DIMENSION OF COMPACT GROUPS AND THEIR REPRESENTATIONS

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In Pontrjagin’s theory of duality for compact abelian groups, the following theorem is known:¹)

Let \( G \) be a compact abelian group, \( G^* \) the dual group. Then the topological dimension of \( G \), in the sense of Lebesgue, is equal to the rank of discrete abelian group \( G^* \).

It was Prof. T. Tannaka who has called my attention to the lack of corresponding theorem in non-commutative case.

I intend to give, in this note, a theorem of this kind in the following form:

THEOREM A. Let \( G \) be an arbitrary compact group, \( G^\wedge \) the aggregate of continuous finite dimensional representations of \( G \), \( C[G^\wedge] \) the algebra over the complex numbers \( C \) generated by the coefficients of representations in \( G^\wedge \), i.e., the “representative ring” of \( G \) in the sense of C. Chevalley². Then the topological dimension of \( G \), in the sense of Lebesgue, is equal to the transcendental degree of \( C[G^\wedge] \) over \( C \).

Another form of corresponding theorem, which may be true, is the following:

THEOREM B. Let \( \overline{G} \) be the space consisting of conjugate classes of a compact group \( G \), \( G^\kappa \) the characters of representations in \( G^\wedge \), \( C[G^\kappa] \) the algebra over \( C \) generated by \( G^\kappa \). Then the topological dimension of \( \overline{G} \) is equal to the transcendental degree of \( C[G^\kappa] \) over \( C \).

In spite of its natural formulation, I cannot prove this theorem at present and merely justified it for connected compact Lie groups.

1. Notations. We shall use the following notations for an arbitrary compact group \( G \):

\( n(G) \): the topological dimension of \( G \) in the sense of Lebesgue.

\( n(G^\wedge) = \langle C[G^\wedge] : C \rangle \): the transcendental degree of “representative ring” \( C[G^\wedge] \) over the complex number field \( C \).

\( r(G) \): the topological dimension of the space \( G \) consists of conjugate classes of \( G \). In case the group \( G \) is a connected compact Lie

1) L. PONTRJAGIN, Topological groups (1939), p.148 Example 49.
2) C. CHEVALLEY, Theory of Lie groups I (1946), p.188.
group, \( r(G) \) is the rank of \( G \) in the sense of H. Hopf i.e., dimension of a maximal abelian subgroup in \( G \) by the well known "principal axis theorem".

\[
\text{\( r(G^*) = \langle C[G^*] : C \rangle \)}: \text{the transcendental degree of "characterring" \( C[G^*] \) over} C.
\]

2. **Auxiliary theorems.**

**Theorem 1** \( n(G) = 0 \) if and only if \( n(GA) = 0 \).

**Proof.** Assume \( n(G) = 0 \), then any \( D \in G\wedge \) maps \( G \) onto a 0-dimensional Lie group, i.e., a finite group. By a suitable coordinate transformation every coefficient \( d_{ij}(x) \) of \( D \) becomes algebraic, therefore \( n(G\wedge) = 0 \). On the other hand, if \( n(G\wedge) = 0 \), then every coefficient \( d_{ij}(x) \) of \( D \in G\wedge \) is algebraic; in particular its character \( \sum d_{ii}(x) \) is algebraic. Hence, by a theorem of Weil\(^3\), \( D(G) \) is a finite group. Since \( G \) has sufficiently many representations, this means that \( n(G) = 0 \), q.e.d.

**Theorem 2** \( G \) is connected if and only if every element in \( C[G\wedge] \) is constant or transcendental.

**Proof.** Assume \( G \) be not connected and put \( G_0 \) for the connected component containing the identity \( 1 \). Then \( G/G_0 \) is a 0-dimensional group and \( C[(G/G_0)\wedge] \subseteq C[G\wedge] \). By preceding theorem there exists a non-constant algebraic element in \( C[(G/G_0)\wedge] \) and a priori in \( C[G\wedge] \).

Conversely, if \( C[G\wedge] \) contains a non-constant algebraic element \( f(x) \); then \( f(x) \) is a finite valued continuous function on \( G \). Therefore \( G \) cannot be connected, q.e.d.

3. **Proof of Theorem A.** The proof is accomplished by a series of elementary lemmas.

**Lemma 1.** \( n(G) \leq n(G\wedge) \).

**Proof.** Assume first \( G \) be a compact Lie group, then \( G \) has a faithful representation \( D(x) \in G\wedge \). Since \( G \) has a neighborhood of the identity homeomorphic to the euclidean \( n \)-space \( \mathbb{R}^n \) \((n = n(G))\), it follows that among the coefficients \( d_{ij}(x) \) of \( C(x) \) there exist \( n \) topologically, hence algebraically independent elements. Therefore \( n(G\wedge) \geq n(G) \).

Next \( G \) be arbitrary, there exists, for any finite number \( n^* \leq n(G) \), a sufficiently small invariant subgroup \( H \subseteq G \) such that \( G/H \) is a Lie group and \( n(G/H) \geq n^* \).

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3) These theorems 1, 2 are founded independently by Y. KAWADA. His results are published in Japanese periodical "Shijo-Sugaku-Danwakai". Weil's theorem quoted in the proof is in C.R. Paris 198, 1739-42; 199, 180-2(1934).

4) e.g., CHEVALLEY, I.c.\(^*\) p.211.

5) e.g., PONTRJAGIN, I.c.\(^*\) p.211 F). Separability assumption is not essential in this proof.
Obviously \( n(G^\wedge) \geq n(G/\ll)^\wedge \geq n(G/\ll) \geq n^* \). This means \( n(G^\wedge) \geq n(G) \). q.e.d

**Lemma 2.** If \( G \) is connected \( C[G^\wedge] \) has no zero-divisors.

**Proof.** Let \( f_1, f_2 \in C[G^\wedge] \) be \( f_1(x)f_2(x) = 0 \) everywhere on \( G \). We must show that at least one of \( f_1, f_2 \) is zero everywhere. Since the problem concerns two elements \( f_1, f_2 \in C[G^\wedge] \) it is sufficient to assume that \( G \) is a Lie group. Now \( f_1, f_2 \) are analytic functions on \( G \), hence, by a property of analytic functions, at least one of \( f_1, f_2 \) is zero in a sufficiently small neighborhood of the identity.

Since \( G \) is connected this holds everywhere.

**Lemma 3.** For the proof of \( n(G) \geq n(G^\wedge) \) it is sufficient to assume that \( G \) is connected.

**Proof.** Let \( G_0 \) be the connected component containing \( 1 \). At first it holds obviously \( n(G) \geq n(G_0) \). We show that \( n(G_0^\wedge) \geq n(G_0) \). For this we put \( n(G_0^\wedge) = n \) and assume \( n \) is finite. Take \( n+1 \) arbitrary elements \( f_1, \ldots, f_{n+1} \in C[G^\wedge] \) and a sufficiently small invariant subgroup \( \ll \) such that \( H = G/\ll \) is a Lie group and \( f_1, \ldots, f_{n+1} \) are functions on \( H \). If \( H_0 \) is the component in \( H \), \( H_0 = G_0/\ll \). \( C[H_0^\wedge] \subseteq C[G^\wedge] \) hence \( n(H_0^\wedge) \leq n \).

If \( H = \sum_{i=1}^h s_i H_0 \) is a coset decomposition of \( H \) by \( H_0 \), the set of elements in \( C[H^\wedge] \) which vanish on \( s_i H_0 \) constitutes an ideal \( \Psi_i \) in \( C[H^\wedge] \) such that \( C[H_0^\wedge]/\Psi_i \cong C[H_0]^\wedge \) has no zero-divisors by **Lemma 2** and its transcendental degree \( n(H_0^\wedge) \leq n \). Hence there exist \( h = [H:H_0] \) polynomials \( P_i \) such that

\[
P_i(f_1, \ldots, f_{n+1}) \in \Psi_i \quad (i = 1, 2, \ldots, h).
\]

Since \( \bigcap_{i=1}^h \Psi_i = 0 \), \( P_i(f_1, \ldots, f_{n+1}) = 0 \). This means that \( f_1, \ldots, f_{n+1} \) are algebraically dependent i.e., \( n(G^\wedge) \leq n \). q.e.d.

**Lemma 4.** If \( G \) is connected \( n(G) \geq n(G^\wedge) \).

**Proof.** Let \( n \leq n(G^\wedge) \) be a finite number, we want to show that \( n(G) \geq n \).

We take \( D_0 \in G^\wedge \) such that, among the coefficients \( d_{ij}(x) \) of \( D_0 \), there exist \( n \) algebraically independent elements in \( C[G^\wedge] \). Put \( \ll = \ker(D_0) = 1 \). Then \( H = G/\ll \) is a Lie group with \( D_0 \) as a faithful representation. Hence by Kampen’s theorem\(^6\) coefficients of \( D(x), D(x) \) generate the algebra \( C[G^\wedge] \). Let \( M(H) \) be the associated algebraic group of \( H \), then by definition the point \( (d_{ij}(x), d_{ij}(x)) \) in complex 2r-space \( C^{2r} \), where \( r = \deg D \), is a generic point of \( M(H) \) over a suitable field \( k \). Therefore \( M(H) \) is the set of specializations of the point \( (d_{ij}(x), d_{ij}(x)) \) over \( k \) and complex dimension of \( M(H) = <C[H^\wedge] : C> \geq n \).

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\(^6\) e.g., **CHEVALLEY**, L.c.\(^3\), p.193-4.

\(^7\) For the definition and properties of associated algebraic group used in the following see **CHEVALLEY**, L.c.\(^3\) pp. 194-202.
On the other hand, if \( h = n(H) \) is the dimension of \( H \), then \( M(H) \) is homeomorphic to \( H \times \mathbb{R}^h \), therefore \( 2h \geq 2n \), i.e., \( h \geq n \).

More precisely, any \( A \in H_1 = M(H) \cap \mathfrak{l}(r) \) (unitary restriction) is associated with \( a \in H \) by

\[
A(f) = f(a) \quad \text{for} \quad f \in \mathbb{C}[H^\wedge]
\]

(duality theorem). \( \mathbb{C}[H^\wedge] \subset \mathbb{C}[G^\wedge] \) and \( A \) is a representation of \( \mathbb{C}[H^\wedge] \). We shall show that \( A \) can be extended continuously on \( \mathbb{C}[G^\wedge] \); continuity means that \( A \to 1 \) implies convergency of extension \( \tilde{A} \):

\[
\tilde{A}(f) \to f(1) \quad \text{for} \quad f \in \mathbb{C}[G^\wedge].
\]

Since \( H_1 \) has a neighborhood of the identity homeomorphic to \( \mathbb{R}^h \), this continuous one to one image in \( G_{\mathbb{A}} = G \) has dimension \( h \). Hence \( n(G) \geq h \geq n \).

Consider couples \((F_i, A_i)\) consisting of a sub-algebra \( F_i \) generated by a set of representations \( \{D_1, D_2, D_3, \ldots\} \) in \( G^\wedge \) and continuous extensions \( A_i \) on \( F_i \) of every \( A \in H_1 \).

\[
(F_i, A_i) \leq (F_2, A_2)
\]

means \( F_i \subset F_2 \) and each \( A_i \) coincides on \( F_i \) with unique \( A_2 \). Then all couples \((F_i, A_i)\) satisfy condition of Zorn's lemma and there exists a maximal couple \((F_\infty, A_\infty)\). We must show \( F_\infty = \mathbb{C}[G^\wedge] \). Otherwise there would exists \( D \in G^\wedge \) such that at least one coefficient of \( D \) or \( D \) does not belong to \( F_\infty \). Take one of such coefficient \( d_i(x) = f \) and define

1) \( A'_i(f) = f(1) \) if \( f \) is transcendental over \( F_\infty \).

2) If \( f \) is algebraic over \( F_\infty \), take an irreducible equation satisfied by \( f \) (since \( \mathbb{C}[G^\wedge] \) is without zero-divisors by Lemma 2):

\[
f g_{a_1} + f g_{a_2} + \cdots + g_0 = 0 \quad (g_i \in F_\infty).
\]

By assumption \( A \to 1 \) implies \( A_\infty(g_i) \to g_i(1) \), there exists a root of equation

\[
X^n A_\infty(g_{a_1}) + X^{a_2} A_\infty(g_{a_2}) + \cdots + A_\infty(g_0) = 0
\]

such that \( A \to 1 \) implies \( \alpha \to f(1) \). We define then

\[
A'_i(f) = \alpha.
\]

Thus we can extend \( A_\infty \) to the algebra \( \mathbb{C}[F_\infty, D, \bar{D}] \) as an algebra-representation with continuity preserved.

Now consider a direct product

\[
\mathfrak{B} = \bigotimes \mathfrak{G}_1 (r_i, D_i) \times \mathfrak{G}_2 (r_i, D_i) \times \cdots \mathfrak{G}_m (r_i, D_i) \times \mathfrak{G}_m (r_i, D_i)
\]

where \( \mathfrak{G}_i (r_i, D_i) = GL(\mathbb{C}[D_i]) \) means complex general linear group of degree \( r_i D_i \). In this product algebra-representations of \( C_\infty F_\infty, D, \bar{D} \) constitute a generalized algebraic group \( \mathfrak{B} \) in the sense that its elements are defined by an infinity of algebraic equations. \( M \in \mathfrak{M} \) implies \( M' = M^{\perp} \in \mathfrak{Y} \) and the subset
satisfying conjugate condition is precisely \[ \mathcal{M} \cap [U(r(D_1)) \times \cdots]. \] In particular above \( A' \) determines

\[ M = A_\omega(D_1) \times A_\omega(D_1) \times \cdots \times A_\omega(D) \times A_\omega(D') \]

which is an element in \( \mathcal{M} \) such that on \( F_\omega \) its components are unitary. Now decompose \( M \) into a unitary matrix \( M_1 \) and a positive definite hermitian matrix \( M_2: M = M_1 \cdot M_2 \) continuously. It is easy to verify that \( M_1 \in \mathcal{M} \) again. Define \( A_\omega \) on \( D \) by

\[ M_1 = A_\omega(D_1) \times A_\omega(D_1) \times \cdots \times A_\omega(D) \times A_\omega(D'), \]

then \( A_\omega \) is an algebra-representation of \( C[F_\omega, D, D] \) preserving conjugate condition. This would imply \( (C[F_\omega, D, D], A_\omega) \) contrary to the hypothesis. q.e.d.

**Remark.** After completion of above proof of Theorem A, I found another proof of Theorem A for separable compact groups by using a result of A. Weil\(^8\) which states that, if \( G \) is a compact separable group, \( U \) an invariant subgroup such that \( G/U \) is a Lie group, then \( n(G) \geq n(G/U) \). Since \( n(G/U) \geq n(G) \) for sufficiently small subgroup \( U \) and \( n(G) = \lim_{U \to 0} n(G/U) \). On the other hand \( n(G^0) = \lim_{U \to 0} n(G(U^0)) \) is obvious. First part of the proof of Lemma \( 4 \) gives a proof of \( n(G/U) = n((G/U)^0) \), therefore \( n(G) = n(G^0) \).

4. **Proof of Theorem B for connected compact Lie groups.**

Every group considered in this section are assumed to be connected compact Lie group.

**Lemma 5.** If \( \overline{G} \) is a finite sheeted covering group of \( G \), then \( \pi(\overline{G}) = \pi(G) \), \( \pi(\overline{G}^0) = \pi(G^0) \).

**Proof.** \( \pi(\overline{G}) = \pi(G) \) is obvious by Hopf’s definition of rank. \( \pi(\overline{G}^0) \geq \pi(G^0) \) is a consequence of \( \overline{G}^0 \supseteq G^0 \). Now let \( D(x) \) be an irreducible representation in \( \overline{G}^0 \) and \( \chi(x) \) be the character of \( D(x) \). Put \( G = \overline{G}/N \) with \( N \) as a finite central subgroup of \( \overline{G} \). By Schur’s lemma,

\[ D(x) = \lambda(x) \cdot 1 \quad (x \in N), \]

where \( \lambda(x) \) is a root of unity such that \( \lambda(x)^n = 1 \) if \( n \) denotes the order of \( N \). Hence the representation

\[ \overline{D}(x) = \cdots \times \frac{D(x)}{n} \]

maps \( N \) into \( 1 \), i.e., this is a representation of \( G = \overline{G}/N \). This means \( x^* \subseteq G^* \), therefore, every character \( x \in G^* \) is algebraic over \( G^* \). Hence \( (\overline{G}^0) \leq \pi(G^0) \), q.e.d.

**Lemma 6.** If \( G \) is a direct product of \( G_1 \) and a central subgroup \( G_2 \) of \( G \), then \( \pi(G) = \pi(G_1) + \pi(G_2) \), \( \pi(G^0) = \pi(G_1^0) + \pi(G_2^0) \).

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PROOF. Let $T_1$ be a maximal torus of $G_1$, $T_2 = G_2$, then $T_1 \times T_2$ is a torus in $G$; hence $r(G_1) + r(G_2) \leq r(G)$. On the other hand if $T$ is a maximal torus in $G$, then, since $G_2$ is central, $G_2 \subseteq T$. As $T/G_2$ is a torus in $G_1 = G/G_2$, dimension of $T/G_2 \leq r(G_1)$. Thus dimension of $T = r(G) \leq r(G_1) + r(G_2)$.

Next, every irreducible representation of $G$ is a Kronecker product of irreducible representations of $G_1$ and $G_2$. Therefore every irreducible character $x$ of $G$ is a product $x = x_1 x_2$ of characters of $G_1$ and $G_2$, i.e., $r(G^*) \leq r(G_1^*) + r(G_2^*)$. Conversely, if $x_1, \ldots, x_{v_1}$ and $\psi_1, \ldots, \psi_{v_2}$ are algebraically independent characters of $G_1^*$ and $G_2^*$ respectively, then $x_1 \psi_1(i=1, \ldots, v_1, j=1, \ldots, v_2)$ are algebraically independent. For if

$$F(x_1 \psi_1, \ldots, x_{v_1} \psi_1, \ldots, x_{v_2} \psi_2) = 0$$

is a polynomial in $r_{v_2}$ arguments, it can be written in the form:

$$\sum_{x_1, \ldots, x_{v_2}} F_{x_1, \ldots, x_{v_2}}(x_1, \ldots, x_{v_1}) \psi_1^{x_1} \cdots \psi_2^{x_{v_2}} = 0$$

where $F_{x_1, \ldots, x_{v_2}}(x_1, \ldots, x_{v_1})$ are polynomials in $x_1, \ldots, x_{v_1}$. If we fix $x \in G_1$ then $F_{x_1, \ldots, x_{v_2}}(x_1(u), \ldots, x_{v_2}(x))$ is a complex number $= 0$ by hypothesis on $\psi$'s. This implies by hypothesis on $x$'s. Hence the equation $F \equiv 0$, and $r(G^*) \geq r(G_1^*) + r(G_2^*)$. q.e.d.

As is well known, every connected compact Lie group $G$ has a finite sheeted covering group $\tilde{G}$ such that

$$\tilde{G} = G_1 \times G_2$$

where $G_1$ is a simply connected semi-simple compact Lie group and $G_2$ a torus. Hence by Lemmas 5, 6, it is sufficient to prove $r(G) = r(G^*)$ for simply connected semi-simple compact Lie groups. In the following let $G$ be such a group.

**Lemma 7.** $r(G) \leq r(G^*)$.

**Proof.** There exists one to one correspondence between representations of $G$ and those of its Lie algebra $\mathfrak{g}$. Every irreducible representation of $\mathfrak{g}$ is determined by a highest weight $A$, which can be written uniquely by Cartan basis $A_1, \ldots, A_r$, $r = r(\mathfrak{g})$, as

$$A = m_1 A_1 + \cdots + m_r A_r \quad (m_i \text{ integers } \geq 0).$$

Conversely to every such weight $A$, there exists unique irreducible representation of $\mathfrak{g}$ having $A$ as highest weight. Let $D_1, \ldots, D_r$ and $x_1, \ldots, x_r$ be the irreducible representations of $\mathfrak{g}$ and characters of $G$ respectively corresponding to the weights $A_1, \ldots, A_r$.

We show that $C[G^*] = C[x_1, \ldots, x_r]$. Take an irreducible character $x \in G^*$ such that its weight is

9) e.g. PONTRJAGIN, loc. cit. p. 282 THEOREM 87.
\[ A = m_1 A_1 + \cdots + m_r A_r \quad (m_i \text{ integers } \geq 0) \]

and that if \( A' < A \) then the character \( x' \) with highest weight \( A' \) is contained in \( C[x_1, \ldots, x_r] \). Irreducible representation of \( G \) which has \( A \) as its highest weight is contained in the Kronecker product

\[
\prod_{i=1}^{m_1} D_i \times \cdots \times \prod_{j=1}^{m_r} D_j
\]

as the irreducible representation with highest weight \( A \) (Cartan composite). Hence

\[ x + x' + x'' + \cdots = x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} \]

where \( x', x'', \ldots \) are characters with highest weight \( A', A'', \ldots < A \). By hypothesis on \( x, x', x'', \ldots \in C[x_1, \ldots, x_r] \), hence \( x \in C[x_1, \ldots, x_r] \) and \( C[G^*] \subseteq C[x_1, \ldots, x_r] \) by an inductive argument. \( \text{q.e.d.} \)

**Lemma 8.** \( r(G) \leq r(G^*) \).

**Proof.** We show that the characters \( x_1, \ldots, x_r \) corresponding to a Cartan basis \( A_1, \ldots, A_r \) of highest weights are algebraically independent. Let \( F(x_1, \ldots, x_r) = 0 \) be a polynomial. If \( \mathfrak{h} \) is a maximal abelian subalgebra of the Lie algebra \( \mathfrak{g} \) of \( G \), then

\[ x_i(x) = \exp \lambda_i \mathfrak{h} + \exp \lambda'_i \mathfrak{h} + \cdots + \lambda''_i \mathfrak{h}, \text{ etc.} \]

where \( x = \exp \mathfrak{h} \) (\( \mathfrak{h} \in \mathfrak{h} \)). Inserting into the polynomial \( F = \sum a_{n_1} \cdots a_{n_r} x_1^{n_1} \cdots x_r^{n_r} \), we see that highest term exists in the sum

\[ \sum a_{n_1} \cdots a_{n_r} \exp (n_1 A_1(h) + \cdots + n_r A_r(h)). \]

Now if \( n_1 A_1 + \cdots + n_r A_r \) is highest, then \( a_{n_1 \cdots n_r} = 0 \). By repeated application of this argument we arrive at \( F \equiv 0 \), i.e., \( r(G^*) \geq r = r(G) \). \( \text{q.e.d.} \)

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