Setting Control Limits in a Multivariate Exponentially Weighted Moving Likelihood Control Chart

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Abstract:

A control chart is one of the representative methodologies used in statistical process control. A multivariate control chart is used to detect changes in multiple variables. Ueno and Nagata (2018) proposed a multivariate exponentially weighted moving likelihood (MEWML) control chart aimed at detecting minor changes in the mean vector and variance-covariance matrix. Although the MEWML control chart can accurately detect the change when the variance increases, it is unable to detect the change when it decreases. A reduction in variance implies that the procedure is moving to a better state. Therefore, detecting the change can lead to an improvement in the procedure. In this paper, we detect reductions in variance by considering both the upper and lower control limits. Through a simulation, we demonstrate that reduction in variance can be detected by using the lower control limit.

Keywords

Average run length, change point detection, control limit, multivariate control chart, statistical process control

1. Introduction

A control chart is one of the representative methodologies used in statistical process control. A multivariate control chart is used to detect changes in multiple variables.

Hotelling (1947) proposed that the Hotelling's T^2 control chart aimed at detecting changes in the mean vector. However, the Hotelling's T^2 control chart cannot detect small changes. A multivariate cumulative sum (MCUSUM) control chart by Woodall and Ncube (1985) can be used to detect small changes. The MCUSUM control chart was further studied by Healy (1987) and Pignatiello and Runger (1990).

The exponentially weighted moving average (EWMA)-type multivariate control chart is one of the multivariate control charts used when small changes continuously occur. There are many applications of EWMA type multivariate control charts in various fields such as quality control and medical care (e.g., Yu et al., 2011; Morton et al., 2001). Lowly et al. (1992) proposed a multivariate exponentially weighted moving average (MEWMA) control chart aimed at detecting changes in the mean vector. Hawkins and Maboudou-Tchao (2008) proposed a multivariate exponentially weighted moving covariance matrix (MEWMC) control chart aimed at detecting changes in the variance-covariance matrix. Zhang et al. (2010) proposed a control chart based on the likelihood ratio using exponential weighted moving average estimators of the mean vector and covariance matrix, which they called the ELR control chart. Ueno and Nagata (2018) proposed the multivariate exponentially weighted moving likelihood (MEWML) control chart that weights log-likelihood to be the statistic aimed to detect minor changes in the mean vector and variance-covariance matrix.

Although the MEWML control chart can accurately detect the change when the variance increases, it is unable to detect change when it decreases. A reduced variance implies that the procedure is becoming better than before the change point. For example, if the manufacturing process of a product is controlled and the variation in the size of the product is smaller than before the change point, the process can be said to have improved. If a change to a better state is found, it can be a benchmark for improvement. Therefore, detecting the change can lead to an improvement in the procedure. In this study, we detect reductions in variance by considering both the upper and

lower control limits. By setting the lower control limit line, the control chart can detect changes even when the value of the final statistic is small. Further, the performance of the analysis procedure was evaluated based on simulation and real data analysis.

The remainder of this paper is structured as follows. Section 2 describes the previous research. Section 3 presents the proposed procedure. Section 4 performs Monte Carlo simulation and Section 5 analyzes real data to evaluate the performance of the proposed procedure. Section 6 presents the conclusions.

2. Previous research

Ueno and Nagata (2018) proposed the MEWML control chart that weights log-likelihood to be the statistic aimed to detect minor changes in the mean vector and variance-covariance matrix. Change detection in MEWML control chart is performed by the following procedure.

First, when the process is in control, it is assumed that p -dimensional vector $\mathbf{x}$ follows the multivariate normal distribution $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$. Where $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ are the population mean vector and population variance-covariance matrix, respectively. Then, it is assumed that both parameters are known.

Next, assuming that the observation variable at time t ($t = 1, 2, \dots$) is $\mathbf{x}_t$, it is assumed that $\mathbf{x}_t$ follows the multivariate normal distribution shown in equation (1). When the process is in control, $\mathbf{x}_t$ follows the distribution of the upper formula, and when the process is out of control, $\mathbf{x}_t$ follows the distribution of the lower formula

$$\begin{cases} \mathbf{x}_t \sim N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) & (\text{In control}) \\ \mathbf{x}_t \sim N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) & (\text{Out of control}) \end{cases} \quad (1)$$

Next, whitening is performed in the multivariate normal distribution of equation (1). Specifically, $\mathbf{u}_t = \mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu}_0)$ is established using $\mathbf{A}$ which satisfies $\mathbf{A}\boldsymbol{\Sigma}_0\mathbf{A}^T = \mathbf{I}_p$. As a result, equation (1) becomes equation (2).

$$\begin{cases} \mathbf{u}_t \sim N_p(\mathbf{0}_p, \mathbf{I}_p) & (\text{In control}) \\ \mathbf{u}_t \sim N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2) & (\text{Out of control}) \end{cases} \quad (2)$$

And, equation (3) is defined as follows.

$$l(\mathbf{x}_t) = p \log(2\pi) + \mathbf{u}_t^T \mathbf{u}_t. \quad (3)$$

Equation (3) is weighted using λ ($0 \leq \lambda \leq 1$) to define equation (4). Where the lower equation corresponds to the weighted moving average of the negative log likelihood, Ueno and Nagata (2018) called this control chart as the MEWML control chart.

$$L(\mathbf{x}_t) = (1 - \lambda)L(\mathbf{x}_{t-1}) + \lambda l(\mathbf{x}_t), \quad (4)$$

where $L(\mathbf{x}_t) = l(\mathbf{u}_t = \mathbf{0}_p) = p \log(2\pi)$ holds when $t = 0$.

Then, judging by the chart statistic defined by equation (5).

$$MEWML_t = L(\mathbf{x}_t) - L(\mathbf{x}_0). \quad (5)$$

3. Proposed procedure

A MEWML control chart has the limitation that it cannot detect changes in which the variance decreases. A MEWML control chart, where the variance changes slightly after the 100th term is shown in Figure 1.

Figure 1: MEWML control chart when the variance decreases

If the variance decreases, the value of the final statistic becomes small and consequently it is not possible to detect the change with the conventional upper control limit line alone. One way to detect such changes is to set a lower control limit line. By setting the lower control limit line, the control chart can detect changes even when the value of the final statistic is small. Further, Triantafyllopoulos (2006) is an example of a multivariate control chart that draws both upper and lower control limit lines.

In this paper, in addition to the upper control limit line used, the lower control limit line is also drawn to detect the reduction in variance. The specific proposed procedure is as follows.

1. We set the upper and lower control limit lines by Monte Carlo simulation. However, it is set such that the type I error is 1%. Where, the final statistic of MEWML control chart, as well as the final statistic of other EWMA type multivariate control chart, makes it difficult to determine the value of the control limit line exactly. Therefore, we used Monte Carlo simulations to obtain approximate values.

2. Execute the change detection according to the procedure of change detection of MEWML control chart described in section 2.

4. Monte Carlo simulation

4.1 Simulation settings

We simulate the situation where the variance decreases and verify the performance of the proposed procedure. The data to be generated is based on equation (6).

$$\begin{cases} \mathbf{u}_t \sim N_2(\mathbf{0}_2, \mathbf{I}_2) & (t = 1, 2, \dots, 100) \\ \mathbf{u}_t \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right) & (t = 101, 102, \dots, 5000) \end{cases} \quad (6)$$

By changing $(\mu_1, \mu_2, \rho, \sigma_1, \sigma_2)$, we set the following four patterns of change: only the variance decreases; the mean increases and the variance decreases; the correlation increases; and the variance decreases, the mean and the correlation increase, and the variance decreases. The data of these four patterns are generated and simulated.

The values of λ in equation (4) are 0.1, 0.3, 0.5, 0.7, and 0.9. The larger the value of λ , the more important it is to emphasize the t period data.

4.2 Evaluation criterion

We use ARL as an evaluation criterion. In the field of multivariate control charts, ARL is often used as an evaluation criterion because the purpose is to detect a change point quickly. ARL represents the period before the

control limit is exceeded.

ARL before the change point is called in control ARL (IC-ARL), whereas that after the change point is called out of control ARL (OC-ARL). In general, larger values of IC-ARL and smaller values for OC-ARL are considered better. In the simulation, the value of IC-ARL is fixed and the control limit line is drawn; OC-ARL is used as the evaluation criterion. Currently, 200, 370 and 500 are often taken as the values of IC-ARL.

In this chapter, we set IC-ARL = 200 and set the control limit line. And we simulate it 10000 times, and we use the average of OC-ARL as the evaluation criterion.

4.3 Experimental result

Table 1 shows the OC-ARL of the MEWML control chart when λ is varied. Group 1 represents changes when only variance decreases; group 2 represents changes when the mean increases and variance decreases; group 3 represents changes when the correlation increases and variance decreases; and group 4 represents changes when the mean and correlation increase, and variance decreases. From Table 1, it is observed that changes can be detected in all simulation patterns. By defining the lower control limit line, the changes in variance can be detected. Further, it can be seen that the value of OC-ARL is very large when $\lambda = 0.1$.

A MEWML control chart for $(\mu_1, \mu_2, \rho, \sigma_1, \sigma_2) = (0, 0, 0, 0.7, 0.7)$ is shown in Figure 2. Although it is clear that the final statistics become smaller after the 100th period, it takes time to exceed the lower control limit line. It is thought that the weight for data in period t is small, and it takes time to accumulate the degree of change.

Table 1: OC-ARL of MEWML control chart when λ is varied

group	μ_1	μ_2	ρ	σ_1	σ_2	λ				
						0.1	0.3	0.5	0.7	0.9
1	0.0	0.0	0.0	0.9	0.9	2073.62	68.88	45.62	41.65	40.82
	0.0	0.0	0.0	0.7	0.7	424.56	16.43	14.83	16.54	20.50
	0.0	0.0	0.0	0.5	0.5	27.98	6.39	5.46	5.89	7.81
2	0.3	0.3	0.0	0.7	0.7	2248.32	25.39	21.90	23.78	28.17
	0.6	0.6	0.0	0.7	0.7	2570.71	123.48	74.39	64.31	59.59
	0.9	0.9	0.0	0.7	0.7	45.97	50.83	56.65	60.75	61.30
3	0.0	0.0	0.3	0.7	0.7	357.84	15.50	13.77	15.30	19.11
	0.0	0.0	0.6	0.7	0.7	237.20	14.18	12.10	12.84	15.92
	0.0	0.0	0.9	0.7	0.7	137.64	11.87	9.26	8.95	9.73
4	0.3	0.3	0.3	0.7	0.7	1628.77	23.10	19.51	20.74	24.39
	0.6	0.6	0.3	0.7	0.7	1287.68	75.08	47.25	42.25	40.83
	0.3	0.3	0.6	0.7	0.7	814.07	19.39	15.61	16.03	18.59

Figure 2: MEWML control chart when only the variance decreases

5. Real data analysis

5.1 Data set overview

We use the data used by Hawkins and Maboudou-Tchao (2008). This data is a 40-period continuous measurement of 4 items: systolic rate, diastolic rate, heart rate, and arterial pressure. Because this data is after the change point from the first period, it is important to find how quickly changes can be detected. Further, the population mean vector and population variance-covariance matrix before the change point are known. The population mean vector, population variance covariance matrix, and matrix used for whitening are shown in equation (7), (8), and (9), respectively.

$$\mu_0 = (126.61, 77.48, 80.95, 97.97)^T, \quad (7)$$

$$\Sigma_0 = \begin{pmatrix} 15.04 & 8.66 & 10.51 & 12.04 \\ 8.66 & 5.83 & 5.56 & 7.50 \\ 10.51 & 5.56 & 15.17 & 8.79 \\ 12.04 & 7.50 & 8.79 & 10.57 \end{pmatrix}, \quad (8)$$

$$\mathbf{A} = \begin{pmatrix} 0.26 & 0.00 & 0.00 & 0.00 \\ -0.63 & 1.09 & 0.00 & 0.00 \\ -0.38 & 0.21 & 0.36 & 0.00 \\ -0.45 & -1.03 & -0.13 & 1.42 \end{pmatrix}. \quad (9)$$

5.2 Evaluation criterion

We use ARL as an evaluation criterion. We drew the control limit line at IC-ARL = 500 from the parameters shown in equation (7), (8), and (9), and confirmed the change in MEWML statistics at $\lambda = 0.1, 0.3, 0.5, 0.7$, and 0.9 . Section 5 draws a control limit line at IC-ARL = 200. However, if we use 200 for this data, a change is detected immediately and the difference is hard to understand; so, 500 is used.

Figure 3: MEWML control chart when changing λ

5.3 Results and discussion

Figure 3 shows MEWML control chart at $\lambda = 0.1, 0.3, 0.5, 0.7$, and 0.9 . In Figure 3, starting from the top left, $\lambda = 0.1, 0.3, 0.5, 0.7$, and 0.9 , with the black line representing the control limit line.

At $\lambda = 0.1$ and 0.3 , the first control period is judged to be after the change point by the lower control limit line. In addition, the statistic value takes a large value in the 25th period, and it can be seen that the smaller the value of λ , the more the likelihood that the 25th period and subsequent ones are judged after the change point. If the value of λ is small, the weight of the data before the t period is large, so it is considered to be excellent for continuous change detection. On the other hand, the value of λ is large, the weight of the data in t period is large, so the value of the final statistic increases when data with a large degree of change is observed. In this reason, it can be said that the smaller the value of λ , the greater the number of continuously detected change points from the 25th period, since the effect of outliers occurring in the 25th period remains.

6. Conclusion

In this paper, we utilized a MEWML control chart, one of the EWMA-type multivariate control charts, and discussed how to set the control limit line. The conventional MEWML control chart cannot detect a reduction in variance, which is its limitation. Therefore, in this paper, we proposed a procedure to detect a reduction in variance by setting the lower control limit line. Through Monte Carlo simulations, we confirmed that the reduction in variance can be detected at some larger values λ . In addition, through real data analysis, we showed that the proposed procedure is practicable.

Future issues include setting hyperparameters λ . In this study, the simulation was performed with λ set to $0.1, 0.3, 0.5, 0.7$, and 0.9 . Determining the optimal value of λ by performing further simulations is necessary. In addition, in this study, we simulated only the MEWML control chart. We believe that its performance should be compared with other control charts.

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