OUTRADIi OF THE TEICHMÜLLER SPACES OF FUCHSIAN GROUPS OF THE SECOND KIND

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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1. Introduction. Let $o(\mathcal{F})$ be the outradius of the Teichmüller space $T(\mathcal{F})$ of a Fuchsian group $\mathcal{F}$. Then $o(\mathcal{F})$ is strictly greater than 2 (Earle [5]) and not greater than 6 (Nehari [7]). A Fuchsian group is said to be of the first kind (resp. second kind) if its region of discontinuity is not connected (resp. connected). If $\mathcal{F}$ is a finitely generated Fuchsian group of the first kind, then $o(\mathcal{F})$ is strictly less than 6 ([9]). Recently the authors proved, by using a basic result on the stability of finitely generated Fuchsian groups (Bers [3]), that $o(\mathcal{F})$ is equal to 6 for a finitely generated Fuchsian group $\mathcal{F}$ of the second kind ([10]). In this paper we give an alternative proof of it, which works also for an infinitely generated Fuchsian group of the second kind.

THEOREM. If $\mathcal{F}$ is a Fuchsian group of the second kind, then $o(\mathcal{F})$ is equal to 6.

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2. Definitions. Let $\mathbb{D}$ be the open unit disc and $\mathbb{D}^*$ be the exterior of $\mathbb{D}$ in the Riemann sphere $\hat{\mathbb{D}}$. For each function $f$ which is conformal in $\mathbb{D}^*$ let $\{f, z\}$ be the Schwarzian derivative of $f$, that is, $\{f, z\} = (f''/f')' - (1/2)(f''/f')^2$. Let $\mathcal{F}$ be a Fuchsian group keeping $\mathbb{D}$ invariant. A quasiconformal automorphism $w$ of $\hat{\mathbb{D}}$ is said to be compatible with $\mathcal{F}$ if $w \circ \gamma \circ w^{-1}$ is a Möbius transformation for each $\gamma \in \mathcal{F}$. Let $w$ be a quasiconformal automorphism of $\hat{\mathbb{D}}$ which is compatible with $\mathcal{F}$ and which is conformal in $\mathbb{D}^*$. The Teichmüller space $T(\mathcal{F})$ of $\mathcal{F}$ is the set of the Schwarzian derivatives $\{w, z\}$ of such $w$'s restricted to $\mathbb{D}^*$. Let $\lambda(z) = (|z|^2 - 1)^{-1}$ be a Poincaré density of $\mathbb{D}^*$. For a function $\phi$ defined in $\mathbb{D}^*$ let $||\phi|| = \sup_{z \in \mathbb{D}^*} \lambda(z)^{-1} |\phi(z)|$. The outradius $o(\mathcal{F})$ of $T(\mathcal{F})$ is defined to be $\sup ||\phi||$, where the supremum is taken over all $\phi$ in $T(\mathcal{F})$.

3. Lemmas. In this section we state two lemmas without proof.
Lemma 1 is due to Chu [4]. Lemma 2 is proved in §§5–6. Let \( k(z) = z + z^{-1} \). Then \( k \) maps \( \Delta^* \) conformally onto \( \hat{\Delta} \) with the closed real segment \([-2, 2]\) removed. Let \( S_r \) be the circle of radius \( r > 1 \) around the origin. Then the image of \( S_r \) under \( k \) is the ellipse

\[
E_r: \frac{\zeta^2}{(r + r^{-1})^2} + \frac{\eta^2}{(r - r^{-1})^2} = 1 ,
\]

where \( \zeta = k(z) \) and \( \zeta = \xi + \eta \sqrt{-1} \).

For two Jordan loops \( J_1 \) and \( J_2 \) in the finite complex plane \( \mathbb{C} \) we define the Fréchet distance \( \delta(J_i, J_j) \) as \( \inf \max_{0 \leq t \leq 1} |z_i(t) - z_j(t)| \), where the infimum is taken over all possible parametrizations \( z_i(t) \) of \( J_i \) \((i = 1, 2)\).

**Lemma 1 (Chu [4]).** For each positive \( \varepsilon \) there exist constants \( r_1 > 1 \) and \( d_1 > 0 \) so that if \( E_{r_1} = k(S_{r_1}) \) and if \( J \) is a Jordan loop in \( \mathbb{C} \) with \( \delta(J, E_{r_1}) \leq d_1 \), then a conformal mapping \( f \) of \( \Delta^* \) onto the exterior of \( J \) satisfies \( \|f(z)|f(z)|^{-1} - 1| \geq 6 - \varepsilon \).

Denote by \( \mu[w] \) the complex dilatation of a quasiconformal mapping \( w \).

**Lemma 2.** Let \( \Gamma \) be a Fuchsian group of the second kind keeping \( \Delta \) invariant. Then for each \( r > 1 \) and \( d > 0 \) there exist a sequence \( \{\sigma_n\}_{n=1}^\infty \) of Möbius transformations and a sequence \( \{F_n\}_{n=1}^\infty \) of quasiconformal automorphisms of \( \hat{\Delta} \) which satisfy the following.

\[
\begin{align*}
(3.1) & \quad F_n \circ \gamma = \gamma \circ F_n \quad \text{for all} \quad \gamma \in \Gamma . \\
(3.2) & \quad F_n \circ \sigma_n(\infty) \in \Delta^* . \\
(3.3) & \quad \lim_{n \to \infty} \|\mu[F_n^{-1} \mid \Delta^*]|F_n^{-1} \circ \sigma_n(\infty) = 0 . \\
(3.4) & \quad \delta(\sigma_n^{-1} \circ F_n(\partial \Delta), E_r) \leq d .
\end{align*}
\]

**4. Proof of Theorem.** For each \( \varepsilon > 0 \) let \( r_1 \) and \( d_1 \) be the constants in Lemma 1. Let \( \{\sigma_n\}_{n=1}^\infty \) and \( \{F_n\}_{n=1}^\infty \) be sequences of Möbius transformations and quasiconformal automorphisms, respectively, obtained from Lemma 2 for \( r = r_1 \) and \( d = d_1 / 2 \).

Set \( \nu_n(z) = \mu[F_n^{-1} \mid \Delta](z) \) for \( z \in \Delta \) and \( = 0 \) for \( z \notin \Delta^* \). Let \( w_n \) be the \( \nu_n \)-conformal automorphism of \( \hat{\Delta} \) which sends \( F_n \circ \sigma_n(0) \), \( F_n \circ \sigma_n(1) \), and \( F_n \circ \sigma_n(\infty) \) to 0, 1 and \( \infty \), respectively (Ahlfors [1, p. 98]). Then \( w_n \) is compatible with \( \Gamma \) by (3.1) and the quasiconformal automorphism \( W_n = w_n \circ F_n \circ \sigma_n \) of \( \hat{\Delta} \) keeps 0, 1, and \( \infty \) fixed. Since \( W_n(\infty) = \infty \), (3.2) implies \( w_n^{-1}(\infty) = F_n \circ \sigma_n \circ W_n^{-1}(\infty) = F_n \circ \sigma_n(\infty) \in \Delta^* \). Hence \( w_n \) maps \( \Delta^* \) conformally onto the exterior of \( w_n(\partial \Delta) \). Since both \( \mu[w_n \mid \Delta] \) and \( \mu[w_n^{-1} \circ F_n^{-1} \mid \Delta] \) are equal to \( \nu_n \mid \Delta, \mu[W_n \mid \sigma_n^{-1} \circ F_n^{-1} \circ \Delta] \) vanishes ([1, p. 9]). Hence
\[ \|\mu[W_n]\|_\infty = \|\mu[F_n|\sigma_{-1}^{-1}F_n^{-1}(A^*)]\|_\infty = \|\mu[F_n|F_n^{-1}(A^*)]\|_\infty = \|\mu[F_n^{-1}|A^*]\|_\infty. \]

Therefore \( \lim_{n \to \infty} \|\mu[W_n]\|_\infty = 0 \) by (3.3). By a result on quasiconformal mappings (Ahlfors-Bers [2, Lemma 17]), we see the existence of a positive integer \( n_1 \) so that
\[ |W_{n_1}(z) - z| \leq d_i/2 \]
for all \( z \) with \( \text{dist}(z, E_{r_1}) \leq d_i/2 \). This shows
\[ \delta(w_{n_1}(\partial A), \sigma_{-1}^{-1}F_n^{-1}(\partial A)) \leq d_i/2. \]

Hence this together with (3.4) implies that \( \delta(w_n(\partial A), E_{r_n}) \leq d_i \). Now Lemma 1 shows \( \|\{w_n|A^*, z\}\|_\infty > 6 - \varepsilon \). Recall that \( \{w_n|A^*, z\} \in T(\Gamma) \).

Then we see \( o(\Gamma) > 6 - \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, \( o(\Gamma) \geq 6 \). On the other hand \( o(\Gamma) \leq 6 \) (Nehari [7]). Therefore \( o(\Gamma) = 6 \). This completes the proof of Theorem.

5. A sequence of quasiconformal mappings. Let \( \{\delta_n\}_{n=1}^{\infty}(\subset (0,1)) \) be a decreasing sequence with \( \lim_{n \to \infty} \delta_n = 0 \). Let \( V_n = \{z \in \mathbb{C}; |z| < \delta_n\} \). Let \( j_n \) be a smooth closed Jordan arc in \( \text{Cl} V_n \) which joins \(-\delta_n \) to \( \delta_n \). Set \( l_n = [-1, -\delta_n \cup j_n \cup (\delta_n, 1] \). Let \( U \) and \( L \) be the upper and lower half-planes, respectively. Let \( B = \{z \in \mathbb{C}; |\text{Re} z| < 1, 0 < |\text{Im} z| < 1\} \). Then both \( \alpha_n = l_n \cup (L \cap \partial A) \) and \( \beta_n = l_n \cup (U \cap \partial B) \) are Jordan loops. Denote by \( A_n \) and \( B_n \) the interiors of \( \alpha_n \) and \( \beta_n \), respectively. Let \( A = \{z \in L; |z| < 1\} \) and \( C = \{z \in L; 1 < |z| < 2\} \). Let \( \Omega \) be the interior of \( \text{Cl}(A \cup B \cup C) \). The purpose of this section is to prove the following lemma.

**Lemma 3.** There exists a sequence of quasiconformal automorphisms \( \{G_n\}_{n=1}^{\infty} \) of \( \Omega \) with \( G_n(z) = z \) for all \( z \in \partial \Omega \) which satisfy the following.

(i) \( G_n(l_n) = \delta U \cap \text{Cl} A \) and \( G_n(A_n) = A \).

(ii) \( \lim_{n \to \infty} \|\mu[G_n^{-1}|\Omega \cap L]\|_\infty = 0. \)

It is known that every quasiconformal mapping between Jordan domains can be extended to a homeomorphism between their closures (Lehto-Virtanen [6, p. 42]). Therefore from now on a quasiconformal mapping of a Jordan domain \( D \) onto another means a homeomorphism of \( \text{Cl} D \) which is quasiconformal in \( D \).

Let \( f_n \) be the conformal mapping which maps \( A_n \) onto \( A \) and which keeps 1, \(-1\) and \(-\sqrt{-1}\) invariant. Let \( R_n \) be the annulus \( \{z \in \mathbb{C}; \delta_n < |z| < \delta_n^{-1}\} \). Then by the reflection principle \( f_n|A_n \cap R_n \) can be continued analytically to \( R_n \) beyond the unit circle and beyond the real line. Thus \( f_n \) has a conformal extension to \( A_n \cup R_n \), for which by abuse of language we use the same letter \( f_n \). Before proving Lemma 3, we prove Lemmas 4-6 which play essential roles in the proof of Lemma 3.
**Lemma 4.** The sequence \( \{f_n\}_{n=1}^\infty \) converges to the identity transformation uniformly in \( R_\ast \).

**Proof.** Each \( f_n \) fixes 1, -1 and \(-\sqrt{-1}\). Hence \( \{f_n\}_{n=m}^\infty \) is a normal family in \( R_m \) (Lehto-Virtanen [6, p. 73]). By a diagonal argument we obtain a subsequence \( \{f_{n_i}\}_{i=1}^\infty \) of \( \{f_n\}_{n=1}^\infty \) which converges uniformly in \( R_{n_i} \), in particular, in \( R_1 \) to a conformal mapping \( f_\infty \) of \( \bigcup_{i=1}^\infty R_{n_i} = C - \{0\} \) ([6, p. 74]). Since \( f_\infty \) can be extended to a conformal automorphism of \( \hat{C} \) and since \( f_\infty \) fixes 1, -1 and \(-\sqrt{-1}\), \( f_\infty \) is the identity transformation. By the same reasoning as above any other convergent subsequence of \( \{f_n\}_{n=1}^\infty \) than \( \{f_{n_i}\}_{i=1}^\infty \) also converges to the identity transformation uniformly in \( R_\ast \), and so does the sequence \( \{f_n\}_{n=1}^\infty \) itself. \( \text{q.e.d.} \)

**Lemma 5.** There exists a quasiconformal mapping \( g_n \) of \( B_n \) onto \( B \) so that \( g_n(z) = f_n(z) \) for all \( z \in l_n \) and \( g_n(z) = z \) for all \( z \in \beta_n - l_n \).

**Proof.** Put \( q_n(z) = f_n(z) \) if \( z \in l_n \) and \( =z \) if \( z \in \beta_n - l_n \). Then \( q_n \) is a homeomorphism of a Jordan loop \( \beta_n \) onto another \( \hat{B} \). For each point \( p \) of \( \beta_n \) we shall show the existence of an open subarc \( J_p \) of \( \beta_n \) containing \( p \) such that \( q_n|J_p \) has a quasiconformal extension to \( \hat{C} \). Then by a theorem of Rickman ([8, Theorem 4]) \( q_n \) has a quasiconformal extension \( g_n \) to \( \hat{C} \). Since \( g_n \) is sense-preserving, \( g_n \) maps \( B_n \) onto \( B \).

First let \( p \in \beta_n \cap U \). Then \( \beta_n \cap U \) is an open subarc of \( \beta_n \) containing \( p \) and \( q_n|\beta_n \cap U \) has a quasiconformal extension to \( \hat{C} \), which is the identity mapping. Secondly, let \( p \in l_n - \{\pm 1\} \). Then \( l_n - \{\pm 1\} \) is an open subarc of \( \beta_n \). Since both \( \alpha_n \) and \( \partial A \) consist of finitely many smooth arcs which meet pairwise at non-zero angles, they are quasicircles (Lehto-Virtanen [6, p. 104]). Hence \( f_n \) can be extended to a quasiconformal automorphism \( f_n \) of \( \hat{C} \) (Ahlfors [1, p. 75]). In particular \( q_n|l_n - \{\pm 1\} \) has a quasiconformal extension \( f_n \) to \( \hat{C} \). Finally, let \( p = \pm 1 \). Let \( b_n \in (\delta_n, 1) \) and let \( N_n = \{z \in C; b_n < p \cdot \Re z < b_n^{-1}, |\Im z| < 1/2\} \). Then \( \beta_n \cap N_n \) is an open subarc of \( \beta_n \) containing \( p \). Set \( u_n(z) = f_n(\Re z) + \sqrt{-1} \Im z \) if \( b_n < p \cdot \Re z < b_n^{-1} \), \(-z - pb_n + f_n(pb_n) \) if \( p \cdot \Re z \leq b_n \), and \(-z - pb_n^{-1} + f_n(pb_n^{-1}) \) if \( p \cdot \Re z \geq b_n^{-1} \). Then \( u_n \) is a quasiconformal extension of \( q_n|\beta_n \cap N_n \) to \( \hat{C} \). \( \text{q.e.d.} \)

**Lemma 6.** There exists a quasiconformal automorphism \( h_n \) of \( C \) so that \( h_n(z) = f_n(z) \) for \( z \in \partial C \cap \partial A \) and \( =z \) for \( z \in \partial C \cap A^* \) and that \( \lim_{n \to \infty} ||\mu[h_n]\|_\infty = 0 \).

**Proof.** For \( \theta \in [-\pi, 0] \) define \( \varphi_n(\theta) \in [-\pi, 0] \) as \( f_n(\exp(\sqrt{-1}\theta)) = \exp(\sqrt{-1}\varphi_n(\theta)) \). Set \( h_n(\rho \exp(\sqrt{-1}\theta)) = \rho \exp(\sqrt{-1}\{\rho - 1}\theta + (2 - \rho)\varphi_n(\theta)) \),
where $\rho \in [1, 2]$ and $\theta \in [-\pi, 0]$. Then $h_n$ is a homeomorphism of $\text{Cl } C$ onto itself with $h_n(z) = f_n(z)$ for $z \in \partial C \cap \partial D$ and $h_n(z) = z$ for $z \in \partial C \cap \partial D^\star$. For $z = \rho \exp(\sqrt{-1}\theta) \in C$ it holds that
\[
|\mu[h_n](z)| = \frac{|\rho(h_n)_p(z) + \sqrt{-1}(h_n)_q(z)|}{|\rho(h_n)_p(z) - \sqrt{-1}(h_n)_q(z)|} = |(2 - \rho)(1 - \psi'(\theta)) + \sqrt{-1}\rho(\theta - \psi(\theta))| \\
\times |\rho + (2 - \rho)\psi'(\theta) + \sqrt{-1}\rho(\theta - \psi(\theta))|^{-1}.
\]
By Lemma 4 $\lim_{n \to \infty} \psi_n(\theta) = \theta$ and $\lim_{n \to \infty} \psi'_n(\theta) = 1$ uniformly on $(-\pi, 0)$. Hence we see $\lim_{n \to \infty} \|\mu[h_n]\| = 0$. q.e.d.

**Proof of Lemma 3.** Define $G_n(z) = f_n(z)$ if $z \in \text{Cl } A_n$, $= g_n(z)$ if $z \in \text{Cl } B_n$ and $= h_n(z)$ if $z \in \text{Cl } C$. Then Lemma 3 follows from Lemmas 5 and 6. q.e.d.

**6. Proof of Lemma 2.** Let $r$ and $s$ be real numbers with $r > 1$ and $0 < s < r + r^{-1}$. Let $T$ be the vertical line in $\hat{C}$ passing through $s$. Then $E_r$ and $T$ intersect at exactly two points $\zeta \in U$ and $\bar{\zeta} \in L$. Let $I$ be the bounded closed subarc of $T$ joining $\zeta$ to $\bar{\zeta}$. Let $P$ be the component of $\hat{C} - T$ containing the origin. Denote by $J$ the Jordan loop $(E_r \cap P) \cup I$. Let $Q$ be the interior of the circle with the diameter $I$. Note that both $T$ and $P$ depend on $s$, and $\zeta$, $I$, $J$ and $Q$ all depend on both $r$ and $s$.

**Proof of Lemma 2.** Fix $s \in (0, r + r^{-1})$ sufficiently near to $r + r^{-1}$ so that
\[\text{diam } Q \leq d/2\]
and
\[\delta(J, E_r) \leq d/2,
\]
where diam $Q$ denotes the Euclidean diameter of $Q$.

First we construct $\{\sigma_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$. Let $\tau_n$ be a Möbius transformation such that $\tau_n(P) = U$ and $\tau_n(Q) = \hat{C} - \text{Cl } V_n$, where $V_n$ is the open ball $\{z \in C; |z| < \delta_n \}$ defined at the beginning of §5. Then $j_n = \tau_n(E_r \cap \text{Cl } P)$ is a smooth closed Jordan arc in $\text{Cl } V_n$ joining $-\delta_n$ to $\delta_n$. Let $\{G_n\}_{n=1}^\infty$ be the sequence of quasiconformal automorphisms of $\Omega$ in Lemma 3. Let $D_0$ be a Dirichlet fundamental region for $\gamma$ in $\Delta$. Since $\Gamma$ is of the second kind, $D_0$ has free sides. Let $D$ be the union of $D_0$, the region obtained from $D_0$ by reflection in $\partial D$ and the free sides of $D_0$. Let $\sigma$ be a Möbius transformation such that $\sigma(U) = \Delta$ and $\sigma(\text{Cl } \Omega) \subset D$. Define
\[F_n = \begin{cases} \gamma \circ \sigma \circ G_n \circ \gamma^{-1} & \text{in } \gamma \circ \sigma(\Omega) \\ \text{the identity mapping in } \hat{C} - \bigcup_{\gamma \in \Gamma} \gamma \circ \sigma(\Omega) \end{cases}
\]
and $\sigma_n = \sigma \circ \tau_n$. Then $F_n$ is a homeomorphism of $\hat{C}$ onto itself which is quasiconformal off $\partial \hat{D}$. Hence $F_n$ is a quasiconformal automorphism of $\hat{C}$ (Lehto-Virtanen [6, p. 45]).

Secondly, we prove (3.1), (3.2) and (3.3). By (6.3) we see $F_n \circ \gamma = \gamma \circ F_n$ for all $\gamma \in \Gamma$. Since $j_n - \{ -\delta_n, \delta_n \} = \tau_n(P \cap E_n) \subset \tau_n(P \cap (\hat{C} - \text{Cl} \ Q)) = U \cap V_n$ and since $\tau_n(\infty) \subset \tau_n(T - I) \subset \tau_n(T \cap (\hat{C} - \text{Cl} \ Q)) = \partial U \cap V_n$, the point $\tau_n(\infty)$ belongs to $A_n$. Then by Lemma 3(i) and (6.3) we see $F_n \circ \sigma_n(\infty) = F_n \circ \gamma \circ \tau_n(\infty) \in F_n \circ \sigma(A_n) = \sigma \circ G_n(A_n) \subset \sigma(L) = \hat{D}^*$. Since by (6.3) $|\mu(F_n^{-1}|\hat{D}^*)|_{\infty} = |\mu(F_n^{-1}|\hat{D}^* \cap D)|_{\infty} = |\mu(G_n^{-1}|\hat{D} \cap L)|_{\infty}$, Lemma 3(ii) shows $\lim_{n \to \infty} |\mu(F_n^{-1}|\hat{D}^*)|_{\infty} = 0$.

Finally, we prove (3.4). It follows from Lemma 3(i) and (6.3) that

$$\sigma_n^{-1} \circ F_n^{-1}(\partial \hat{D}) = \tau_n^{-1} \circ \sigma^{-1} \circ F_n^{-1}(\partial \hat{D}) \subset \tau_n^{-1}(l_n \cup (\hat{C} - Q))$$

$$\subset \tau_n^{-1}(j_n \cup (\hat{C} - \text{Cl} \ V_n)) = (E_n \cap \text{Cl} \ P) \cup Q \subset J \cup Q.$$  

Hence by (6.1) $\delta(\sigma_n^{-1} \circ F_n^{-1}(\partial \hat{D}), J) \leq \frac{d}{2}$. This together with (6.2) yields that $\delta(\sigma_n^{-1} \circ F_n^{-1}(\partial \hat{D}), E_n) \leq \frac{d}{2}$. Now we complete the proof of Lemma 2 and hence that of Theorem.

REFERENCES