TWO THEOREMS ON THE RIEMANN SUMMABILITY

HIROSHI HIROKAWA AND GEN-IICHIRÔ SUNOUCHI

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1. The series $\sum_{v=1}^{\infty} a_v$ is said $(R_1)$-summable to zero if the series

$$ F(t) = \sum_{v=1}^{\infty} \frac{s_v}{v} \sin vt, $$

where $s_v = \sum_{v=1}^{n} a_v$, converges in some interval $0 < t < t_0$, and if $F(t)$ tends to zero as $t$ tends to zero.

The series $\sum_{v=1}^{\infty} a_v$ is said $(R,1)$-summable to zero if the series

$$ G(t) = \sum_{v=1}^{\infty} a_v \frac{\sin vt}{vt} $$

converges in some interval $0 < t < t_0$, and if $G(t)$ tends to zero as $t$ tends to zero.

Recently, one of the present authors [2] proves the following theorem:

**THEOREM A.** Suppose that

$$ s_v = o(n), $$

and

$$ a_v = O(n^{-\alpha}), $$

where $0 < \alpha < 1$. Then the series $\sum_{v=1}^{\infty} a_v$ is $(R_1)$-summable to zero.

**THEOREM B.** Under the assumptions of Theorem A, the series $\sum_{v=1}^{\infty} a_v$ is $(R,1)$-summable to zero.

The object of this paper is to generalize the above theorems.

**THEOREM 1.** Let $s_3^n$ be the $(C,\beta)$-sum of $\sum_{n=1}^{\infty} a_n$. Then, if

$$ s_3^n = o(n^{\beta \alpha}), $$

and

$$ \sum_{v=n}^{\infty} \frac{|a_v|}{v} = O(n^{-\alpha}), $$

...
where $0 < \alpha < 1$, $0 \leq \beta$, the series $\sum_{n=1}^{\infty} a_n$ is $(R_\beta)$-summable to zero.

**Theorem 2.** Under the assumptions of Theorem 1, the series $\sum_{n=1}^{\infty} a_n$ is $(R_1)$-summable to zero.

2. Proof of Theorem 1.

Firstly, we shall show that the series (1) is convergent for all $t$. If we put $r_n = \sum_{\nu=1}^{n-1} |a_\nu|$, then $|a_n| = n(r_n - r_{n+1})$.

Since, by (4),

$$\sum_{\nu=1}^{n} |a_\nu| = \sum_{\nu=1}^{n} r_\nu - n r_{n+1} = O\left(\sum_{\nu=1}^{n} \nu^{-\alpha}\right) + O(n^{1-\alpha}) = O(n^{1-\alpha}),$$

we have

(5) $s_n = O(n^{1-\alpha}).$

Hence

(6) $\sum_{p=n}^{\infty} \frac{|s_p|}{p^2} = O(n^{-\alpha}).$

Furthermore, by (4), (6),

(7) $\sum_{\nu=1}^{\infty} \frac{S_\nu}{\nu} - \frac{S_{\nu+1}}{\nu + 1} = \sum_{\nu=1}^{\infty} \frac{s_\nu - s_{\nu+1}}{\nu} + \left(\frac{1}{\nu} - \frac{1}{\nu + 1}\right) s_{\nu+1} \\
\leq \sum_{\nu=1}^{\infty} \frac{|a_\nu|}{\nu} + \sum_{\nu=1}^{\infty} \frac{|s_{\nu+1}|}{\nu^2} = O(n^{-\alpha}).$

Using the Abel's lemma, we have

(8) $\sum_{\nu=1}^{\infty} \frac{S_\nu}{\nu} \sin \nu t = \sum_{\nu=1}^{\infty} \left(\frac{S_\nu}{\nu} - \frac{S_{\nu+1}}{\nu + 1}\right) T_\nu(t) + \frac{S_m}{m} T_m(t) - \frac{s_n}{n} T_{n-1}(t),$

where

$$T_\nu(t) = \left\{\cos t - \cos\left(n + \frac{1}{2}\right) t\right\}/2 \sin \frac{1}{2} t.$$

Since

$$\left|\sum_{\nu=1}^{\infty} \frac{S_\nu}{\nu} \sin \nu t\right| < 2t^{-1} \sum_{\nu=1}^{\infty} \left|\frac{S_\nu}{\nu} - \frac{S_{\nu+1}}{\nu + 1}\right| + 2t^{-1} \left(\frac{|s_m|}{m} + \frac{|s_n|}{n}\right)$$

if $t \neq 0$, by (5), (7), the series (1) is convergent. If $t = 0$, this fact is evident. Thus the series (1) is convergent for all $t$.

Given a positive number $\delta$, put $M = \lfloor \delta^{-1/\alpha} \rfloor$, and

$$\sum_{\nu=1}^{\infty} \frac{S_\nu}{\nu} \sin \nu t = \left(\sum_{1}^{M} + \sum_{M+1}^{\infty}\right) = U(t) + V(t),$$

say. Then we have
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\[ |V(t)| \leq 2t^{-1} \sum_{\nu=1}^{\infty} \frac{|s_{\nu}|}{\nu} \left| \frac{s_{\nu+1}}{\nu+1} \right| + 2t^{-1} \frac{|s_{T}|}{M} \]

(9)  

\[ = O(t^{-1} M^{-a}) + O(t^{-1} M^{-a}) = O(t^{-1} \varepsilon) \]

\[ \leq O(\varepsilon), \]

by (5), (7), (8).

Nextly, we show that \( U(t) = o(1) \). Putting \( \lfloor \beta \rfloor = \gamma \), by repeated use of Abel's transformation \( \gamma \) times, we have

\[ U(t) = \sum_{\nu=1}^{M-\gamma} s_{\nu} \Delta_{\nu}^{\gamma}(t) + s_{M-\gamma+1} \Delta_{M-\gamma+1}^{\gamma}(t) + \ldots \]

(10)

\[ \ldots + s_{M-1} \Delta_{M-1}^{\gamma}(t) + s_{M} \Delta_{M}^{\gamma}(t) \]

\[ = W(t) + \sum_{\nu=1}^{\gamma} U_{\nu}(t), \]

say, where

\[ \Delta_{\nu}^{\gamma}(t) = \sin nt/n, \ \Delta_{\nu}^{k}(t) = \Delta_{\nu+1}^{k-1}(t) - \Delta_{\nu}^{k+1}(t), \]

and

\[ U_{\nu}(t) = S_{M-\nu+1} \Delta_{M-\nu+1}^{\gamma}(t). \]

Since

(11 a)  

\[ \Delta_{k}^{2k}(t) = (-1)^{k+1} \int_{0}^{t} \left( \sin \frac{t}{2} \right)^{2k} \cos(n+k)t \ dt, \]

(11 b)  

\[ \Delta_{k}^{2k+1}(t) = (-1)^{k+1} \int_{0}^{t} \left( \sin \frac{t}{2} \right)^{2k+1} \sin \left( n + \frac{2k+1}{2} \right) t \ dt, \]

for \( k = 0, 1, 2, \ldots, \), we have

(12)  

\[ \Delta_{k}(t) = O(t^{2k}/n) \]

by the second mean value theorem. From (3), (5), using the Riesz convexity theorem [1], we have

(13)  

\[ s_{\nu} = O((n^{-a})^{1-\nu/\beta} (\nu^{\beta} \nu^{\beta})), \quad (\nu = 1, 2, 3, \ldots). \]

Hence by (11 a), (12),

\[ U_{\nu}(t) = O(M((1-a)(\beta-\nu)+\nu\beta)/\beta) \]

\[ = O(t^{-(3-a\beta-\nu+\nu\beta)/2}) \]

\[ = O(t^{-(1-a)/\beta}) \]

\[ = o(1) \]

for \( \nu = 1, 2, 3, \ldots, \gamma \). Thus

(14)  

\[ \sum_{\nu=1}^{\gamma} U_{\nu}(t) = o(1). \]

Nextly, we shall prove that \( W(t) = o(1) \). By the well-known formula

(15)  

\[ s_{\nu}^{\beta} = \sum_{n=0}^{\nu} (-1)^{n} \left( \frac{\beta-\gamma}{\nu-n} \right) s_{n}^{\beta}, \quad (s_{0} = 0), \]
where \( \binom{m}{n} = \frac{m(m-1)\ldots(m-n+1)}{n!} \) and \( \binom{0}{0} = 1 \), we have

\[
W(t) = \sum_{\nu=1}^{M-\gamma} s_{\nu} \Delta_{\nu}^\gamma(t)
\]

\[
= \sum_{\nu=1}^{M-\gamma} \left\{ \sum_{r=0}^{\nu} (-1)^{r-n} \binom{\beta - \gamma}{\nu - r} s_{\nu} \right\} \Delta_{\nu}^\gamma(t)
\]

\[
= \sum_{\nu=1}^{M-\gamma} s_{\nu} \sum_{r=0}^{\nu} (-1)^{r-n} \binom{\beta - \gamma}{\nu - r} \Delta_{\nu}^\gamma(t).
\]

Here, we consider the two case, that is, i) \( \gamma \) is even; ii) \( \gamma \) is odd.

i). By (11, a), we have

\[
W(t) = \sum_{n=0}^{M-\gamma} \sum_{\nu=n}^{M-\gamma} (-1)^{r-n} \binom{\beta - \gamma}{\nu - n} \int_0^t \sin \left( \frac{t}{2} \right)^\gamma \cos \left( \nu + \frac{\gamma}{2} \right) t dt
\]

\[
= \sum_{n=0}^{M-\gamma} \sum_{\nu=n}^{M-\gamma} (-1)^{r-n} \binom{\beta - \gamma}{\nu - n} \cos \left( \nu + \frac{\gamma}{2} \right) t \left( \sin \frac{t}{2} \right)^\gamma dt
\]

\[
= \sum_{n=0}^{M-\gamma} \sum_{\nu=n}^{M-\gamma} (-1)^{r-n} \binom{\beta - \gamma}{\nu - n} \cos \left( \nu + \frac{\gamma}{2} \right) t \left( \sin \frac{t}{2} \right)^\gamma dt.
\]

Since

\[
\sum_{\nu=0}^{\infty} (-1)^{\nu} \binom{\beta - \gamma}{\nu} \cos \left( \nu + \frac{\gamma}{2} \right) t
\]

\[
= R \left\{ \sum_{\nu=0}^{\infty} (-1)^{\nu} \binom{\beta - \gamma}{\nu} e^{int} \sin \left( \nu + \frac{\gamma}{2} \right) t \right\}
\]

\[
= 2^{\gamma-1} \left( \sin \frac{t}{2} \right)^{\beta-\gamma} \cos \left( \frac{\beta}{2} + n \right) t + \frac{\beta - \gamma}{2} \pi \right},
\]

we write \( W(t) \) in the form

\[
W(t) = \sum_{n=0}^{M-\gamma} \sum_{\nu=n}^{M-\gamma} (-1)^{r-n} \binom{\beta - \gamma}{\nu - n} \int_0^t \cos \left( \frac{t}{2} \right)^\gamma \cos \left( \nu + \frac{\gamma}{2} \right) t \left( \sin \frac{t}{2} \right)^\gamma dt
\]

\[
- (-1)^{\nu} 2^\nu \int_0^t \sum_{r=M-\gamma-n+1}^{\infty} (-1)^{r-n} \binom{\beta - \gamma}{\nu - n} \cos \left( \nu + \frac{\gamma}{2} \right) t \left( \sin \frac{t}{2} \right)^\gamma dt
\]

\[= W_1(t) + W_2(t),
\]

say. By second mean value theorem

\[
\int_0^t \left( \sin \frac{t}{2} \right)^\gamma \cos \left( \frac{\beta}{2} + n \right) t + \frac{\beta - \gamma}{2} \pi \right) dt = O(t^\xi/n),
\]

If \( \gamma = 0 \), we put \( U(t) = W(t) \).
and then
\[ W_1(t) = o\left( \sum_{n=0}^{N-\gamma} n^{\frac{\gamma}{2}} \frac{t^{3}}{n} \right) \]
\[ = o(M^{\beta + 1}) \]
\[ = o(e^{-bt^2}) \]
\[ = o(1). \]  

(18)

Now we have
\[ W_2(t) = o\left( \sum_{n=0}^{N-\gamma} n^{\frac{\gamma}{2}} \sum_{\nu=N-\gamma+1}^{N-\gamma+1} \nu^{-\gamma+1} t^{\gamma} \frac{1}{\nu + n} \right) \]
\[ = o\left( \frac{(M-\gamma)}{M-\gamma+1} \sum_{n=0}^{N-\gamma+1} (M-\gamma n+1)^{-\beta+\gamma} t^{\gamma} \right) \]
\[ = o\left( (t^{M^{\beta+1} \sum_{n=1}^{M-\gamma+1} n^{-\beta+\gamma}}) \right) \]
\[ = o(t^{(M^{\beta+1} \sum_{n=1}^{M-\gamma+1} n^{-\beta+\gamma}}) \]
\[ = o(1). \]  

(19)

Thus, from (14), (18), (19)
\[ W(t) = o(1). \]

Therefore, given arbitrarily fixed \( \epsilon > 0 \), from (9)
\[ |F(t)| \leq |U(t) + V(t)| \leq \epsilon \quad (t \to 0). \]

Since \( \epsilon \) is arbitrarily small,
\[ F(t) \to 0 \quad \text{as} \quad t \to 0. \]

Thus, if \( \gamma \) is even, the proof is complete.

ii) Nextly, we consider the 2nd case, i.e., \( \gamma \) is odd. The proof is similar to the former case. In this case, if we replace (11, a) by (11, b), we get
\[ W(t) = \sum_{n=0}^{M-\gamma} s_{n} \left\{ (-1)^{\gamma+1} \frac{1}{2} t^{\gamma} \sum_{\nu=0}^{M-\gamma-n} (-1)^{\gamma} \left( \frac{\beta - \gamma}{\nu} \right) \sin \left( \nu + n + \frac{\gamma}{2} \right) t \right\} \]

and similar as (17)
\[ \sum_{\nu=0}^{\infty} (-1)^{\nu} \left( \frac{\beta - \gamma}{\nu} \right) \sin \left( \nu + n + \frac{\gamma}{2} \right) t \]
\[ = I \left[ \sum_{\nu=0}^{\infty} (-1)^{\nu} \left( \frac{\beta - \gamma}{\nu} \right) e^{nt} e^{(n+\gamma)\frac{t}{\nu}} \right] \]
\[ = 2^{\beta-\gamma} \left( \sin \frac{t}{2} \right)^{\beta-\gamma} \sin \left\{ \left( \frac{\beta}{2} + n \right)t + \frac{\beta - \gamma}{2} \pi \right\} \]

Therefore, we can proceed the proof similarly as in former case. Thus, the proof of theorem is complete.
3. Proof of Theorem 2.
The method of proof is similar to the former section. We first show the series (2) is convergent for all positive \( t < t_0 \).

Since, by (4),

\[
|G(t)| \leq \frac{1}{t} \sum_{n=1}^{\infty} \frac{|a_n|}{n} < +\infty,
\]

the series (2) is convergent for all such \( t \).

Nextly, choose \( M \equiv \left( \frac{1}{t_0^\varepsilon} \right)^{1/\gamma} \) and write

\[
G(t) = \sum_{n=1}^{\infty} a_n \frac{\sin nt}{nt} = \left( \sum_{n=1}^{M} + \sum_{n=M+1}^{\infty} \right) = U(t) + V(t),
\]

say. Then, by (4),

\[
|V(t)| \leq t^{-1} \sum_{n=M+1}^{\infty} \frac{|a_n|}{n} = O(t^{-1} t^\varepsilon) \leq \varepsilon.
\]

Putting \([\beta] = \gamma\), by repeated use of Abel's transformation \((\gamma + 1)\) times, we have

\[
U(t) = t^{-1} \left( \sum_{n=1}^{M} a_n \frac{\sin nt}{n} \right)
\]

\[
= t^{-1} \left\{ \sum_{\nu=1}^{M-\gamma-1} s_{\nu}^{\gamma} \Delta_{\nu}^{\gamma+1} (t) + s_{M-\gamma}^{\gamma} \Delta_{M-\gamma}^{\gamma} (t) + \ldots \right. \]

\[
\quad + \left. s_{M-1}^{1} \Delta_{M-1}^{1} (t) + s_{M}^{1} \Delta_{M}^{1} (t) \right\}
\]

\[
= t^{-1}(W(t) + U(t)),
\]

say, where \( \Delta_{\nu}^{\gamma}(t) \) is same in \( \S 2 \).

In the same method in \( \S 2 \) we obtain, by (12), (13),

\[
U_{\nu}(t) = O\left( M^{1-\alpha} \frac{1}{M} \right) = O(M^{-\alpha}) = O(t^\varepsilon) \leq \varepsilon t.
\]

Now, we shall prove that \( W(t) = o(t) \). Using (15),

\[
W(t) = \sum_{n=0}^{\nu-1} s_{n}^{\beta} \sum_{\nu-n}^{\nu-1} (-1)^{\nu-n} \left( \frac{\beta - \gamma}{\nu - n} \right) \Delta_{\nu-n}^{\gamma+1} (t).
\]

Dividing the method into two case as in \( \S 2 \), we shall prove the case in which \( \gamma \) is odd. Using (17),

\[
W(t) = \sum_{n=0}^{\nu-1} s_{n}^{\beta} \left\{ (-1)^{\nu+1/2} \nu+1 \right\} \sum_{\nu-n}^{\nu-1} (-1)^{\nu-n} \left( \frac{\beta - \gamma}{\nu} \right) \cos \left( \nu + n + \frac{\gamma+1}{2} \right) t_{0}^\nu
\]
\[
\sum_{n=0}^{M-\gamma-1} s_n \{ (-1)^{(\gamma+1)/2} \sum_{\nu=0}^{\beta \gamma} (-1)^\nu \left[ \left( \frac{\beta}{2} + n \right) t + \frac{\beta - \gamma - 1}{2} \pi \right] \cos \left( \frac{\beta - \gamma}{2} \right) \sin \left( \frac{\beta}{2} + n \right) \} dt
\]

therefore, similarly as (18), (19), we obtain

\[ W(t) = o(t). \]

If we use the same method as in §2, we obtain

\[ G(t) \to 0 \quad \text{as} \quad t \to 0 \]

The case in which \( \gamma \) is even is similar.
Thus, the theorem is proved.

References


Faculty of Engineering, Gifu University, Gifu,
Mathematical Institute, Tohoku University, Sendai.