ESTIMATES FOR OPERATORS IN WEIGHTED L^{p,q}-SPACES

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Abstract. We give conditions on pairs of weight functions for which a certain operator defined on $\Omega \subseteq \mathbb{R}^2_+$ is bounded between weighted Lorentz spaces. The result is applied to obtain weighted estimates for the Laplace transform.

Introduction. Let $f$ and $w$ be nonnegative functions on $\mathbb{R}^+$. Then the distribution function of $f$ relative to the measure $w(x)dx$ is defined by

$$f_w(s) = \int_{\{x : |f(x)| > s\}} w(x)dx = w\{x : |f(x)| > s\},$$

where $s > 0$ and the decreasing rearrangement of $|f|$ relative to $w(x)dx$ is obtained by $f^*(t) = \inf\{s : f_w(s) \leq t\}$ (see e.g. [4; 6, Chapter V]). Further if $0 < p, q \leq \infty$, then the weighted Lorentz spaces $L^{p,q}(w)$ are defined by

$$L^{p,q}(w) = \{f : \|f\|_{L^{p,q}(w)} < \infty\},$$

where

$$\|f\|_{L^{p,q}(w)} = \begin{cases} (q/p) \int_0^\infty \left[ t^{1/p} f^w(t) \right]^{q} t^{-1} dt \right]^{1/q}, & 0 < p, \quad q < \infty \\ \sup_{t>0} t^{-1/p} f^w(t), & 0 < p \leq \infty, \quad q = \infty. \end{cases}$$

In case either $1 < p < \infty$ and $1 < q < \infty$ or $p = q = \infty$, $L^{p,q}(w)$ is a Banach space with norm equivalent to the quasi-norm $\|f\|_{L^{p,q}(w)}$.

Clearly if $w \equiv 1$ then $L^{p,q}(w) = L^{p,q}$, where $L^{p,q}$ are the usual Lorentz spaces. We denote by $L^p_*(\Omega)$, $0 < p \leq \infty$, the space of weighted measurable functions $f$ for which $\|f\|_{L^p_*(\Omega)} = \|w^{1/p} f\|_{L^p(\Omega)}$ is finite, where $\|\cdot\|_{L^p(\Omega)}$ denotes the usual Lebesgue norm. Note that $L^{p,q}(w) = L^p_*(\mathbb{R}^+)$. If $1 < p < \infty$, $1 \leq q \leq \infty$ and $1/p + 1/p' = 1 = 1/q + 1/q'$, then

$$C^{-1} \|f\|_{L^{p,q}(w)} \leq \sup_{\|g\|_{L^{p',q'}(w)} < 1} \left| \int fg w dx \right| \leq C \|f\|_{L^{p,q}(w)}$$

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Throughout, $p'$ denotes the conjugate index of $p$ and is related to $p$ by $p + p' = pp'$ with $p' = +\infty$ if $p = 1$. Similarly for other letters. Further, constants are denoted by $C$ and may be different at different appearances but are always independent of the function in question. $\mathbb{Z}$ denotes the set of integers.

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**LEMMA 1.** Suppose $0 < p$, $q < \infty$. Then

(a) $\|f\|_{L^q(w)} = q \int_0^\infty f''(s)^{q/p} w(s)^{-1} ds$.

(b) If $f$ is non-increasing on $\mathbb{R}^+$ and if $g(x) = \int_0^x w(s) ds < \infty$, whenever $w(x) > 0$, then

$$||f||_{L^p(w)} = (q/p) \int_0^\infty f(x)^{q/p} g(x)^{q/p - 1} w(x) dx.$$ 

Part (a) is due to Sawyer [5, Lemma 1] and Part (b) follows essentially along the same lines as in [5, Lemma 1].

**PROOF.** Part (a) follows, on evaluating the two integral of $qs^{q-1}(q/p)t^{q/p - 1}$ over the set $\{(t, s) : 0 < s < f''(t), 0 < t\}$. By permuting the $s$ integration first we obtain the right hand side of (1) to the $q$-th power, i.e.,

$$\int_{\{(t, s) : 0 < s < f''(t), 0 < t\}} qs^{q-1}(q/p)t^{q/p - 1} ds dt = q \int_0^\infty \left( \int_0^{f''(s)} (q/p)t^{q/p - 1} dt \right) ds
= q \int_0^\infty s^{q-1} f''(s)^{q/p} ds .$$

Hence, Part (a) follows.

(b) is established by evaluating the two iterated integrals of

$$qx^{q-1}(q/p)g(y)^{q/p - 1} w(y) ,$$

over the set $M = \{(x, y) : 0 < x < f(y); 0 < y\}$. Performing the $x$ integration over $M$ on (3) first yields the right hand side of (b)

$$\int_0^\infty \left[ q \int_0^{f(y)} x^{q-1} dx \right] (q/p)g(y)^{q/p - 1} w(y) dy = (q/p) \int_0^\infty f(y)^{q/p} g(y)^{q/p - 1} w(y) dy$$

and performing the $y$ integration first yields the right hand side of (a)
where

$$\chi_{(0,f(y))}(x) = \begin{cases} 
1, & \text{if } x \in (0, f(y)) \\
0, & \text{if } x \notin (0, f(y)).
\end{cases}$$

But $M = \{(x, y): x > 0, 0 < y < S(x)\}$ where $S(x) = \inf\{y: f(y) \leq x\}$. Therefore, the right hand side of (4) is equal to

$$\int_0^\infty x^{q-1}dx \int_{(0,S(x))} g(y)^{q/p-1}w(y)dy$$

But

$$f_w(x) = w(\{y: f(y) > x\}) = \int_{\{y: f(y) > x\}} w(y)dy = \int_0^{S(x)} w(y)dy = g(S(x)).$$

So that, from (5) and (6) we obtain

$$\|f\|_{L^p,w} = q \int_0^\infty x^{q-1}f_w(x)^{q/p}dx = (q/p) \int_0^\infty f(x)^q g(x)^{q/p-1}w(x)dx.$$
implies

\[(8) \quad \sup_{x>0} \left( \int_0^x w(t)dt \right)^{1/p} \|k(x, \cdot)/v\|_{L^r(v)} = C < \infty.\]

Conversely, the condition

\[(9) \quad \sup_{x>0} \left( \int_0^x w(t)dt \right)^{1/p} \|h/v\|_{L^r(v)(k(x, \cdot)/v)} = C < \infty\]

implies (7).

**Proof.** Let us prove (7) implies (8). Let \(f\) be a non-negative function. Then

\[(Kf)_w(\xi) = \int_{\{t \in (0, \infty): (Kf)(t) > \xi\}} w(t)dt \geq \int_{\{t \in (0, \infty): (Kf)(t) > \xi\}} w(t)dt\]

which implies

\[(10) \quad (Kf)_w(\xi) \geq \int_0^x w(t)dt, \quad \text{for} \quad 0 < \xi < (Kf)(x).\]

Inequalities (7) and (10) together with Lemma 1 yield

\[\|f\|_{L^r(v)} \geq C^{-1} \|Kf\|_{L^p(v)} = C^{-1} \left[ \int_0^\infty q^{q-1} (Kf)_q(t)dt \right]^{1/q}\]

\[\geq C^{-1} \left[ \int_0^\infty w(t)dt \left( \int_0^x w(t)dt \right)^{q/p} q^{-q-1} dt \right]^{1/q}\]

\[= C^{-1} \left( \int_0^x w(t)dt \right)^{1/p} \left( \int_0^\infty (k(x, t)/v(t)) f(t)v(t)dt \right).\]

Thus

\[\sup_{x>0} \left( \int_0^x w(t)dt \right)^{1/p} \|k(x, \cdot)/v\|_{L^r(v)} = C < \infty,\]

which proves that (7) implies (8).

Conversely, fix \(f \geq 0\) in \(L^r(v)\), \(s \leq r\) and choose \(\{x_j\}\) such that

\[(Kf)(x_j) = \int_0^\infty k(x_j, t)f(t)dt = 2^{-j}\]

for all \(j \in \mathbb{Z}\). Then, \(\{x_j\}_{j \in \mathbb{Z}}\) is an increasing sequence of positive numbers (note that

\[(Kf)(x)\]

is decreasing)

\[2^{-j+1} = 2^{-j} - 2^{-j+1} = (Kf)(x_j) - (Kf)(x_{j+1}) = \int_0^\infty (k(x_j, t) - k(x_{j+1}, t)) f(t)dt.\]
By Hölder’s inequality and Lemma 1 we obtain

\begin{equation}
\|Kf\|_{L^{p,q}(\omega)} = (q/p) \sum_j \int_{x_j}^{x_j+1} (Kf)^q(x) \, g(x)^{q/p-1} \, \omega(x) \, dx
\end{equation}

\[\leq (q/p) \sum_j (Kf)^q(x_j) \int_0^{x_j} g(x)^{q/p-1} \omega(x) \, dx\]

\[= 2^q \sum_j \left( \int_0^{x_j} g(x) \, dx \right)^{q/p} \left\{ \int_0^\infty [k(x_j, t) - k(x_{j+1}, t)] (h(t)/v(t))(v(t)/h(t)) \, f(t) \, dt \right\}^q\]

\[\leq 2^q \sum_j \left( \int_0^{x_j} g(x) \, dx \right)^{q/p} \|h/v\|_{L^{r,s}((k(x_j, \cdot) - k(x_{j+1}, \cdot))/v/h)} \|f\|_{L^{r,s}((k(x_j, \cdot))v/h)} .\]

Since \( k(x_j, x) - k(x_{j+1}, x) \leq k(x_j, x) \) we obtain

\[\|h/v\|_{L^{r,s}((k(x_j, \cdot) - k(x_{j+1}, \cdot))/v/h)} \leq \left[ \int_0^\infty s^{t^{-1}} \left( \int_{x \in (0, \infty) : f(x) > t} (k(x, x)/h(x))v(x) \, dx \right)^{s/r} \, dt \right]^{1/s} = \|h/v\|_{L^{r,s}((k(x_j, \cdot))/v/h)} .\]

From this estimate, (9) and Minkowski’s inequality the previous inequality (11) shows that

\[\|Kf\|_{L^{p,q}(\omega)} \leq C \sum_j \|f\|_{L^{r,s}((k(x_j, \cdot) - k(x_{j+1}, \cdot))/v/h)} \]

\[= C \sum_j \left[ \int_0^\infty s^{t^{-1}} \left( \int_{x \in (0, \infty) : f(x) > t} (k(x_j, x) - k(x_{j+1}, x))v(x)/h(x) \, dx \right)^{s/r} \, dt \right]^{q/s}\]

\[\leq C \left[ \sum_j \left[ \int_0^\infty s^{t^{-1}} \left( \int_{x \in (0, \infty) : f(x) > t} (k(x_j, x) - k(x_{j+1}, x))v(x)/h(x) \, dx \right)^{s/r} \, dt \right]^{p/s} \right]^{q/r}\]

\[\leq C \left[ \int_0^\infty s^{t^{-1}} \left( \sum_j (k(x_j, x) - k(x_{j+1}, x))v(x)/h(x) \, dx \right)^{s/r} \, dt \right]^{q/s}\]

\[\leq C \left[ \int_0^\infty s^{t^{-1}} \left( \sum_j v(x) \, dx \right)^{s/r} \, dt \right]^{q/s} = C \|f\|_{L^{r,s}((v))} .\]

where the second, the third and the last inequalities follow from \( r \leq q \), Minkowski’s inequality and

\[\sum_j (k(x_j, x) - k(x_{j+1}, x)) \leq Ch(x) ,\]

respectively. This completes the proof of the theorem.

We now state and prove the 2-dimensional weighted Lebesgue inequality for the \( K \)-operator for \( 1 < p \leq q < \infty \).
THEOREM 3. Suppose $1 < p \leq q < \infty$ and that $k(x, y) = \prod_{i=1}^{2} k_i(x_i, y_i)$, $h(x) = \prod_{i=1}^{2} h_i(x_i)$, $0 \leq k_i(x_i, y_i) \leq C h_i(y_i)$, $i = 1, 2$. Define

$$(Kf)(x) = \int_{\mathbb{R}^2} k(x, y)f(y)dy,$$

so that there exists a sequence $\{x_j\}_{j \in \mathbb{Z}}$ satisfying

$$\int_{0}^{\infty} k_1(x_j, y_1)(K_2 f)(y_1, x_2) dy_1 = 2^{-j}, \quad \int_{0}^{\infty} k_2(x_j, y_2)f(y_1, y_2)dy_2 = 2^{-j}$$

where

$$(K_2 f)(y_1, x_2) = \int_{0}^{\infty} k_2(x_2, y_2)f(y_1, y_2)dy_2$$

for all $f(y_1, y_2) \geq 0$ with $(\int_{\mathbb{R}^2} w(x)f(x)^q dx)^{1/p} < \infty$. Suppose further that $w(x) = w(x_1, x_2)$ is a non-negative function defined on $\mathbb{R}^2$ and satisfies

$$(12) \quad \sup_{s > 0} \left( \int_{0}^{\infty} w(x_1, x_2)^{q/p} dx \right)^{1/q} \left( \int_{0}^{\infty} k_1(s, x_1)h^{1-p/q}(x_1)w(x_1, x_2)^{1-p} dx_1 \right) = C < \infty$$

for $i = 1, 2$. Then

$$(13) \quad \left( \int_{\mathbb{R}^2} w(x)^{q/p}(Kf)^q(x) dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^2} w(x)f(x)^q dx \right)^{1/p}.$$

PROOF. Suppose first that (12) holds. Then

$$(14) \quad \int_{\mathbb{R}^2} w(x)^{q/p}(Kf)^q(x) dx = \int_{\mathbb{R}^2} w(x)^{q/p} \left[ \int_{0}^{\infty} \int_{0}^{\infty} k_1(x_1, y_1)k_2(x_2, y_2)f(y_1, y_2)dy_1 dy_2 \right]^{q} dx$$

$$= \int_{\mathbb{R}^2} w(x)^{q/p} \left[ \int_{0}^{\infty} k_1(x_1, y_1) \left( \int_{0}^{\infty} k_2(x_2, y_2)f(y_1, y_2)dy_2 \right) dy_1 \right]^{q} dx$$

$$= \int_{\mathbb{R}^2} w(x)^{q/p} \left[ \int_{0}^{\infty} k_1(x_1, y_1)(K_2 f)(y_1, x_2) dy_1 \right]^{q} dx,$$

where

$$(K_2 f)(y_1, x_2) = \int_{0}^{\infty} k_2(x_2, y_2)f(y_1, y_2)dy_2.$$

Now, fix $f \geq 0$ in $L^p_c(\mathbb{R}^2)$ and a positive increasing sequence $\{x_j\}_{j \in \mathbb{Z}}$ and define

$$\int_{0}^{\infty} k_1(x_j, y_1)(K_2 f)(y_1, x_2) dy_1 = 2^{-j}.$$

Then
By combining (14) and (15) one obtains
\[
\int_{\mathbb{R}^+} w(x)^{q/p}(Kf)^q(x) dx
= \int_0^\infty \sum_j \left( \int_{x_{j-1}}^{x_j} w^{q/p}(x_1, x_2) \left[ \int_0^\infty k_1(x_1, y_1)(K_2f)(y_1, x_2) dy_1 \right]^q dx_1 \right) dx_2
\leq \int_0^\infty \left( \sum_j \int_0^\infty k_1(x_{j-1}, y_1)(K_2f)(y_1, x_2) dy_1 \right)^q \int_{x_{j-1}}^{x_j} w^{q/p}(x_1, x_2) dx_1 \right) dx_2
\leq 2^{2q} \int_0^\infty \sum_j \left( \int_{x_{j-1}}^{x_j} w^{q/p}(x_1, x_2) dx_1 \right)^q \left( \int_0^\infty \left[ k_1(x_1, y_1) - k_1(x_{j-1}, y_1) \right] \right)
	imes (h_1^{1/p}(y_1)/w(y_1, x_2))(K_2f)(y_1, x_2)/h_1^{1/p}(y_1))w(y_1, x_2) dy_1 \right)^q dx_2.
\]

Hölder's inequality is applied to the second integral of the previous integral and one gets
\[
\int_{\mathbb{R}^+} w(x)^{q/p}(Kf)^q(x) dx \leq 2^{2q} \int_0^\infty \sum_j \left( \int_{x_{j-1}}^{x_j} w^{q/p}(x_1, x_2) dx_1 \right)^q \left[ \int_0^\infty \left[ k_1(x_1, y_1) - k_1(x_{j-1}, y_1) \right] \right)
	imes (h_1^{1/p}(y_1)/w(y_1, x_2))(K_2f)(y_1, x_2)/h_1^{1/p}(y_1))w(y_1, x_2) dy_1 \right)^q dx_2.
\]

Again since \( k_1(x_1, y_1) - k(x_{j-1}, y_1) \leq k_1(x_1, y_1) \) and on applying (12)
\[
\sup_{s > 0} \left( \int_0^s w^{q/p}w(x_1, x_2) dx_1 \right)^{1/q} \left( \int_0^\infty k_1(s, y_1)h_1(y_1)^{1/p}w(y_1, x_2)^{1/p}dy_1 \right)^{1/p} \leq C,
\]
for fixed \( x_2 \) one gets

\[
\int_{\mathbb{R}^+} w(x)^{q/p}(Kf)^q(x) dx \leq C \int_0^\infty \left[ \int_0^\infty \left[ \sum_j \left( k_1(x_1, y_1) - k_1(x_{j-1}, y_1) \right) \right)
	imes (K_2f)(y_1, x_2)/h_1^{1/p}(y_1))w(y_1, x_2) dy_1 \right)^q dx_2
\leq C \int_0^\infty \left[ \int_0^\infty (K_2f)^p w(y_1, x_2) dy_1 \right]^{q/p} dx_2.
\]
where the last inequality follows from
\[ \sum_j (k_1(x_j, y_i) - k_1(x_{j+1}, y_i)) \leq C(h_i(y_i)). \]

Again choose a positive increasing sequence \( \{x_i\}_{i \in \mathbb{Z}} \) such that
\[ \int_0^\infty k_2(x, y) f(y_1, y_2) dy = 2^{-j}. \]

Then
\[ 2^{-(j+1)} = \int_0^\infty [k_2(x, y_2) - k_2(x_{j+1}, y_2)] f(y_1, y_2) dy. \]

By Minkowski’s inequality for \( p \leq q \) applied to the last integral of (16) and on proceeding as in the beginning of the proof we obtain
\[
\int_{R^d} w(x)^{q/p} (Kf)(x) dx \leq C \left( \int_0^\infty \left( \int_0^\infty (Kf)(y_1, x_2) w^{q/p}(y_1, x_2) dy_1 \right)^{p/q} dy_2 \right)^{q/p}.
\]

Next, from Hölder’s inequality, it follows that
\[
(17) \quad \int_{R^d} w(x)^{q/p} (Kf)(x) dx \leq C \left[ \sum_j \left( \int_0^\infty \left( \int_0^\infty k_2(x, y) f(y_1, y_2) dy_2 \right)^{q/p} dy_1 \right)^{q/p} \left( \int_0^\infty \left( \int_0^\infty k_2(x, y_2) f(y_1, y_2) dy_2 \right)^{q/p} dy_1 \right)^{q/p} \right].
\]

Here, we used \( k_2(x, y_2) - k_2(x_{j+1}, y_2) \leq k_2(x, y_2) \) in the second integral of the last
inequality.

On applying (12),
\[ \sup_{s > 0} \left( \int_0^s w^{q/p}(y_1, x_2)dx_2 \right)^{1/q} \left( \int_0^\infty k_2(s, y_2)h^{p/p}(y_2)w^{1-p}(y_1, y_2)dy_2 \right)^{1/p'} \leq C, \]
for fixed \( y_1 \) and
\[ \sum_j [k_2(x_j, y_2) - k_2(x_{j+1}, y_2)] \leq h_2(y_2) \]
to the above inequality (17) one obtains
\[ \int_{R^*_2} w(x)^{q/p}(Kf)^q(x)dx \leq C \left( \int_0^\infty \int_0^\infty w(y_1, y_2)f^{p}(y_1, y_2)dy_2dy_1 \right)^{q/p} = C \left( \int_{R^*_2} w(x)f(x)dx \right)^{q/p}, \]
which proves the theorem.

Here, we apply Theorems 2 and 3 to the Laplace transforms

\[ (Lf)(x) = \int_0^\infty e^{-xt}f(t)dt, \quad x > 0 \]

and

\[ (Lf)(x) = \int_{R^*_2} e^{-\langle x, y \rangle}f(y)dy, \quad x \in R^*_2, \]

respectively, where \( \langle x, y \rangle = x_1y_1 + x_2y_2, x_i, y_i \in R^*_+, i = 1, 2 \), complementing those results obtained in [1], [3].

**Theorem 4.** Suppose \( 1 < s < r < q < \infty, 0 < p < \infty \). If \( w, v \) are non-negative weight functions on \( R^*_+ \), then

\[ \|Lf\|_{L^p(w)} \leq C\|f\|_{L_{s}^{\infty}(0)}, \quad \text{for all } f \geq 0 \]
implies

\[ \sup_{x > 0} \left( \int_0^x w(t)dt \right)^{1/p} \|e^{-x/v}\|_{L_{s}^{\infty}(0)} = C < \infty. \]

Conversely, the condition

\[ \sup_{x > 0} \left( \int_0^x w(t)dt \right)^{1/p} \|1/v\|_{L_{s}^{\infty}(e^{-x-w})} = C < \infty \]
implies (20).

The proof follows from Theorem 2 and (18) with \( K = L \).
THEOREM 5. Suppose $1 < p \leq q < \infty$. If $w$ and $v$ are non-negative weights and satisfy

$$
\sup_{s > 0} \left( \int_{0}^{s} w(x_1, x_2)^{q/p} dx_1 \right)^{1/q} \left( \int_{0}^{\infty} e^{-sx} w(x_1, x_2)^{1/p'} dx_1 \right)^{1/p'} = C < \infty,
$$

$i = 1, 2$. Then,

$$
\left( \int_{\mathbb{R}^2_+} w(x)^{q/p} (Lf)^q(x) dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^2} w(x)f(x)^p dx \right)^{1/p}.
$$

Again, the proof follows from (19) and Theorem 3 with $K = L$.

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