^{-1} \), respectively. The curve \( \tilde{S}_i \cup \tilde{S}_j \) is a level curve of \( \sigma^* f \). We may suppose that \( \tilde{S}_i \) intersects \( B_i \). Denote by \( F_{e_i}, F_{e_j} \) the level curves of \( \sigma^* f \) with the value \( c_i, c_j \), respectively. Let \( K_i \) be the component of \( F_{e_i} \) intersecting \( B_i \) for each \( i \). Let \( \tau: M \to \tau(M) \) be the composite of \( \sigma \)-processes which contracts the curve \( (F_{e_i} \setminus K_i) \cup (F_{e_j} \setminus K_j) \cup \tilde{S}_i \). Then \( (\tau(B_i)) = (\tau(F_{e_i})) = (\tau(F_{e_j})) = 0 \). By Proposition 2(ii), \( \tau(M) \) is biregularly isomorphic to \( P \times P \). On the other hand \( (\tau(B_i)) = 1 \), which contradicts the fact that the self-intersection number of an algebraic curve on \( P \times P \) is even. Hence we have proved our assertion.
The function $f$ has only one reducible level curve $L$, which consists of two prime curves $S_1$, $S_2$. At least one of them is of $C$-type. We suppose that $S_1$ is of $C$-type. If $f$ has an irreducible multiple level curve of $C^*$-type, denote it by $S_3$. If $f$ has no irreducible multiple level curve of $C^*$-type, denote by $S_3$ an arbitrary regular level curve. Let $t_i$ be an irreducible homogeneous polynomial which defines $S_i$ ($i = 1, 2, 3$). Consider the rational functions $g_2 = t_2^{t_2}/t_1^{t_1}$, $g_3 = t_3^{t_3}/t_1^{t_1}$, where coprime positive integers $t_1$, $t_2$ satisfy $t_2 \geq t_1$ and coprime positive integers $t_1$, $t_3$ satisfy $t_3 \geq t_1$. Since the restriction $f|_V$ of $f$ to $V = P^2 \setminus (S_1 \cup S_2 \cup S_3)$ is of proper direct $C^*$-type, Theorem 1 shows that the restrictions $g_2|_V$, $g_3|_V$ are of proper direct $C^*$-type. By the classification in §1.4 in this chapter, $g_2$ and $g_3$ are of $C$-type on $P^2$. Since $g_2$ is of $C$-type on $P^2$, the restriction $g_2|_{S_3}$ of $g_2$ to $S_3$ is a rational function of degree one on $S_3$. Hence each level curve of $g_2$ intersects $S_3$ transversally at a point and is of order one. Since $g_3$ is of $C$-type on $P^2$, the restriction $g_3|_{S_2}$ of $g_3$ to $S_2$ is a rational function of degree one on $S_2$. Hence each level curve of $g_3$ intersects $S_2$ transversally at a point and is of order one. We obtain $t_2 = t_3 = 1$. Hence the restriction of $g_2$ to each level curve $L$ of $g_3$ with a finite value is a rational function of degree one on $L$. The restriction of $g_2$ to each level curve $L'$ of $g_2$ with a finite value is a rational function of degree one on $L'$. Hence the mapping $\theta$ defined by $\theta(p) = (g_2(p), g_3(p))$, $p \in P^2 \setminus S_1$, is a biregular isomorphism of $P^2 \setminus S_1$ onto $C^2$. By Lemma 7, $S_1$ is an algebraic curve of degree one. By Proposition 1, we obtain the following.

**Proposition 5.** If a rational function $f$ of $C^*$-type on $P^2$ has two indetermination points, then $f$ belongs to $\mathcal{F}_1$. In an inhomogeneous coordinate $(x, y)$ of $P^2$, $f$ is written as $f = \Lambda(T_1/T_2)$ for coprime integers $m$, $n$, and conversely. Here $T_1(x, y)$, $T_2(x, y)$ are polynomials of $x$, $y$ such that $(x', y') = (T_1(x, y), T_2(x, y))$ is an algebraic automorphism of $C^2$ and $\Lambda(z)$ is a rational function of $z$. If both $T_1$ and $T_2$ are of degree one, then $(m, n) \neq (1, 1)$.

3. **Transformation group defined by $C^*$**. Here we give an alternative proof of the first half of Proposition 5. We also obtain a result on an analytic transcendental automorphism of the complement of an algebraic curve on $P^2$. Let $S$ be a prime curve of $C$-type of $f$. The restriction $f|_V$ of $f$ to $V = P^2 \setminus S$ has only one point $p_i$ of indetermination. Each level curve of $f|_V$ is irreducible and of $C^*$-type. Let $S_a$ be a level curve of order $\nu$ of $f$ with the value $a$. Let $\mu_a: S_a \to C^*$ be an analytic isomorphism of $S_a$ onto $C^* = \{w \mid 0 < |w| < 1\}$ which maps a neighbourhood of
p₁ into a neighbourhood of the origin \( w = 0 \). For a fixed non-zero complex number \( c \), we consider an analytic automorphism \( \phi_c \) of \( \mathbb{C}^* \) defined by \( \phi_c(w) = c^w \). The transformation \( T_c = \mu_c \circ \phi_c \circ \mu_c \) of \( S_a \) is independent of choice of \( \mu_c \). The mapping \( T_c \) of \( V \setminus \{p₁\} \) into itself defined by \( T_c(p) = T_c(p_c) \), \( p \in V \setminus \{p₁\} \), is bijective. We prove the following.

**Lemma 8.** The mapping \( T_c \) is an analytic automorphism of \( V \setminus \{p₁\} \).

**Proof.** To see that \( T_c \) is holomorphic in a neighbourhood of \( S_a \), we may suppose that \( a = 0 \). For a sufficiently small positive number \( r \), the punctured disc \( \Gamma_r = \{ z \mid z \neq 0, |z| < r \} \) does not contain a critical value. Consider the tube \( V_r = \{ p \in V \setminus \{p₁\} \mid f(p) \in \Gamma_r \} \). Denote by \( \tilde{V}_r \) the domain of existence of the function \( (f|_{V_r})^{1/n} \). Denote by \( \tilde{\omega} : \tilde{V}_r \to V_r \) the canonical projection. The set \( S_0 = \tilde{\omega}^{-1}(S_0) \) is an irreducible curve on \( V_r \) which is a \( n \)-sheeted covering surface of \( S_0 \) with no branch point and with no relative boundary. Hence the domain \( \tilde{V}_r \) is analytically isomorphic to \( \Gamma_r \times \mathbb{C}^* \). Since \( \tilde{\omega}^{-1} \circ T_c \circ \tilde{\omega} \) defines an analytic automorphism of \( \tilde{V}_r \setminus S_0 \) whose analytic continuation to \( \tilde{V}_r \) is still holomorphic on \( \tilde{V}_r \), \( T_c|_{V_r} \) is holomorphic on \( V_r \), which proves the lemma.

By putting \( T_c(p₁) = p₁, T_c \) defines an analytic automorphism of \( V \). Suppose that \( |c| > 1 \). Let \( U \) be a solid sphere with the center \( p₁ \). Since \( V = \lim_{n \to \infty} T_c^n(U) \) for \( n \)-times iteration \( T_c^n \) of \( T_c \), \( V \) is simply connected. Hence, by Lemma 7, \( S \) is a curve of degree one. Thus we have proved the first half of Proposition 5.

We also obtain the following.

**Corollary to Lemma 8.** Let \( C \) be an algebraic curve in \( \mathbb{P}^2 \). Suppose that the complement \( \mathbb{P}^2 \setminus C \) has a regular rational function of \( \mathbb{C}^* \)-type any level curve of which is irreducible and of \( \mathbb{C}^* \)-type. Then \( \mathbb{P}^2 \setminus C \) has an analytic transcendental automorphism.

4. **Exposition of examples.** In this section, we give simple examples of the graphs \( \Sigma(f) \) of rational functions of \( \mathbb{C}^* \)-type with one point of indetermination, which will help the reader to understand the proof of Theorem 2.

Class (C): Figure 53.
Class (D): Figure 54.
Class (E): Figure 55.

5. **Class (A).** Let \( f \) be a primitive rational function of \( \mathbb{C}^* \)-type on \( \mathbb{P}^2 \) belonging to the Class A in §1.4 in this chapter. Let \( \sigma : M = \mathbb{P}^2 \to \mathbb{P}^2 \) be the minimal resolution of the indetermination points of \( f \). We determine
the graph of $\Sigma(f)$ and $f$ in another way, which shows a way to construct functions of $C^*$-type with multiple prime curves of higher order.

We suppose that the level curve of $f$ with the value 0 satisfies the condition (b)(i) in $\S$ 1.4. As was seen in $\S$ 2.2, we may suppose that each level curve of $f$ with a finite non-zero value is of order one. Denote by $S_1, S_2$ the two prime curves of $f$ with the value zero and by $S_3$ the level curve of $f$ with the value $\infty$. Let $\tilde{S}_i$ be the proper image of $S_i$ under the mapping $\sigma^{-1}$ for each $i$. Let $B_1, B_2$ be the basic sections of $\sigma^*f$. Then $(B_1^2) = (B_2^2) = -1$. The level curve $F_0$ of $\sigma^*f$ with the value 0 intersects each $B_i$ $(i = 1, 2)$ at a simple point of $F_0$ transversally. The component of $F_0$ intersecting $B_i$ is of order one. At least one of $\tilde{S}_1$ and $\tilde{S}_2$ is an exceptional curve of the first kind.

(1) The case where $(S_1^2) = (S_2^2) = -1$. Lemma 3(ii) shows $F_0 = \tilde{S}_1 \cup \tilde{S}_2$. We may suppose that $(\tilde{S}_1 \cdot B_1) = (\tilde{S}_2 \cdot B_2) = 1$. Suppose that the level curve $F_\infty$ of $\sigma^*f$ with the value $\infty$ is irreducible. Denote by $\tau$ the $\sigma$-process contracting $S_1$. Then $(\tau(\tilde{S}_1^2)) = (\tau(B_1^2)) = 0$. Hence, by Proposition 2(i), $\tau(M)$ is biregularly isomorphic to $P \times P$. On the other hand, $(\sigma(B_2^2)) = -1$, a contradiction. Hence $F_\infty$ must be reducible. The graph of $F_\infty$ is that in Lemma 3(iii). By Lemma 2, the graph of $\Sigma(f)$ must be as in Figure 56. From this, we see that $S_1$ and $S_2$ are of order one and

![Figure 56](image)

that $S_3$ is of order two. Since $(\sigma(\tilde{S}_1^2)) = (\sigma(\tilde{S}_2^2)) = (\sigma(\tilde{S}_3^2)) = 1$, the curves $S_1, S_2, S_3$ are algebraic curves of degree one on $P^2$. Let $(X_1 : X_2 : X_3)$ be a homogeneous coordinate of $P^2$ such that $\tilde{S}_1 = \{X_1 = 0\}$ $(i = 1, 2, 3)$. Under the inhomogeneous coordinate $(x, y) = (X_1/X_3, X_2/X_3)$ of $P^2$, $f$ is written as $f = axy$ for a non-zero constant $a$.

(2) The case where $(S_1^2) \neq (S_2^2)$. Suppose that $(S_1^2) = -1$. The closure of $F_0 \setminus \tilde{S}_2$ consists of two connected components each of which intersects a basic section of $\sigma^*f$. Hence $F_0$ satisfies the condition in Lemma 3(iii). Therefore the graph of $\Sigma(f)$ is as in Figure 57. By
Lemma 2, we have $q_r = q'_r$. Let $\tau$ be the composite of $\sigma$-processes which contracts $B_1$ and $B_2$. The graph of $\tau(\Sigma(f) \cup \tilde{S}_1 \cup \tilde{S}_2 \cup \tilde{S}_3)$ has the same property as the graph of $\Sigma(f)$ and is shorter than it, from which, using Proposition 2, we obtain another rational function belonging to the Class (A). The loci of zeros and poles of this new function are the same as those of $f$. Repeating these processes, our case (2) is reduced to the former case (1). Hence $f$ is written as $f = ax^m y^n$ for positive integers $m$, $n$ and for a non-zero constant $a$. Since $f$ is primitive, $m$ and $n$ must be coprime.

\[ \text{Figure 57.} \]

§ 3. Non-existence of a rational function of torsional $C^*$-type on $P^2$.

1. Let $f$ be a primitive rational function of torsional $C^*$-type on $P^2$. Let $S_1$, $S_2$ be two irreducible level curves of $C$-type of $f$. The other level curves of $f$ are irreducible and of $C^*$-type. Let $\sigma : M \to P^2$ be the minimal resolution of the indetermination point $p_0$ of $f$ by $\sigma$-processes. The basic section $B$ of $\sigma^*f$ satisfies $(B^2) = -1$. The restriction $\sigma^* f|_B$ of $\sigma^* f$ to $B$ is a rational function of degree two on $B$. Hence the curve $B$ is a two-sheeted ramified covering surface over $P$ with the projection $\sigma^* f$. By Lemma 3(iii), we see that the two ramification points of this covering surface are over the points $f(S_1)$ and $f(S_2)$. Denote by $F_i$ the level curve of $\sigma^* f$ with value $f(S_i)$ ($i = 1, 2$) and by $S_i$ the proper image of $S_i$ under the mapping $\sigma^{-1}$.

Suppose that each level curve of $\sigma^* f$ is irreducible. Then $B = \sigma^{-1}([p_0])$ and $\sigma$ is the blowing-up at $p_0$. The curve $F_i$ must be tangent to $B$ with

\[ \text{Figure 58.} \]
order one. Hence \( S_i = \sigma(F_i) \) must be singular at \( p_0 \). Since the multiplicity of \( S_i \) at \( p_0 \) is two and \( (F_i)^2 = 0 \), we have \((\tilde{S}_i)^2 = 4\). This means that \( S_i \) is an algebraic curve of degree two. Hence \( S_i \) has no singular point, a contradiction. Hence at least one level curve of \( \sigma^*f \) must be reducible.

Suppose that a level curve \( F_3 \) of \( \sigma^*f \) with value \( a \), different from \( f(S_1) \) and \( f(S_2) \), is reducible. Since the level curve \( S_3 \) of \( f \) with value \( a \) is of \( C^*-\)type, by Lemma 3 (iii), it must be multiple. Since \( B \) intersects at most two other components of \( \Sigma(f) \), each level curve of \( \sigma^*f \) with value different from \( f(S_3) \) is irreducible. The graph of \( \Sigma(f) \) is as in Figure 58, where \( \tilde{S}_i \) denotes the proper image of \( S_i \) under the mapping \( \sigma^{-1} \). It means that \( \Sigma(f) \) has the property \((P)\), which contradicts the fact that \( \Sigma(f) \) is an exceptional curve. Hence each level curve of \( \sigma^*\text{-type} \) of \( f \) must be of order one. The curve \( \Sigma(f) \) consists of \( B \), the closure \( \Sigma_i \) of \( F_i \setminus \tilde{S}_i \) and the closure \( \Sigma_z \) of \( F_i \setminus S_i \).

2. If \( \Sigma_i \neq \emptyset \), then a component \( K_i \) of \( \Sigma_i \) intersects \( B \) at a point transversally. Suppose that \( K_i \) and \( S_i \) are prime curves of order one of \( \sigma^*f \) and \( \tilde{S}_i \) intersects \( B \) at \( K_i \cap B \) transversally. By Lemma 3 (i), the closure of \( F_i \setminus \tilde{S}_i \) is an exceptional curve of the first kind, which contradicts the fact that \( \sigma \) is minimal. Hence \( K_i \) is a prime curve of order two of \( \sigma^*f \) and \( \tilde{S}_i \cap B = \emptyset \).

Since \( F_i \) is reducible, \( \tilde{S}_i \) is only one exceptional component of the first kind of \( F_i \). Let \( \tau_i: M \to \tau_i(M) \) be the \( \sigma \)-process which contracts \( \tilde{S}_i \). Since \( S_i \) is of \( C \)-type, \( \tau_i(F_i) \) has only one exceptional component of the first kind. Let

\[
M \xrightarrow{\tau_1} M_1 \xrightarrow{\tau_2} \cdots \xrightarrow{\tau_k} M_{k-1} \xrightarrow{\tau_k} M_k
\]

be a sequence of \( \sigma \)-processes \( \tau_j \) \((j = 1, 2, \cdots, k)\) which contracts a component of \( \tau_{j-1} \circ \tau_{j-2} \circ \cdots \circ \tau_1 \circ \tau_i(F_i) \). Set \( \overline{\tau}_j = \tau_j \circ \tau_{j-1} \circ \cdots \circ \tau_2 \circ \tau_1 \). For a

Figure 59.

Figure 60.
sufficiently large $j$, the image $\overline{\tau}_j(F_i)$ contains only components of order one. Since $(\overline{\tau}_j(B) \cdot \overline{\tau}_j(F_i)) = 2$ for each $j$, there exists $j$ such that $\overline{\tau}_j(B)$ is smooth and $\overline{\tau}_j(B)$ intersects two components $C_1$, $C_2$ of order one of $\overline{\tau}_j(F_i)$ transversally at the point $\overline{\tau}_j(S_i)$. Figure 59 gives a sketch of $\overline{\tau}_j(\Sigma(f))$. Hence the graph of $B \cup \Sigma_i$ is not linear. Since $\Sigma(f)$ is an exceptional curve of the first kind containing only one irreducible exceptional component of the first kind, at least one of the graphs of $B \cup \Sigma_i$ and $B \cup \Sigma_2$ is linear. Hence at least one of $\Sigma_i$ and $\Sigma_2$ must be empty. So we may suppose that $\Sigma_2 = \emptyset$. By the fact stated in §3.1, $\Sigma_i$ is not empty.

3. We suppose $i = 1$ in the former subsection. By Lemma 3(i), we know that $(C_i^*') = (C_i^*') = -1$ and $\overline{\tau}_j(F_i^*) = C_i \cup C_2$. Set $C_i^* = \tau_{j}^{-1}(\overline{\tau}_j(S_i^*))$. Denote by $C_i'$, $C_i''$ the proper transform of $C_i$, $C_2$ under the mapping $\tau_{j}^{-1}$, respectively. The graph of $\overline{\tau}_{j-1}(\Sigma(f))$ is as in Figure 60. The image $\overline{\tau}_{j-1}(S_i^*)$ must be a point because $S_i^*$ is of C-type. Suppose that the point $\overline{\tau}_{j-1}(S_i^*)$ is neither $C_i' \cap C_i''$ nor $C_i' \cap C_i''$. Denote by $C_i'''$ the proper transform of $C_i$ under the mapping $\overline{\tau}_{j-1}$. Since $C_i^{'''}$ is a component of $F_i$ of order one, by Lemma 3(i), the closure of $F_i \setminus C_i^{'''}$ is an exceptional curve of the first kind. On the other hand, the proper transform of $C_i' \cup C_i'' \cup C_i'''$ under the mapping $\overline{\tau}_{j-1}$ is the last branch of $\Sigma(f)$. So $F_i$ must be exceptional, a contradiction. Hence the point $\overline{\tau}_{j-1}(S_i^*)$ is $C_i' \cap C_i''$ or $C_i' \cap C_i''$. We may suppose that $\overline{\tau}_{j-1}(S_i^*) = C_i' \cap C_i''$. The graph of $\overline{\tau}_{j-1}(\Sigma(f))$ is as in Figure 61. Since $\Sigma(f)$ is an exceptional curve of the first kind, three components of $\Sigma(f)$ near $B$ must be as in Figure 62(a). Suppose that the graph of $\Sigma_i$ is linear. By Lemma 2, the graph of $\Sigma(f)$ must be as in Figure 62(b). On the other hand, $\tau_{i}(F_i) = \tau_{i}(\Sigma_i)$ has the graph in Lemma 3(iii), which

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (c) at (0,0) {$C_i$};
  \node (c) at (2,0) {$C_i'$};
  \draw[thick] (c) -- (c);
  \node at (1,0) {$1$};
  \node at (2,0) {$1$};
  \node at (0,1) {$2$};
  \node at (0,2) {$3$};
  \node at (1,1) {$2$};
  \node at (1,2) {$1$};
  \node at (2,1) {$2$};
  \node at (2,2) {$3$};
\end{tikzpicture}
\caption{(a) $C_i$, (b) $C_i'$}
\end{figure}
is impossible. Hence $\Sigma_i$ has at least one diverging component. Since $F_i$ has the property $(P)$, the graph of $\Sigma(f)$ is as in Figure 63(a) or (b).

Suppose that $\Sigma_i$ has only one diverging component. Since $\Sigma(f)$ is an exceptional curve of the first kind, the graph of $\Sigma(f)$ is as in Figure 64(a) or (b). The portion in the parenthesis may not exist. This contradicts the fact that $F_i$ has the property $(P)$. Hence $\Sigma_i$ has at least two diverging components. Since $F_i$ has the property $(P)$, the graph of $\Sigma(f)$ must be as in Figure 65(a) or (b). In Figure 65, the mark labeled $T(p)$ represents the diagram in Figure 66.

Suppose that $\Sigma_i$ has two diverging components. Since $\Sigma(f)$ is an exceptional curve of the first kind, the graph of $\Sigma(f)$ is as in Figure 67. This contradicts the fact that $F_i$ has the property $(P)$. Repeating these
processes, we see that $\Sigma(f)$ must have the graph with infinite length as in Figure 68.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig67}
\caption{Figure 67.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig68}
\caption{Figure 68.}
\end{figure}

It is a contradiction. Thus we have proved the following theorem.

**Theorem 3.** There exists no rational function of torsional $C^*$-type on the two-dimensional complex projective space.

**References**


