NORMS OF HANKEL OPERATORS AND UNIFORM ALGEBRAS, II

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Abstract. Let $H^\omega$ be an abstract Hardy space associated with a uniform algebra. Denoting by $(f)$ the coset in $(L^\omega)^{-1}/(H^\omega)^{-1}$ of an $f$ in $(L^\omega)^{-1}$, define $\|(f)\| = \inf\{\|g\|^{-1}; g \in (f)\}$ and $r_0 = \sup\{(f)\|; (f) \in (L^\omega)^{-1}/(H^\omega)^{-1}\}$. If $r_0$ is finite, we show that the norms of Hankel operators are equivalent to the dual norms of $H^1$ or the distances of the symbols of Hankel operators from $H^\omega$. If $H^\omega$ is the algebra of bounded analytic functions on a multiply connected domain, then we show that $r_0$ is finite and we determine the essential norms of Hankel operators.

0. Introduction. Let $X$ be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on $X$, and let $A$ be a uniform algebra on $X$. For $\tau \in M_A$, the maximal ideal space of $A$, set $A_\tau = \{f \in A; \tau(f) = 0\}$. Let $m$ be a representing measure for $\tau$ on $X$.

The abstract Hardy space $H^p = H^p(m)$, $1 \leq p \leq \infty$, determined by $A$ is defined to be the closure of $A$ in $L^p = L^p(m)$ when $p$ is finite and to be the weak* closure of $A$ in $L^\infty = L^\infty(m)$ when $p$ is infinite. Put $H^p_\tau = \{f \in H^p; \int_X f d\mu = 0\}$, $K^p = \{f \in L^p; \int_X f d\mu = 0 \text{ for all } g \in A_\tau\}$ and $K^p_\tau = \{f \in K^p; \int_X f d\mu = 0\}$. Then $H^p_\tau \subset K^p$ and $H^p \subset K^p_\tau$.

Let $Q^{(1)}$ be the orthogonal projection from $L^2$ to $(H^2)^\perp = \bar{K}^2_\tau$ and $Q^{(2)}$ the orthogonal projection from $L^2$ to $\bar{K}^2_\tau$. For a function $\phi$ in $L^\infty$ we denote by $M_\phi$ the multiplication operator on $L^2$ determined by $\phi$. As in the previous paper [14], two generalizations of the classical Hankel operators are defined as follows. For $\phi$ in $L^\infty$ and $f$ in $H^2$

$$H_{\phi}^{(j)} f = Q^{(j)} M_\phi f \quad (j = 1, 2).$$

If $A$ is a disc algebra and $\tau(f) = \tilde{f}(0)$, where $\tilde{f}$ denotes the holomorphic extension of $f$ in $A$, then $\tau$ is in $M_A$. The normalized Lebesgue measure $m$ on the unit circle $T$ is a representing measure for $\tau$. Then $H^2$ is the classical Hardy space and $H^2_\tau = K^2_\tau$. Hence $H_{\phi}^{(1)} = H_{\phi}^{(2)}$ and it is the classical Hankel operator $H_{\phi}$. It is well known that

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(a) \( \| H_\phi \| = \| \phi + H^\infty \| \)
and
(b) \( \| H_\phi \|_* = \| \phi + H^\infty + C(T) \|_* \),
where the essential norm \( \| H_\phi \|_* \) of \( H_\phi \) is the distance to the compact operators. (a) is due to Nehari (cf. [16, Theorem 1.3], [15]), while (b) is due to Adamyan, Arov and Krein (cf. [16, p. 6], [2]). (b) yields Hartman's result (cf. [16, Theorem 1.4], [11]) to the effect that
(c) \( H_\phi \) is compact if and only if \( \phi \) is in \( H^\infty + C(T) \).

In the previous paper [14] we considered the generalizations of (1). The main idea was to consider Hankel operators on \( vH^2 \) for every non-negative invertible function \( v \) in \( L^\infty \), avoiding a factorization theorem of \( H_1 \). Namely, if \( h \) is in \( H^1 \) and \( \text{int} \{ |h| \text{d}m \leq 1 \} \), then \( h = fg \), \( f \in H^2 \) and \( g \in H^1_0 \) where \( \text{int} \{ |f|^2 \text{d}m \leq 1 + \varepsilon \} \) and \( \text{int} \{ |g|^2 \text{d}m \leq 1 + \varepsilon \} \) for some \( \varepsilon > 0 \). Let \( v \) be a nonnegative function in \( L^\infty \) with \( v^{-1} \) in \( L^\infty \). Let \( Q_\phi^{(1)} \) be the orthogonal projection from \( L^2 \) onto \( (vH^2)^\perp = v^{-1}K_0^2 \) and \( Q_\phi^{(2)} \) the orthogonal projection from \( L^2 \) onto \( v^{-1}H_0^2 \). If \( v \) is a constant function, then \( Q_\phi^{(j)} = Q^{(j)} \) (\( j = 1, 2 \)). For \( \phi \in L^\infty \) and \( f \in vH^2 \), \( H_\phi^{(j)*} \) is the operator defined by
\[
H_\phi^{(j)*} f = Q_\phi^{(j)} M_\phi f, \quad (j = 1, 2).
\]
If \( v \) is a nonzero constant, then \( H_\phi^{(j)*} = H^{(j)}_\phi \) (\( j = 1, 2 \)). Put \( (L^\infty)^{\perp 1} = \{ v \in L^\infty : v^{-1} \in L^\infty \} \). The following theorem was shown in the previous paper [14] and gives (a).

**Generalized Nehari's Theorem I.** Let \( \phi \) be a function in \( L^\infty \), then
\[
\sup \{ \| H_\phi^{(2)} v \phi \| ; v \in (L^\infty)^{\perp 1} \} = \| \phi + K^\infty \| .
\]
If \( K^\infty \) is dense in \( K^1 \), then
\[
\sup \{ \| H_\phi^{(1)} v \phi \| ; v \in (L^\infty)^{\perp 1} \} = \| \phi + H^\infty \| .
\]

We now show two lemmas which will be used in later sections. Let \( P_\phi \) be the orthogonal projection from \( L^2 \) onto \( vH^2 \). If \( v \) is a constant function, we shall write \( P_\phi = P \).

**Lemma 1.** Let \( \phi \) be a function in \( L^\infty \). Then for any \( v \) and \( u \) in \( (L^\infty)^{\perp 1} \) and for \( j = 1, 2 \)
\[
\| H_\phi^{(j)} - H_\phi^{(2)} v \phi \| \leq \| \phi \|_\infty (\| v^{-1} \|_\infty + \| u^{-1} \|_\infty) \| v - u \|_\infty .
\]

**Proof.** Since \( \| M_\phi f \|_2 \geq \| v^{-1} \|_\infty \| f \|_2 \) for all \( f \in H^2 \), by [7, Lemma 1.1.]
\[
\| P_\phi - P_u \| \leq (\| v^{-1} \|_\infty + \| u^{-1} \|_\infty) \| v - u \|_\infty .
\]
Similarly, \( \| Q_\phi^{(2)} - Q_\phi^{(2)} \| \leq (\| v^{-1} \|_\infty + \| u^{-1} \|_\infty) \| v - u \|_\infty . \) Hence for \( j = 1, 2 \)
LEMMA 2. Let $\phi$ be a function in $L^\infty$. If $H_{\phi}(j)(j=1, 2)$ is compact, then $H_{\phi}v(j)(j=1, 2)$ is compact for any $v$ in $(L^\infty)^{-1}_\omega$.

**Proof.** For any $f \in vH^2$ and $g \in v^{-1}K^2_0$, $(H_{\phi}(1)v(f), g) = (vH_{\phi}(1)v^{-1}f, g)$. Hence $H_{\phi}v(f) = Q(1)v(MvH_{\phi}(1)v^{-1}f)$ for any $f \in vH^2$. The proof for $H_{\phi}(2)v$ is similar. This implies the lemma.

Let $N_\tau$ denote the set of representing measures for $\tau$ on $X$. In this paper we sometimes will impose the following two conditions on $\tau$:

(1) $N_\tau$ is finite dimensional and $n = \dim N_\tau$.

(2) $m$ is a core measure of $N_\tau$.

Let $N_\tau^\infty$ be the real annihilator of $A$ in $L_\tau^\infty$. Then $\dim N_\tau^\infty = n$ and $A + \hat{A} + N_\tau^\infty$ is weak*-dense in $L^\infty$, where $N_\tau^\infty = N_\tau^\circ + iN_\tau^\circ$ (cf. [10, p. 109]). $L^2 = H^2 \oplus H_0^2 \oplus N_\tau^\circ$. Set $\mathcal{E} = \exp N_\tau^\circ$; then $\mathcal{E}$ is a subgroup of $(L^\infty)^{-1}_\omega$. Moreover, together with (1) we often will make the following stronger conditions (3) on $\tau$ instead of (2).

(3) $m$ is a unique logmodular measure of $N_\tau$.

Then the linear span of $N_\tau^\infty \cap \log((H^\infty)^{-1})$ is $N_\tau^\circ$ (cf. [10, p. 114]). Choose $h_1, \ldots, h_n \in (H^\infty)^{-1}$ so that $\{\log|h_j|\}_{j=1}^n$ is a basis for $N_\tau^\circ$. Put $\omega_j = \log|h_j|$ $(1 \leq j \leq n)$ and $\mathcal{E}_0 = \{\exp(\sum_{j=1}^n \sigma_j \omega_j); 0 \leq \sigma_j \leq 1\}$. Then $\mathcal{E}_0 \subset \mathcal{E}$. The following theorem was shown in the previous paper [14].

**Generalized Nehari’s Theorem II.** Assume the assumptions (1) and (3) on $\tau$. Let $\phi$ be a function in $L^\infty$, then

$$\sup_{v \in \mathcal{E}_0} \|H_{\phi}v\| = \|\phi + H^\infty + N_\tau^\infty\|$$

and

$$\sup_{v \in \mathcal{E}_0} \|H_{\phi}v\| = \|\phi + H^\circ\|.$$

Moreover the supremums in both equalities are attained.

In Section 1, $\gamma_0$, which is defined in Abstract, is studied. Under the assumptions (1) and (2) we determine when $\gamma_0$ is finite. In Section 2, we give examples of concrete uniform algebras to which results in this paper can apply. Moreover $\gamma_0$ is calculated in some examples. In Section 3, if $\gamma_0$ is finite, we show that $\|H_{\phi}v\|$ (resp. $\|H_{\phi}v\|$) is equivalent to $\|\phi + H^\circ\|$ (resp. $\|\phi + K^\infty\|$). In Section 4 we give applications of results in Section 3 to weighted norm inequalities for conjugation operators and invertible Toeplitz operators in uniform algebras. In Section 5 we determine the
essential norms of Hankel operators in the case of (I) in Section 2. In Section 6 we consider the relationship between \( \gamma_0 \) and the factorization theorem of \( H_+ \). In Section 7 we consider the relationship between generalized Nehari's Theorem and Arveson's distance formula for nest algebras.

1. Quotient group and a constant. Denoting by \( (f) \) the coset in \( (L^\omega)^{-1}/(H^\omega)^{-1} \) of an \( f \) in \( (L^\omega)^{-1} \), define

\[
\| (f) \| = \inf \{ \| g \|_\infty \| g^{-1} \|_\infty ; g \in (f) \}
\]

and

\[
\gamma_0 = \sup \{ \| (f) \| ; (f) \in (L^\omega)^{-1}/(H^\omega)^{-1} \} .
\]

Then \( \| (f)(h) \| \leq \| (f) \| \| (h) \| \) and, in general, \( \gamma_0 \) can be finite or infinite. Let \( L^\omega_R \) be the space of real-valued functions in \( L^\omega \). Let \( \log \| (H^\omega)^{-1} \| \) be the lattice in \( L^\omega_R \) consisting of the elements of the form \( \log |f|, f \in (H^\omega)^{-1} \). There is a natural map of \( (L^\omega)^{-1}/(H^\omega)^{-1} \) onto \( L^\omega_R/\log \| (H^\omega)^{-1} \| \) which sends \( (f) \) to \( (\log |f|) \). Define \( \| (\log |f|) \| = \inf \{ \| \log |f| + \log |g| \|_\infty ; g \in (H^\omega)^{-1} \} \) and \( \gamma_1 = \sup \{ \| (\log |f|) \| ; (\log |f|) \in L^\omega_R/\log \| (H^\omega)^{-1} \| \} \).

**Proposition 1.** \( \| (f) \| = \exp 2 \| (\log |f|) \| \) and \( \gamma_0 = \exp 2 \gamma_1 \).

**Proof.** It suffices to show that \( \| (f) \| = \exp 2 \| (\log |f|) \| \) for all \( f \in (L^\omega)^{-1} \). Pick such an \( f \).

\[
\| (f) \| = \inf \{ \| fg \|_\infty \| g^{-1} \|_\infty ; g \in (H^\omega)^{-1} \}
\]

\[
= \exp \inf \{ \text{ess. sup} \{ \log |f| + \log |g| \} - \text{ess. inf} \{ \log |f| + \log |g| \} \} .
\]

Since the constants are in \( (H^\omega)^{-1} \), this last quantity can be rewritten as

\[
= \exp 2 \inf \{ \text{ess. sup} \{ \log |f| + \log |g| \} \} = \exp 2 \| (\log |f|) \| .
\]

The proof of Proposition 1 is parallel to that of Proposition 2.2 in [17]. Rochberg [17] considered \( (H^\omega)^{-1}/\exp H^\omega \) instead of \( (L^\omega)^{-1}/(H^\omega)^{-1} \). If \( A \) is a disc algebra, then \( \| (f) \| = 1 \) for any \( (f) \in (L^\omega)^{-1}/(H^\omega)^{-1} \) and so \( \gamma_0 = 1 \) because of Proposition 1 and \( L^\omega_R = \log \| (H^\omega)^{-1} \| \). Let \( \text{crls} \log \| (H^\omega)^{-1} \| \) denote the closed real linear span of \( \log \| (H^\omega)^{-1} \| \).

**Lemma 3.** (1) If \( v = \sum_{j=1}^n t_j \log |h_j| \) with \( 0 \leq t_j \leq 1 \) and \( h_j \in (H^\omega)^{-1} \) (\( 1 \leq j \leq n \)), then \( \sup \{ \| (tv) \| ; -\infty < t < \infty \} < \infty \). (2) If \( v \in L^\omega_R \) is not in \( \text{crls} \log \| (H^\omega)^{-1} \| \), then \( \sup \{ \| (tv) \| ; -\infty < t < \infty \} = \infty \).

**Proof.** (1) \( \| (tv) \| = \| (\sum_{j=1}^n t_j \log |h_j|) \| = \| (\sum_{j=1}^n (t_j - [t_j]) \log |h_j|) \| \), where \([\cdot]\) is the greatest integer function. Hence \( \sup \{ \| (tv) \| ; -\infty < t < \infty \} < \infty \).

(2) There exists a positive constant \( s \) such that \( \| tv + \text{crls} \log \| (H^\omega)^{-1} \| \|_\infty \geq s \| tv \|_\infty \) for any \( t \). Hence \( \sup \{ \| (tv) \| ; -\infty < t < \infty \} = \infty \).

**Theorem 2.** Suppose \( \tau \) satisfies the conditions (1) and (2). Then
m is a unique logmodular measure if and only if $\gamma_0$ is finite.

**Proof.** By Proposition 1 it is sufficient to show that $m$ is a unique logmodular measure if and only if $\gamma_1$ is finite. $m$ is a unique logmodular measure if and only if $\text{crls} \log |(H^\infty)^{-1}| = L_0^\infty$ (see [10, p. 114]). By this and (2) in Lemma 3, if $m$ is not a unique logmodular measure, then $\gamma_1 = \infty$.

Suppose $m$ is a unique logmodular measure. If $v \in L_0^\infty$, then $v = u_0 + \log |g|$ with $u_0 \in N_0$ and $g \in (H^\infty)^{-1}$ (cf. [10, p. 109]). Moreover, we can choose $u_0 \in \log E_0$ (see Introduction). By (1) in Lemma 3 $\gamma_1$ is finite and in fact $\gamma_1 \leq \sup \{|\sum_{j=1}^n s_j u_j|; 0 \leq s_j \leq 1\}$.

2. Concrete uniform algebras. (I) Let $Y$ be a compact subset of the plane, and let $R(Y)$ be the uniform closure of the set of rational functions in $C(Y)$. We regard $R(Y)$ as a uniform algebra on its Shilov boundary, the topological boundary $X$ of $Y$. Suppose the complement $Y^c$ of $Y$ has a finite number $n$ of components and the interior $Y^o$ of $Y$ is a nonempty connected set. Let $A = R(Y)|X$; then $M_A = Y$. If $\tau \in M_A$ is in $Y^o$ and $m$ is a harmonic measure, then $m$ is a unique logmodular measure of $N_\tau$ and $\dim N_\tau = n < \infty$ [10, p. 116]. Then $N_\infty \subset C(X)$ By Theorem 2, $\gamma_0$ is finite. Let $X = X_0 \cup X_1 \cup \cdots \cup X_n$, where $X_0$ is the “outside” component of $X$ and $X_1, \ldots, X_n$ are the “inside” components of $X$.

Define $v_j \in L_0^\infty$ to be 1 on $X_j$ and 0 on $X \setminus X_j$ $1 \leq j \leq n$. Then $\gamma_1 = \sup \{|\sum_{j=1}^n s_j v_j|; -\infty < t_j < \infty$ and $1 \leq j \leq n\}$.

(II) In (I) let $Y$ be the annulus $\{r \leq |z| \leq 1\}$. Then $\gamma_0 = r^{-1/2}$. Since $(L_0^\infty/ \text{the uniform closure of } Re H^\infty)$ has dimension one, we get $\gamma_1 = \sup \inf \{\|t \log |z| - (\text{Re } f + n \log |z|)|_\infty; f \in H^\infty \text{ and } n \text{ ranges over all integers}\}$.

For any integer $n$

$$\inf \{\|t - n \log |z| - \text{Re } f\|_\infty; f \in H^\infty\}$$

$$= |t - n| \log r^{-1} \inf \{\|\chi_E - \text{Re } f\|_\infty; f \in H^\infty\} = \frac{1}{2} |t - n| \log r^{-1},$$

where $\chi_E = 0$ on $|z| = 1$ and $\chi_E = 1$ on $|z| = r$. Thus

$$\gamma_1 = \sup \inf \frac{1}{2} |t - n| \log r^{-1} = \frac{1}{4} \log r^{-1}.$$  

We shall show that

$$\inf \{\|\chi_E - \text{Re } f\|_\infty; f \in H^\infty\} = 1/2.$$  

Choosing $f = 1/2$, the infimum is not greater than 1/2. If the infimum is less than 1/2 then by a theorem of Runge we can show that $\chi_E \in H^\infty$ as in the proof of Theorem in [13, p. 182]. This contradiction implies
that the infimum is just 1/2.

(III) Let \( \mathcal{A} \) be the disc algebra and let \( A \) be a subalgebra of \( \mathcal{A} \) which contains the constants and which has finite codimension in \( \mathcal{A} \). If \( \tau(f) = \hat{f}(0) \) for \( f \in A \) and \( m \) is the normalized Lebesgue measure on the circle \( T \), then it is easy to check that \( m \) is a core point of \( N \), and \( N^\infty \subset C(T) \). If \( A \neq \mathcal{A} \), then \( H^\infty \) is contained properly in the classical Hardy space. Hence \( H^\infty \) is not \( \tau \)-maximal. On the other hand if \( \tau \) has a unique logmodular measure \( m \), then \( H^\infty \) is \( \tau \)-maximal ([9, Theorem 5.5]). This implies that \( m \) is not a unique logmodular measure and hence Theorem 2 implies that \( \gamma_0 \) is infinite.

(IV) The unit polydisc \( U^n \) and the torus \( T^n \) are cartesian products of \( n \) copies of the unit disc \( U \) and of the unit circle \( T \), respectively. \( A(U^n) \) is the class of all continuous complex functions on the closure \( \bar{U}^n \) of \( U^n \) with holomorphic restrictions to \( U^n \). Let \( A = A(U^n) \big| X \) and \( X = T^n \). This is the so-called polydisc algebra. For simplicity we assume \( n = 2 \). Let \( m \) be the normalized Lebesgue measure; then \( m \) is a representing measure for \( \tau \) on \( X \) where \( \tau(f) = f(0) \) and \( 0 \in U^2 \). Suppose \( 1 \leq p \leq \infty \) and \( Z_2^+ = \{(n, m) \in Z^2; n \geq 0 \text{ and } m \geq 0\} \). Then \( H^p = \{f \in L^p; \hat{f}(n, m) = 0 \text{ if } (n, m) \in Z_2^+\} \) and \( K^p = \{f \in L^p; \hat{f}(n, m) = 0 \text{ if } (-n, -m) \in Z_2^+\} \). \( K^\infty \) is dense in \( K^p \). Unfortunately we do not know whether \( \gamma_0 \) is finite or not.

3. Norms of Hankel operators. Assuming \( \gamma_0 \) is finite, we show that \( ||H_\phi^{(1)}|| \) (resp. \( ||H_\phi^{(2)}|| \)) is equivalent to \( ||\phi + H^\infty|| \) (resp. \( ||\phi + K^\infty|| \)).

**Theorem 3.** Let \( \phi \) be a function in \( L^\infty \). Then

\[
||H_\phi^{(1)}|| \leq ||\phi + H^\infty|| \leq \gamma_0||H_\phi^{(2)}||.
\]

If \( K^\infty \) is dense in \( K^1 \), then

\[
||H_\phi^{(1)}|| \leq ||\phi + H^\infty|| \leq \gamma_0||H_\phi^{(2)}||.
\]

**Proof.** Let \( v \in (L^\infty_1)^+ \). If \( \int_X |f|^p v^p dm \leq 1 \) and \( \int_X |g|^p v^{-2} dm \leq 1 \), then

\[
\int_X |f|^p dm \leq ||v^{-2}||_\infty \text{ and } \int_X |g|^p dm \leq ||v^2||_\infty.
\]

Hence

\[
||H_\phi^{(1)}|| = \sup \left\{ \left| \int_X f g \phi dm \right|; f \in H^1, g \in K^\infty, \int_X |f|^p v^p dm \leq 1 \text{ and } \int_X |g|^p v^{-2} dm \leq 1 \right\}
\]

\[
\leq (||v^{-2}||_\infty ||v^2||_\infty)^{1/2} ||H_\phi^{(1)}||.
\]

If \( h \in (H^\infty)^{-1} \), then \( v|h|H^1 = b(vH^1) \) and \( v^{-1} |h|^{-1} K^\phi = b(v^{-1}K^\phi) \) with \( b = ||h||. \) Then \( Q^{(1)}_{M_\phi} = M_\phi Q^{(1)}_M \) and so \( H_\phi^{(1)} = M_\phi Q^{(1)}_M H_\phi \). Hence \( ||H_\phi^{(1)}|| = ||H_\phi^{(1)}||. \) Thus for any \( h \in (H^\infty)^{-1} \)

\[
||H_\phi^{(1)}|| \leq ||v|h|| \|v^{-1}||h^{-1}\|_\infty ||H_\phi^{(1)}||.
\]
It is easy to see that
\[
\sup_{v \in L^\infty} \{ \inf \{ \|v\|_\infty, \|v^{-1}\|_\infty \mid h \in (H^\infty)^{-1} \} \} = \gamma_0.
\]
Now generalized Nehari’s Theorem I shows that \( \|H_f\| \leq \|\phi + H^\infty\| \leq \gamma_0\|H_f\| \). Similarly the inequality for \( H_f^{(2)} \) follows.

4. Applications. In the previous paper [14] we gave applications of generalized Nehari’s Theorems I and II to weighted norm inequalities and invertible Toeplitz operators. In this section we shall give applications of Theorem 3.

Recall \( P \) is the orthogonal projection from \( L^2 \) to \( H^2 \). Let \( \mathcal{S}^{(1)} \) denote \( P \) restricted to \( H^\infty + K_0^\infty \) and \( \mathcal{S}^{(2)} \) denote \( P \) restricted to \( H^\infty + H_0^\infty \). We are interested in knowing when \( \mathcal{S}^{(j)} \) (\( j = 1, 2 \)) is bounded in \( L^2(w) = L^2(wdm) \), where \( w \) is a nonnegative weight function in \( L^1 \). Put
\[
\begin{align*}
(a) & \sup \left\{ \left| \int_X f(g)wdm \right| ; f \in H^\infty, g \in K_0^\infty \text{ and } \int_X |f|^2 wdm = \int_X |g|^2 wdm = 1 \right\} = \rho_1, \\
(b) & \sup \left\{ \left| \int_X f(g)wdm \right| ; f \in H^\infty, g \in H_0^\infty \text{ and } \int_X |f|^2 wdm = \int_X |g|^2 wdm = 1 \right\} = \rho_2.
\end{align*}
\]
Then it is easy to see that \( \|\mathcal{S}^{(j)}\| \leq (1 - \rho_j)^{-1} \). The following lemma is known [19]. We shall give the proof for completeness.

**LEMMA 3.** \( \|\mathcal{S}^{(j)}\|^2 = (1 - \rho_j)^{-1} \) (\( j = 1, 2 \)).

**Proof.** If \( \gamma = \|\mathcal{S}^{(1)}\| < \infty \), then for any real \( t \) and for any \( f \in H^\infty \) and \( g \in K_0^\infty \) we have
\[
t \leq \frac{1}{t} \left( \int_X |f|^2 wdm + \int_X |g|^2 wdm + 2t \text{ Re} \int_X fg wd m \right) \geq 0.
\]
Hence
\[
\int_X |fgwdm|^2 \leq \frac{\gamma^2 - 1}{\gamma^2} \int_X |f|^2 wdm \int_X |g|^2 wdm
\]
and so \( \gamma^2 \geq (1 - \rho_j)^{-1} \). We can prove it for \( j = 2 \) in the same method.

If \( A \) is a disc algebra, then \( \mathcal{S} = \mathcal{S}^{(1)} = \mathcal{S}^{(2)} \) is bounded in \( L^2(w) \) if and only if \( w = |h|^2 \) for some outer function \( h \) in \( H^2 \) and \( \|\phi + H^\infty\| < 1 \) with \( \phi = |h|^2/h^2 \). This result is called Helson-Szego’s theorem [12]. This was generalized to general uniform algebras by the author [14]. The following generalization seems to be better than the previous one.

**COROLLARY 1.** Suppose \( K^\infty \) is dense in \( K^2 \). Let \( w = |h|^2 \) for some function \( h \) in \( H^2 \) such that \( hH^\infty \) is dense in \( H^2 \) and \( hK^\infty \) is dense in \( K^2 \).
Let $\phi = |h|^2/h^2$.

(1) If $\|\phi + H^\omega\| = \rho < 1$ then $\mathcal{F}^{(1)}$ is bounded in $L^2(w)$ and $\|\mathcal{F}^{(1)}\| \leq (1 - \rho^5)^{-1/2}$.

(2) If $\mathcal{F}^{(1)}$ is bounded in $L^2(w)$ and $\|\mathcal{F}^{(1)}\| = \gamma$ then

$$\|\phi + H^\omega\| < \gamma_0 \gamma^{-1}(\gamma^2 - 1)^{1/2}$$

Hence if $\gamma < \gamma_0(\gamma_0 - 1)^{1/2}$ then $\|\phi + H^\omega\| < 1$.

PROOF. (1) $\rho_1 \leq \rho$, since

$$\rho = \|\phi + H^\omega\| = \sup \left\{ \left| \int_X F \phi dm \right| ; F \in K_1, \|F\|_1 \leq 1 \right\}$$

$$\geq \sup \left\{ \left| \int_X \phi g dm \right| ; f \in H^\omega, g \in K_1, \int_X |f|^2 dm = \int_X |g|^2 dm = 1 \right\} = \rho_1.$$}

In the last equality we used the facts that $w = |h|^2$ and that $hH^\omega$ (resp. $hK_1^\omega$) is dense in $H^2$ (resp. $K_1^\omega$).

(2) Since $\rho_1 = \gamma^{-1}(\gamma^2 - 1)^{1/2}$ by Lemma 3 and $\rho_1 = \|H^\omega\|$ by the proof of (1), Theorem 3 implies (2).

$K^\omega$ is dense in $K_1$ if we impose the assumptions (1) and (2) or if $A$ is a polydisc algebra. We have a similar result for $\mathcal{F}^{(2)}$ (or $\|\phi + K^\omega\|$) as in Corollary 1.

For $\phi$ in $L^\omega$ let $T_\phi$ be the operator on $H^\omega$ defined by $T_\phi f = P(M_\phi f)$. The operator $T_\phi$ will be called a Toeplitz operator. We are interested in knowing when $T_\phi$ is left invertible. In case $A$ is a disc algebra, Widom [18] showed that $T_\phi$ is left invertible if and only if $\|\phi + H^\omega\| < 1$. Abrahamse [1] generalized Widom's theorem to the case of (I) in concrete uniform algebras such that $\partial Y$ consists of $n + 1$ non-intersecting analytic Jordan curves. The author [14] generalized it to general uniform algebras. However these generalizations are not sufficient because except in the case of a disc algebra we cannot determine $\phi$ when $T_\phi$ is left invertible.

COROLLARY 2. Suppose $K^\omega$ is dense in $K^1$. Let $\phi$ be a unimodular function in $L^\omega$.

(1) If $\|\phi + H^\omega\| = \rho < 1$, then $\|T_\phi f\|_2 \geq (1 - \rho^5)^{1/2} \|f\|_2$ for any $f$ in $H^2$.

(2) If $\|T_\phi f\|_2 \geq \varepsilon \|f\|_2$ for any $f$ in $H^2$, then

$$\|\phi + H^\omega\| \leq \gamma_0(1 - \varepsilon^5)^{1/2}.$$ 

Hence if $\varepsilon > \gamma_0(\gamma_0 - 1)^{1/2}$, then $\|\phi + H^\omega\| < 1$.

PROOF. Since $\phi$ is a unimodular function, $\|H^\omega_0 f\|_2^2 + \|T_\phi f\|_2^2 = \|f\|_2^2$ for any $f \in H^2$. Theorem 3 and this imply the corollary.

In the case of (I) for concrete uniform algebras, $\|\phi + H^\omega\| < 1$ may
not hold even if $T_\phi$ is left invertible (cf. [1]).

5. Essential norms of Hankel operators. In this section we shall concentrate on concrete uniform algebras, that is, (I) in Section 2 such that $\partial Y$ consists of $n+1$ non-intersecting analytic Jordan curves. Hence $\tau$ satisfies the conditions (1) and (3). Using generalized Nehari’s Theorem II we shall generalize (b) in Introduction to this context.

Let $s = (s_1, s_2, \ldots, s_n) \in I^n = [0, 1] \times \cdots \times [0, 1]$. Then the mapping $s \mapsto \exp(\sum_{j=1}^n s_j u_j)$ is continuous, one-to-one and onto from $I^n$ to $\mathcal{E}_\omega$. Put

$$H_\phi^{(j)} = H_\phi^{(j)*} \quad (j = 1, 2),$$

where $v = \exp(\sum_{j=1}^n s_j u_j)$.

**Lemma 4.** Let $\phi$ be a function in $L^\omega$. Then for $j = 1, 2$ and for any $v$ and $u$ in $\mathcal{E}_\omega$

$$\|H_\phi^{(j)} v - H_\phi^{(j)} u\| \leq \|\phi\|_{\infty} (2 \sup_{v \in \mathcal{E}_\omega} \|v^{-1}\|_\omega) \|v - u\|_\omega.$$

The proof is clear by Lemma 1.

**Lemma 5.** If $\phi$ in $H^\omega + C(X)$, then $H_\phi^{(j)}$ ($j = 1, 2$) is compact for any $v$ in $\mathcal{E}_\omega$.

**Proof.** By Lemma 2 it is sufficient to show that $H_\phi^{(j)}$ is compact for any $\phi \in C(X)$. Let $\phi = (z - a)^{-1}$ for some $a \in Y^\omega$. Then

$$H_\phi^{(j)} f = Q^{(j)} \left[ \frac{f(a)}{z - a} + \frac{f - f(a)}{z - a} \right] = Q^{(j)} \frac{f(a)}{z - a}$$

for any $f \in H^\omega$ because $\{f \in H^\omega : f(a) = 0\} = (z - a)H^\omega$. Hence $H_\phi^{(j)}$ has rank one. Similarly if $\phi = (z - a)^{-n}$ for a positive integer $n$, we can show that $H_\phi^{(j)}$ has rank $n$. For any $\phi \in C(X)$ we can approximate $\phi$ by the following functions: $\sum_{j=0}^n b_j(z - a_j)^{-j}$ where $a_j \in Y^\omega$ and $b_j$ is constant (0 $\leq j \leq n$). Since $\|H_\phi^{(j)}\| \leq \|\phi + H^\omega\|$ and $\|H_\phi^{(j)}\| \leq \|\phi + H^\omega + N_\omega\|$, we can show that $H_\phi^{(j)}$ is compact if $\phi \in C(X)$.

**Theorem 4.** Let $\phi$ be a function in $L^\omega$. Then

$$\sup_{v \in \mathcal{E}_\omega} \|H_\phi^{(1)} v\|_\varepsilon = \sup_{v \in \mathcal{E}_\omega} \|H_\phi^{(2)} v\|_\varepsilon = \|\phi + H^\omega + C(X)\| .$$

Moreover, the suprema in both equalities are attained.

**Proof.** By Lemma 5 it is clear that $\sup\{\|H_\phi^{(j)} v\|_\varepsilon ; v \in \mathcal{E}_\omega\} \leq \|\phi + H^\omega + C(X)\|$ for $j = 1, 2$. We shall show the opposite inequality. Let $F$ be the Ahlfors function for $Y^\omega$ and $\tau \in Y^\omega$. Then $F \in C(X)$ (see [8, p. 114]). For any $v \in \mathcal{E}_\omega$ with $v = \exp(\sum_{j=1}^n t_j u_j)$ and $t = (t_1, t_2, \cdots, t_n) \in I^n$, put

$$f^{(j)}(t, l) = \|H_\phi^{(j)} v\| \quad (l = 0, 1, 2, \cdots; j = 1, 2).$$
Then \( f^{(j)}(t, l) \geq f^{(j)}(t, l + 1) \) and by Lemma 4
\[
|f^{(j)}(t, l) - f^{(j)}(s, l)| \leq ||\phi||_{\infty}(2 \sup_{v \in \mathbb{C}_0} ||v^{-1}||_{\infty})||\exp(\sum_{j=1}^{n} t_{j}n_{j}) - \exp(\sum_{j=1}^{n} s_{j}n_{j})||_{\infty}.
\]
Hence \( \{f^{(j)}(t, l)\}_{l=1}^{\infty} \) is an equicontinuous collection on \( I^n \) and uniformly bounded on \( I^n \). By Ascoli's theorem, there exists a subsequence \( \{f^{(j)}(t, l_i)\}_{i=1}^{\infty} \) of \( \{f^{(j)}(t, l)\}_{l=1}^{\infty} \) and a continuous function \( f^{(j)}(t) \) on \( I^n \) such that
\[
\sup_{t \in I^n} |f^{(j)}(t) - f^{(j)}(t, l_i)| \to 0 \quad (i \to \infty).
\]
Since \( \{f^{(j)}(t, l)\}_{l=1}^{\infty} \) is a decreasing sequence, this actually converges to \( f^{(j)}(t) \) uniformly on \( I^n \). Thus
\[
\lim_{l \to l \to I^n} \sup_{t \in I^n} f^{(j)}(t, l) = \sup_{t \in I^n} f^{(j)}(t) .
\]
By generalized Nehari’s Theorem II, \( \sup\{f^{(1)}(t, l); t \in I^n\} = ||F^{\phi} + H^\phi|| \) and \( \sup\{f^{(2)}(t, l); t \in I^n\} = ||F^{\phi} + H^\phi + N^\phi|| \) and so for \( j = 1, 2 \)
\[
\sup_{t \in I^n} f^{(j)}(t) \geq ||\phi + H^\phi + C(X)|| ,
\]
because the closure of \( \bigcup_{n=1}^{\infty} F^n(H^\phi + N^\phi) \) coincides with the closure of \( \bigcup_{n=1}^{\infty} F^n(H^\phi + N^\phi) \), which is \( H^\phi + C(X) \) (cf. [1, Theorem 1.22]). For any \( t \in I^n \), let \( S_{i} \) denote the multiplication by \( F \) on \( vH^2 \) where \( v = \exp(\sum_{j=1}^{n} t_{j}u_{j}) \). Let \( S_{i}^{*} \) be the adjoint of \( S_{i} \) from \( vH^2 \) to \( v^{-1}K_{i}^2 \) and \( S_{i}^{*} \) the adjoint of \( S_{i} \) from \( vH^2 \) to \( v^{-1}K_{i}^2 \). If \( K_{i}^1 \) is any compact operator from \( vH^2 \) to \( v^{-1}K_{i}^2 \) and \( K_{i}^2 \) is any compact operator from \( vH^2 \) to \( v^{-1}K_{i}^2 \), and \( l \) is positive integer, then for \( j = 1, 2 \)
\[
||H_{i}^{(j)} - K_{i}^{(j)}|| \geq ||(H_{i}^{(j)} - K_{i}^{(j)})S_{i}^{*}|| \geq ||H_{i}^{(j)}S_{i}^{*}|| - ||K_{i}^{(j)}S_{i}^{*}|| .
\]
Since \( (S_{i}^{*}) \to 0 \) strongly, we have \( ||K_{i}^{(j)}S_{i}^{*}|| \to 0 \). Also
\[
H_{i}^{(j)}S_{i} = H_{i}^{(j)} .
\]
Hence we can prove that the suprema are attained as in the proof of generalized Nehari’s Theorem II.
\[
||H_{i}^{(j)} - K^{(j)}|| \geq \lim_{t \to l} ||H_{i}^{(j)}(t, l) = f^{(j)}(t) .
\]
Thus \( ||H_{i}^{(j)}||_e \geq f^{(j)}(t) \) and
\[
\sup_{t \in I^n} ||H_{i}^{(j)}||_e \geq \sup_{t \in I^n} f^{(j)}(t) \geq ||\phi + H^\phi + C(X)|| .
\]
The following theorem is another generalization of (b) in Introduction.

**Theorem 5.** Let \( \phi \) be a function in \( L^\phi \). Then for \( j = 1, 2 \)
\[
||H_{i}^{(j)}||_e \leq ||\phi + H^\phi + C(X)|| \leq \gamma_\phi ||H_{i}^{(j)}||_e .
\]
The proof follows as in the case of a disc algebra (see [16, Theorem 1.4]) if we use Theorem 3 and the Ahlfors function.

6. Factorization theorems. We say $H_0$ has the weak approximate $\gamma$-factorization if $H_0$ satisfies the following property: For any $F$ in $H_0$ and any $\varepsilon > 0$, there exist $\{f_j\}_{j=1}^n$ in $H^2$ and $\{g_j\}_{j=1}^n$ in $H_0^*$ such that

$$\sum_{j=1}^n \|f_j\|_2 \|g_j\|_2 \leq \gamma \|F\|,$$

and

$$\|F - \sum_{j=1}^n f_j g_j\|_1 < \varepsilon.$$

**Proposition 6.** There exists a constant $\gamma$ with $\gamma \geq 1$ such that $\|\phi + K^o\| \leq \gamma \|H_0^{(w)}\|$ for all $\phi$ in $L^\infty$ if and only if $H_0$ has the weak approximate $\gamma$-factorization.

**Proof.** Let $V_\gamma$ be the closure in $L^1$ of the following set:

$$\left\{ \sum_{j=1}^n f_j g_j; f_j \in H^2, g_j \in H_0^* \text{ and } \sum_{j=1}^n \|f_j\|_2 \|g_j\|_2 \leq \gamma \right\}.$$

Put $V^1 = \{F \in H_0^*; \|F\|_1 \leq 1\}$. Then $V_\gamma$ is the closed convex subset in $\gamma V^1$. If $H_0$ has the weak approximate $\gamma$-factorization, then $V^1 \subset V_\gamma$ and so $\|\phi + K^o\| \leq \gamma \|H_0^{(w)}\|$, since

$$\left| \int_X \left( \sum_{j=1}^n f_j g_j \right) \phi dm \right| \leq \|H_0^{(w)}\| \sum_{j=1}^n \|f_j\|_2 \|g_j\|_2.$$

Conversely, suppose $\|\phi + K^o\| \leq \gamma \|H_0^{(w)}\|$. If $H_0$ does not have the weak approximate $\gamma$-factorization, then there exists $F \in V^1$ with $F \notin V_\gamma$. Then by the Hahn-Banach theorem there exists $\phi \in L^\infty$ such that

$$\left| \int_X \phi F dm \right| > \sup \left\{ \left| \int_X \phi f dm \right| : f \in V_\gamma \right\}$$

and so $\|\phi + K^o\| > \gamma \|H_0^{(w)}\|$.

For $K_0$ we can define the weak approximate $\gamma$-factorization and prove Proposition 6 with $H_0^{(w)}$, $H^\infty$ and $K_0$ instead of $H_0^{(w)}$, $K^\infty$ and $H_0$, respectively. In (I) for concrete uniform algebras we have factorization theorems of $H_0$ and $K_0$. M. Hayashi pointed out a factorization theorem of $H_0$. We now give a proof and clarify its relationship with $\gamma_0$.

**Theorem 7.** Suppose $A$ is a concrete uniform algebra (I).

(1) If $f$ is in $H^2$, then there is a $g$ in $H^2$ and an $h$ in $H_0^*$ such that $f = gh$ and $\|g\|_2 \|h\|_1 \leq \gamma_2 \|f\|_1$, where $\gamma_2 = \sup \{\|v^{-1}\|_{\infty} \|v\|_{\infty} : v \in \mathcal{S}_0\}$. In this case $\gamma_2 \geq \gamma_0$.

(2) If $f_1$ is in $K_0$, then there is a $g_1$ in $H^1$ and an $h_1$ in $K_0^*$ such...
that $f_i = g_i h_i$ and $\|g_i\|_2 = \gamma_i \|f_i\|_2$, where $\gamma_i = \gamma_\infty v_0$ and $K^0 = v_0 H^0$.

PROOF. (1) A function $f \in H^0$ is of the form $f = FG^2$ where $F \in H^{\infty}$
with $|F| \in \mathcal{E}$ and $G \in H^2$ [3, p. 138]. If $|F| = \exp(\sum_{j=1}^n t_j u_j)$, let $k = \prod_{j=1}^n (h_j)^{l_j}$
and $l_j = [t_j/2]$. Then $k \in (H^{\infty})^{-1}$. Put $s_j = 2(t_j/2 - [t_j/2])$. Then $q = Fk^{-1} \in H^0$
and $|q| = \exp(\sum_{j=1}^n s_j u_j) \in \mathcal{E}^0 = \{v^2; v \in \mathcal{E}\}$. Let $g = kG$ and $h = qG$.

Then $f = gh$ and

$$\begin{align*}
\int_x |g|^2 dm \int_x |h|^2 dm &= \int_x |q^{-1}| \cdot |f| dm \int_x |q| \cdot |f| dm \\
&\leq \sup\{|q^{-1}| \cdot |q| \cdot \|q\|_{\infty} \cdot \|f\| dm \sqrt{\int_x |f|^2 dm}^2}.
\end{align*}$$

If $u \in (L^\infty)^i$ then $u = v|g|$ with $v \in \mathcal{E}$ and $g \in (H^{\infty})^{-1}$. Hence

$$\gamma_i = \sup\{|v| \cdot |v^{-1} - v| \cdot \|v\| \cdot \|v\|_{\infty} \cdot \|u\|; u \in (L^\infty)^i = \gamma_\infty.$$

(2) A function $f_i \in K^0$ is of the form $f_i = v_i f$ for some $f \in H^0$. Apply
(1) to this $f$, and let $g_i = g$ and $h_i = v_i h$, then $g_i \in H^2$ and $h_i \in K^0$. Now
(2) follows.

(1) of Theorem 7 gives $\|H^{(1)} \leq \|\phi + K^{(1)} \leq \gamma_i \|H^{(1)}\|$ in the case of
(1) for concrete uniform algebras.

(2) of Theorem 7 gives that $\|H^{(1)} \leq \|\phi + H^{(1)} \leq \gamma_\infty \|H^{(1)}\|$. For any
uniform algebra with finite $\gamma_\infty$, Theorem 3 and Proposition 6 show that
both $H^0$ and $K^0$ have the weak approximate $\gamma_\infty$-factorizations.

7. Arveson's distance formula. Let $\mathcal{A}$ be a (possibly non-self-adjoint) algebra of operators on a Hilbert space $\mathcal{H}$, and let $T$ be an arbitrary bounded operator. Then $d(T, \mathcal{A}) \geq \sup_p \|T - TP\|$, where $d(T, \mathcal{A})$ is the distance from $T$ to $\mathcal{A}$ and where the supremum is taken
over the lattice lat $\mathcal{A}$ of all $\mathcal{A}$-invariant projections. Arveson [5, Theorem 1.1.] showed that if $\mathcal{A}$ is a nest algebra (i.e., lat $\mathcal{A}$ is totally ordered) then the equality holds above. Let $\mathcal{A} = L^2$ and $P^{(1)} = 1 - Q^{(1)}$.

Generalized Nehari's Theorem I implies that if $K^{(1)}$ is dense in $K^1$ and
lat $\mathcal{A} \ni P^{(1)}$ for any $v$ in $(L^\infty)^i$, then for any $\phi$ in $L^\infty$

$$d(M_\phi, \mathcal{A}) = \sup_p \|M_\phi - TP\|.$$

Let $l_\mathcal{E}(\mathcal{H})$ be the space of all compact operators on $\mathcal{H}$ and $\mathcal{B}$ the
the norm closure of $\mathcal{A} + l_\mathcal{E}(\mathcal{H})$. Then $d(T, \mathcal{B}) \geq \sup_p \|T - TP\|$. Theorem 4 implies that if lat $\mathcal{A} \ni P^{(1)}$ for any $v$ in $\mathcal{E}$, then

$$d(M_\phi, \mathcal{B}) = \sup_p \|M_\phi - TP\|$$ for any $\phi$ in $L^\infty$. 
References


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