1. Let \( \{T(\xi); 0 \leq \xi < \infty\} \) be a one-parameter semi-group satisfying the following conditions:

(i) \( T(\xi) \) is a bounded linear transformation from a complex \((B)\)-space \( E \) into itself.

(ii) \( T(\xi + \eta) = T(\xi)T(\eta) \), \( T(0) = I \) (the identity transformation).

(iii) \( T(\xi) \) tends to \( I \) strongly as \( \xi \to 0 \), but not uniformly.

We define \( Ax = \lim_{h \to 0} \frac{1}{h} [T(h) - I]x \) whenever the limit exists and the set of elements \( x \) for which \( Ax \) exists, will be denoted by \( D[A] \).

Let us put

\[
R(\lambda; A)x = -\int_0^\infty e^{-\lambda t} T(\xi)x d\xi, \quad \lambda = \sigma + i\tau,
\]

for all \( x \in E \), then the representation holds at least for all \( \lambda \) such that \( R(\lambda) = \sigma > \log M \), where \( M = \sup_{\xi \leq 1} \| T(\xi) \| \).

We shall consider a problem of E. Hille [1] which is stated as follows:

What properties should an operator \( A \) possess in order that it be the infinitesimal generator of a strongly continuous semi-group \( \{T(\xi); 0 \leq \xi < \infty\} \) of bounded linear transformations from a complex \((B)\)-space into itself?

A theorem of E. Hille [1, Theorem 12.2.1] reads as follows.

**Theorem H.** Let \( A \) be a closed linear unbounded operator on \( E \) into itself whose domain is dense in \( E \). Let the spectrum of \( A \) is located in \( R(\lambda) = \sigma \geq 0 \) and suppose that

\[
R(\lambda; A)x = -\int_0^\infty e^{-\lambda t} T(\xi)x d\xi, \quad \lambda = \sigma + i\tau,
\]

where \( \beta \) is a fixed constant, \( \beta \geq 0 \). Then there exists a semi-group \( \{T(\xi); 0 \leq \xi < \infty\} \) such that \( T(\xi) \) satisfies the conditions (i)-(iii), \( \| T(\xi) \| \leq e^{\beta \xi} \) for all \( \xi > 0 \) and that \( A \) is its infinitesimal generator.

The behavior of the norm, \( \| T(\xi) \| \leq e^{\beta \xi} \), is by no means implied by the conditions (i)-(iii) as may be seen later (Example 1). Therefore this theorem is not the perfect solution of the above problem.

In this paper we shall give a necessary and sufficient condition that the
operator \( A \) becomes an infinitesimal generator of a semi-group \( \{T(\xi); 0 \leq \xi < \infty \} \) satisfying the conditions (i)-(iii), in terms of its spectrum and resolvent.

2. We first prove the following theorem.

**Theorem 1.** Let \( \{T(\xi); 0 \leq \xi < \infty \} \) be a one-parameter semi-group satisfying the conditions (i)-(iii). Then

(a) \( A \) is a closed linear operator and \( D[A] \) is dense in \( E \);

(b) for \( \lambda \) such that \( R(\lambda) > \log M \), where \( M = \max \left( \sup_{\xi \in E} \|T(\xi)\|, 1 \right) \), we have

\[
(A - \lambda I)R(\lambda; A)x = x, \quad x \in E,
\]

\[
R(\lambda; A)(A - \lambda I)x = x, \quad x \in D[A];
\]

(c) for \( \sigma > \log M \), we have

\[
\| R(\sigma; A) \|^k \leq M_{\sigma - \log M}^{-k}, \quad (k = 1, 2, 3, \ldots).
\]

**Proof.** For the proof of (a) and (b), see Hille's book [1]. We shall prove (c). Since \( \sup_{\xi \in E} \|T(\xi)\| \leq M \), we have \( \|T(\xi)\| \leq M^{\xi + 1} \). Now

\[
[R(\lambda; A)]^k x = - \int_0^\infty e^{-t\lambda} T(\xi_1) (R(\lambda; A)]^{k-1}) x d\xi_1
\]

\[
= (-1)^k \int_0^\infty e^{-t\lambda} d\xi_1 \int_0^\infty e^{-t\lambda} T(\xi_1 + \xi_2) [R(\lambda; A)]^{k-2} x d\xi_2
\]

\[
= (-1)^k \int_0^\infty e^{-t\lambda} d\xi_1 \int_0^\infty e^{-t\lambda} d\xi_2 \ldots \int_0^\infty e^{-t\lambda} T(\xi_1 + \xi_2 + \ldots + \xi_k) x d\xi_k.
\]

Hence

\[
\| R(\sigma; A) \|^k \leq \int_0^\infty e^{-t\sigma} d\xi_1 \int_0^\infty e^{-t\sigma} d\xi_2 \ldots \int_0^\infty e^{-t\sigma} T(\xi_1 + \xi_2 + \ldots + \xi_k) d\xi_k
\]

\[
\leq M \prod_{i=1}^k \int_0^\infty e^{-t\sigma} M^i d\xi_i = M_{\sigma - \log M}^{-k}.
\]

Q.E.D.

3. We prove now the main theorem.

**Theorem 2.** (i') Let \( A \) be a closed linear unbounded operator on \( E \) into itself whose domain is dense in \( E \). (ii') There is a constant \( M \geq 1 \) such that the spectrum of \( A \) is located in \( R(\lambda) = \sigma \leq \log M \), and such that for \( \sigma > \log M \),

\[
\| R(\sigma; A) \|^k \leq M_{\sigma - \log M}^{-k} \quad (k = 1, 2, \ldots)
\]

where \( R(\sigma; A) \) is the resolvent of \( A \). Then there exists a semi-group \( \{T(\xi); 0 \leq \xi < \infty \} \) such that \( T(\xi) \) satisfies the conditions (i)-(iii) and that \( A \) is its infinitesimal generator.

We have, from the assumption (ii') of the above theorem,
I. MIYADERA

\[(A - \lambda I)R(\lambda; A)x = x, \quad x \in E,
R(\lambda; A)(A - \lambda I)x = x, \quad x \in D[A],\]

for all \(R(\lambda) = \sigma > \log M.\)

For the proof of this theorem\(^2\) we need the following lemmas.

**Lemma 1.** Under the assumption of Theorem 2, \(D[A^2] \) is dense in \(E.\)

This result is due to E. Hille \([1, \text{Lemma 12.2.1}].\)

**Lemma 2.** Under the assumption of Theorem 2, we have

\[
\lim_{n \to \infty} A(-nR(n; A)y) = Ay, \quad y \in D[A^2].
\]

**Proof.** From the result (3) we have \(-nR(n; A)y = y - R(n; A)Ay\)
for all \(z \in D[A^2].\) Hence

\[
A(-nR(n; A)y) = Ay - AR(n; A)Ay.
\]

Since \(y \in D[A^2], \quad Ay \in D[A],\) Therefore we have \(AR(n; A)Ay = R(n; A)Ay.\)

Thus it follows that

\[
\|A(-nR(n; A)y - Ay\| \leq M(n - \log M)^{-1} \|Ay\|. \quad Q.E.D.
\]

**Proof of Theorem 2.** Let us put

\[(4) \quad I_n = -nR(n; A), \quad n > \log M.\]

Then \(I_n\) is a bounded linear transformation from \(E\) into itself and

\[
(5) \quad \|I_n\| = \|nR(n; A)\|^k \leq M[n(n - \log M)^{-1}]^k \quad (k = 1, 2, \ldots),
\]

\[
(6) \quad A_n x = n(I_n - I)x, \quad x \in E.
\]

Since

\[
\exp(\xi A_n)x = \exp(\xi - n\xi)x = \sum_{k=0}^{\infty} \frac{[\xi nI_n]^k}{k!}\exp(-n\xi)x
\]

we have

\[
(7) \quad \|\exp(\xi A_n)\| \leq \sum_{k=0}^{\infty} \frac{\xi^k}{k!} \|I_n\|^k \exp(-n\xi) \leq M \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\xi n^k}{n - \log M}\right) \exp(-n\xi)
= M \exp\left(\frac{\xi \log M}{n - \frac{1}{n} \log M}\right).
\]

By the definition of \(T_n(\xi),\) we have

\[
(8) \quad T_n(\xi)x - x = \int_{0}^{\xi} T_n(\zeta)A_nxd\zeta.
\]

Since \(A_n\) commute with \(T_n(\xi)\) for all integers \(m, n > \log M,\) it follows that

\(^2\) We wish to express our cordial thanks to Prof. K. Yoshida, who has kindly
pointed us, that the proof of our Theorem 2 is similar to his investigation in K.
Yoshida, On the differentiability and the representation of one-parameter semigroup of
\[ \| [T_m(\xi) - T_n(\xi)] x \| = \left\| \int_0^\xi \frac{d}{d\xi} \left[ \exp (\xi - \xi) A m T_m(\xi)x \right] d\xi \right\| \\
= \left\| \int_0^\xi \left[ \exp (\xi - \xi) A m \right] T_m(\xi) (A m - A n) x d\xi \right\| \\
\leq M^2 \exp \left( \frac{\xi \log M}{1 - \frac{1}{n} \log M} + \frac{\xi \log M}{1 - \frac{1}{m} \log M} \right) \xi \| (A m - A n) x \| . \]

By Lemma 2, we have
\[ \lim_{m, n \to \infty} T_m(\xi)x - T_n(\xi)x = 0, \quad x \in D(A^2) . \]
Thus, by (7) and Lemma 1, for all \( \xi > 0 \) and all \( x \in E \)
(9) \[ \lim_{n \to \infty} T_m(\xi)x \]
exists.

We shall now define \( T(\xi) \) as the limit (9). Then we have, by (7),
(10) \[ \| T(\xi) \| \leq M^s \xi, \quad \xi > 0 . \]
It is obvious that \( \{ T(\xi); 0 \leq \xi < \infty \} \) satisfies the conditions (i) and (ii). By
(11) \[ T(\xi)x - x = \int_0^\xi T(\xi) A x d\xi, \quad x \in D(A^2) , \]
Thus it follows that \( \lim_{\xi \to 0} T(\xi)x - x = 0 \) for \( x \in D(A^2) \). By the Banach-
Steinhaus theorem, we have the condition (iii). It follows from (11) that
\( A \) is the infinitesimal generator of \( \{ T(\xi); 0 \leq \xi < \infty \} \). This completes the
proof of Theorem 2.

Let us consider the assumption (1) of Theorem H. If \( \| R(\sigma; A) \| \leq 1/\sigma + \beta/\sigma^2 \), then \( \| R(\sigma; A) \| \leq (\sigma - \beta)^{-1} \) for \( \sigma > \beta \). Hence \( \| R(\sigma; A) \|^k \| \leq (\sigma - \beta)^{-k} \leq e^{\beta (\sigma - \beta)^{-k}} \) for \( \sigma > \beta \). Hence (1) implies (1'). Hence Theorem 2
implies Theorem H.

As a particular case of Theorem 2, we get

**Theorem 3.** Let \( \{ T(\xi); 0 \leq \xi < \infty \} \) be a semi-group satisfying the assumptions (i)-(iii) and let \( \| T(\xi) \| \leq M \) for all \( \xi \geq 0 \). Then
(a') \( A \) is a closed linear unbounded operator on \( E \) into itself whose domain
is dense in \( E ; \)
(b') there is a constant \( M \geq 1 \) such that the spectrum of \( A \) is located in
\( \sigma = R(\lambda) \leq 0 \) and such that for \( \sigma > 0 \)
(12) \[ \| R(\sigma; A) \|^k \| \leq M \sigma^{-k} \] \( (k = 1, 2, \ldots) \),
where \( R(\sigma; A) \) is the resolvent of \( A \).

Conversely, let \( A \) be an operator satisfying (a') and (b'). Then there exists
a semi-group \( \{ T(\xi); 0 \leq \xi < \infty \} \) such that \( T(\xi) \) satisfies the condition (i)-(iii),
\[ T(\xi) \leq M \text{ for all } \xi \geq 0 \text{ and that } A \text{ is its infinitesimal generator.} \]

By Theorem 1 and Theorem 2, we get a perfect solution of E. Hille's problem.

4. We shall extend above theorem to the \( n \)-parameter semi-group.

Let \( E_n \) be a real Euclidean space of \( n \)-dimension with usual metric and \( e_1, e_2, \ldots, e_n \) be its independent unit vectors.

Let \( A_i \) be the infinitesimal generator of \( T(e_i) \). We have then the following theorem.

**Theorem 4.** Let \( \{T(\xi_1, \xi_2, \ldots, \xi_n) : 0 \leq \xi_1 < \infty, 0 \leq \xi_2 < \infty, \ldots, 0 \leq \xi_n < \infty\} \) be an \( n \)-parameter semi-group satisfying the following conditions:

- (ia) \( T(\xi_1, \xi_2, \ldots, \xi_n) \) is a bounded linear transformation from \( E \) into itself;
- (iib) \( T(0,0,\ldots,0) = I \) (the identity transformation);
- (iiia) \( T(\xi_1, \xi_2, \ldots, \xi_n) \) tends to \( I \) strongly as \( (\xi_1, \xi_2, \ldots, \xi_n) \to (0,0,\ldots,0) \), but not uniformly. Then we have

\[ A_i \text{ (}i = 1, 2, \ldots, n\text{)} \text{ is a closed linear unbounded operator on } E \text{ into itself whose domain is dense in } E; \]

- (ib) there is a constant \( M \geq 1 \) such that the spectrum of \( A_i \) is located in \( R(\sigma) = \sigma < \log M \) and that, for all \( 1 \leq i \leq n, \)

\[ \|R(\sigma; A_i)\| \leq M[\sigma - \log M]^{-k}, \sigma > \log M \quad (k = 1, 2, \ldots), \]

where \( R(\sigma; A_i) \) is the resolvent of \( A_i \):

- (iib) \( R(\sigma; A_i)R(\sigma; A_j) = R(\sigma; A_j)R(\sigma; A_i) \) for \( i, j = 1, 2, \ldots \).

Conversely, let \( A_i \) (\( i = 1, 2, \ldots, n \)) be operators satisfying the conditions (ib)-(iib). Then there exists a semi-group \( \{T(\xi_1, \xi_2, \ldots, \xi_n) : 0 \leq \xi_i < \infty, i = 1, 2, \ldots, n\} \) such that \( T(\xi_1, \xi_2, \ldots, \xi_n) \) satisfies the conditions (ia)-(iiia) and all the generators of \( \{T(\xi_1, \xi_2, \ldots, \xi_n) : 0 \leq \xi_i < \infty, i = 1, 2, \ldots, n\} \) are of the form \( A = \sum_{i=1}^{n} \xi_i A_i \).

For the proof, it is sufficient to note that \( \biguplus_{i=1}^{n} D[A_i] \) is dense in \( E \), where \( D[A_i] \) denotes the domain of \( A_i \). Then Theorem 4 is immediately obtained from Theorems 1 and 2.

5. We shall show, by examples, that the conditions (i)-(iii) do not imply \( T(\xi) \leq e^{\eta} \).

**Example 1.** Let \( E = L^\infty([0, \infty); e^{-\sqrt{t}}] \) be the class of all measurable functions on \( (0, \infty) \) such that

\[ \|x\| = \int_0^{\infty} |x(t)| e^{-\sqrt{t}} \, dt < \infty. \]

Then \( E \) is a \( (B) \)-space. Let \( \{T(\xi) : 0 \leq \xi < \infty\} \) be the semi-group of right
translations on $E$. Then $\{T(\xi); 0 \leq \xi < \infty\}$ satisfies the conditions (i)-(iii) and $\|T(\xi)\| = e^{\sqrt{\xi}}$. Hence $T(\xi)$ does not satisfy the Hille's condition $\|T(\xi)\| \leq e^{\beta \xi}$ for any $\beta \geq 0$.

We shall now show that the condition (i') of Theorem 2 cannot be replaced by $\|R(\sigma; A)\| \leq M/\sigma$.

**Example 2.** Let $E^2_n$ ($n = 1, 2, \ldots$) be a sequence of two dimensional complex Euclidean spaces, where the norm of $x_n = (y, z)$ in $E^2_n$ is defined by $\|x_n\| = (|y|^2 + n|z|^2)^{1/2}$. By $E$, we denote the set of all sequences $\{x_n \in E^2_n\}$ such that $\sum \|x_n\| < \infty$, with the norm $\|\{x_n\}\| = \sum \|x_n\|$. We now define a semi-group $T_n(\xi)x_n = x'_n = (y', z')$ from $E^2_n$ into $E^2_n$ such that

\[
y' = \exp[-(n + in\xi)](y \cos n\xi - z \sin n\xi),
\]

\[
z' = \exp[-(n + in\xi)](y \sin n\xi + z \cos n\xi).
\]

It is obvious that $\|T_n(\xi)\| \leq n^{1/2} \exp(-n\xi)$. If we define a semi-group $\{T(\xi); 0 < \xi < \infty\}$ from $E$ into itself by $T(\xi)x_n = \{T_n(\xi)x_n\}$, then we have $\|T(\xi)\| \leq (2e\xi)^{-1/2}$ and $\{T(\xi); 0 < \xi < \infty\}$ is a semi-group satisfying the following conditions [2]:

(i) $T(\xi)$ is strongly measurable for $\xi > 0$.

(ii) $\int_0^\infty \exp(-\lambda \xi) \|T(\xi)\| d\xi$ exists for $\lambda > 0$.

(iii) $\lim_{\lambda \to 0^+} \int_0^\infty T(\xi)x d\xi = x$ for all $x \in E$.

(iv) $\lim\sup_{\xi \to 0^+} T(\xi) = \infty$.

From the conditions (i)-(iv), we can see that the infinitesimal generator $A$ is a closed linear operator, $D[A]$ is dense in $E$ and that the spectrum of $A$ is located in $R(\lambda) = \sigma \leq 0$ [1, § 11.8]. Conditions (ii) means that $T(\xi)$ is strongly $(c, 1)$-ergodic at $\xi = 0$. Thus $T(\xi)$ is strongly Abel-ergodic at $\xi = 0$. Hence there exists a constant $M_0 > 0$ such that $\sigma \|R(\sigma; A)\| \leq M_0$ for $1 \leq \sigma < \infty$ [1, Theorem 14.7.3]. Since $\|T(\xi)\| \leq (2e\xi)^{-1/2}$, we have

\[
\|\sigma R(\sigma; A)\| \leq \sigma \int_0^\infty e^{-\xi} (2e\xi)^{-1/2} d\xi = \sqrt{\frac{\pi}{2e}} \leq \sqrt{\frac{\pi}{2e}} \cdot \left(\frac{\sqrt{\pi}}{2e}\right)^{1/2} \leq 1.
\]

Hence there exists a constant $M > 0$ such that $\sigma \|R(\sigma; A)\| \leq M$ for $0 < \sigma < \infty$. On the other hand, the condition (iv) shows that $T(\xi)$ does not imply the condition (iii).

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