Optimal Replacement Policy for a Deteriorating System
Subjected to a Geometric Brownian Motion

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Abstract:
When system deterioration information is continuously available, it is convenient to use a continuous stochastic process to characterize the deterioration. The gamma process and inverse Gaussian process are usually used to model monotonic deterioration; however, if the system deterioration is measured frequently, the measured amount of deterioration may not be monotonic. A replacement problem is considered here for a system with a non-monotonic amount of deterioration. The deterioration is characterized by a geometric Brownian motion, which can capture deterioration with an increasing rate over time. The system is inspected at equal time intervals, and the deterioration amount can be specified exactly from the inspection results. An optimal replacement policy is derived in accordance with the age and deterioration amount that minimizes the total expected cost over an infinite time horizon. The cost structure includes replacement cost and operating cost, which both increase with age and the amount of deterioration. The optimization problem is formulated as a Markov decision process and provides a set of conditions with which the structural properties of the optimal replacement policy is characterized. The total expected cost is proven to be monotonically non-decreasing in both age and deterioration amount. Moreover, the optimal replacement policy is shown to be a control limit policy in which the optimal control limits do not monotonically increase with either the amount of deterioration or age. A numerical example is given that illustrates the effects of the cost of replacement and the parameters of geometric Brownian motion on the control limit policy.

Keywords
Condition-based maintenance, Control limit policy, Deteriorating system, Markov decision process, Preventive maintenance

1. Introduction
Recent advances in sensing and measuring technology have led to sensors becoming smaller, more sophisticated, and less expensive, making it easier to collect, process, and organize data. This has enabled detailed monitoring of the deterioration of various types of systems with high frequency. An important application of the obtained detailed data, which provide continuous information about system deterioration, is condition-based maintenance. This paper reports the results of an investigation of the optimal replacement policy based on continuous deterioration information for a single-unit system.

A convenient way to analyze detailed monitoring information is to use a continuous stochastic process. Various optimal maintenance policies for deteriorated systems have been proposed that use continuous stochastic processes [1]. Dieulle et al. [4] used the gamma process to model deterioration and proposed an optimal maintenance policy based on a cost criterion. Chen et al. [2] analytically derived the structural properties of an optimal maintenance policy under the inverse Gaussian process. Their deterioration models ([2][4]) are commonly used to analyze continuous data on deterioration. However, the processes they used can describe only monotone deterioration. If the system is measured frequently, the measured increase in deterioration may not be monotone. In that case, a non-monotonic continuous stochastic process is necessary.

Wiener processes and geometric Brownian motion are continuous stochastic processes that can be used to describe non-monotone deterioration. Elwany et al. [5] derived the structural properties of the optimal maintenance
policy for a Wiener process under the assumption that the operating cost of the system is constant. However, in reality, performance may decrease and operating cost may increase as the system deteriorates. Liu et al. [6] derived the same properties as Elwany et al. [5] for a Wiener process under the assumption that the operating cost depends on the operating time of the system and the amount of deterioration. The Wiener process is a non-monotonic stochastic process that can represent data with noise and capture stationary changes in the deterioration rate. However, for systems for which the deterioration rate increases with time, it is preferable to express the deterioration amount using an exponential function. A geometric Brownian motion can be used to model systems with non-stationary deterioration rates ([3][7]).

We examined the replacement policy for a system with deterioration under the assumption that the operation cost depends on the system operation time and the deterioration state. Using a geometric Brownian motion as a deterioration model, we analyzed the properties of a replacement policy that minimizes the total expected discounted cost over an infinite period.

The remainder of this paper is organized as follows. Section 2 summarizes the modeling of a deteriorating system subjected to a geometric Brownian motion. Section 3 formulates the optimization problem for preventive replacement using a Markov decision process and presents several properties of the optimal replacement policy. Section 4 provides a numerical example illustrating our results. Section 5 summarizes the key points and mentions directions for future work.

2. Model

2.1 Notation

\[ S(t) \quad \text{Amount of system deterioration at time } t \]
\[ \tau \quad \text{Time interval between inspections} \]
\[ c_F \quad \text{Corrective replacement cost} \]
\[ c_P \quad \text{Preventive replacement cost} \]
\[ G(t,s) \quad \text{Continue operating cost at time } t \text{ for system deterioration is } s \]
\[ W(k,s) \quad \text{Expected operating cost within time interval } k\tau \leq t < (k + 1)\tau \text{ given } S(k\tau) = s \]
\[ e^{-\gamma \tau} \quad (\gamma > 0) \quad \text{Discount factor} \]

2.2 System deterioration

We consider a system that deteriorates over time. The amount of deterioration, \( S(t) \), is completely observed during inspections that are scheduled at equally spaced time epochs \( t (= \tau, 2\tau, 3\tau, \ldots) \). Interval \( k\tau \leq t < (k + 1)\tau \) is referred to as the \( k \)-th time period. If the cumulative system operating time is \( k\tau \), then \( k \) is defined as the age of the system. We assume that \( S(t) \geq 0 \) follows a geometric Brownian motion model, which is a stochastic process that satisfies

\[ \frac{dS(t)}{S(t)} = \beta dt + \sigma dB(t). \]

Here, \( \beta \) is the mean, \( \sigma \) is the standard deviation, and \( B(t) \) is the error term, which follows a Brownian motion with mean \( 0 \) and variance \( t \). Using Itô’s lemma, we obtain

\[ S(t) = S(0) \cdot \exp \left( \left( \beta - \frac{\sigma^2}{2} \right) t + \sigma B(t) \right), \]

and the probability density function of \( S(t) \) is given as
The optimization problem for maintenance decision making over an infinite horizon is formulated using a total expected discounted cost over the cumulative system operating time $k\tau$ as:

$$f_{t|0}(s|S(0)) = \frac{1}{\sqrt{2\pi} \sigma^2 s} \cdot \exp \left\{ - \frac{\left( \ln \left( \frac{s}{S(0)} \right) - \left( \beta - \frac{\sigma^2}{2} \right) t \right)^2}{2\sigma^2 t} \right\},$$

where $S(0)$ is the initial state of the system.

### 2.3 Maintenance actions and costs

To ensure safe system operation, necessary maintenance is carried out at the beginning of each time period $k\tau$ ($k = 0, 1, 2, 3, \ldots$) in accordance with a pre-determined replacement policy based on deterioration amount $S(k\tau)$. Safety is considered to be insufficient if $S(k\tau)$ exceeds safety limit $l$. We investigated the replacement problem under the assumption that if the deterioration amount exceeds the safety limit during time interval between two successive inspections but is within the limit at the end of the time period, the system is nevertheless considered to be insufficiently safe, the same assumption used in previous studies [5][6].

If the deterioration amount is greater than $l$, corrective replacement is selected as the maintenance action, which incurs cost $c_c$. If the deterioration amount is less than or equal to $l$, the decision maker needs to decide on the basis of the amount of deterioration whether to continue operating the system for another time period or to perform preventive replacement, which incurs cost $c_p$. The corrective replacement cost is assumed to be more than the preventive replacement cost ($c_p > c_c$). If continue operating is selected, operating cost $G(t, S(t))$ depends on the cumulative system operating time $t$ and the amount of deterioration $S(t)$. $G(t, S(t))$ is monotonically non-decreasing in operating time $t$. For simplicity, we assume that $G(t, S(t))$ is a linear function of $S(t)$, as assumed in a previous study [6]. In other words, $G(t, S(t)) = h(t)S(t)$. Since the system’s operating costs may increase rapidly due to aging and serious deterioration, $h(t) > 0$ is monotonically non-decreasing in $t$. This research assumed that the operation cost is incurred only when the system operation time exceeds a specific threshold value $k_c\tau$, where $k_c$ is a constant as in Liu et al. [6]. The expected operating cost $W(k, s)$ for the $k$-th time period can be derived as

$$W(k, s) = \int_{k\tau}^{(k+1)\tau} E[G(t, S(t))|S(k\tau) = s]dt \quad (k \geq k_c)$$

$$= \int_{k\tau}^{(k+1)\tau} \int_0^\infty h(t)u f_{\tau|k\tau}(u|s)du \, dt,$$

where $f_{\tau|k\tau}(u|s)$ is the probability density function of the amount of deterioration at time $t$ given that $S(k\tau) = s$.

### 3. Optimal replacement policy

#### 3.1 Formulation of optimization problem

The optimization problem for maintenance decision making over an infinite horizon is formulated using a discrete-time Markov decision process. We focus on the deterioration amounts at the discretized time epochs. For brevity, let $S_k$ denote the deterioration amount at the $k$-th time epoch instead of $S(k\tau)$; i.e., $S_k = S(k\tau)$, ($k = 0, 1, 2, 3, \ldots$).

First, we consider a finite time horizon $i = 0, 1, 2, \ldots, n$ and let $V_i(k, s)$ be the total expected discounted cost over the remaining $(n - i)$ time periods following time epoch $i$ for current system age $k$ and deterioration amount $s$:

$$V_i(k, s) = \begin{cases} V^C_i(k, s) & s > l \\ \min\{V^C_i(k, s), V^C_{i+1}(k, s)\} & s \leq l \end{cases}$$

where $V_i(k, s) = 0$ holds for any $k$ and $s$. Here,

$$V^C_i(k, s) = c_p + e^{-\gamma \tau} V_{i+1}(0,0)$$

is the total expected discounted cost over $(n - i)$ time periods when corrective replacement is selected for $s > l$.
and
\[ V_i^P(k, s) = c_p + e^{-\gamma T} V_{i+1}(0, 0) \]
\[ V_i^C(k, s) = W(k, s) + e^{-\gamma T} [E[V_{i+1}(k+1, S_{k+1} | k, S_k = s)] ] \]
are respectively the corresponding total expected discounted costs when preventive replacement and continue operating are selected as the optimal action for a system with age \( k \) and deterioration amount \( s \). The total expected cost is discounted exponentially with \( \gamma > 0 \) as the discount rate. From the standard argument of contraction mapping theory [8], \( V_i(k, s) \) can be written as
\[ V(k, s) = \begin{cases} V_i^F(k, s) & s > l \\ \min[V_i^P(k, s), V_i^C(k, s)] & s \leq l' \end{cases} \]
as \( i \) approaches infinity. Here,
\[ V_i^F(k, s) = c_p + e^{-\gamma T} V_i(0, 0), \]
\[ V_i^P(k, s) = c_p + e^{-\gamma T} V_i(0, 0), \]
and
\[ V_i^C(k, s) = W(k, s) + e^{-\gamma T} [E[V(i+1, S_{k+1} | k, S_k = s)]]. \]
The properties of \( V(k, s) \) are helpful to understand the pattern of the optimal replacement policy. Liu et al. [6] presented the monotonicity of \( V(k, s) \) in deterioration amount \( s \) under a deterioration process that follows the Wiener process. For the Wiener process, the random increment during one time interval follows a normal distribution, which is a symmetrical distribution. Therefore, the expected value of \( V(k+1, S_{k+1} | k, S_k = s) \) preserves the monotonicity of the previous deterioration amount \( s \), leading to the monotonicity of \( V(k, s) \), as shown in equations (4) and (5). However, geometric Brownian motion does not have this property. We introduce a necessary definition and some lemmas that are necessary to obtain the monotonicity of \( V(k, s) \) for geometric Brownian motion.

**Definition:** Stochastic Ordering between Two Distribution Functions [9]

For two random variables \( X \) and \( Y \), \( X \) has first-order stochastic dominance over \( Y \) if \( F_x(s) \leq F_y(s) \) for any \( s \).

**Lemma 1**

If \( S_k \) follows a geometric Brownian motion, the variable \( S_k | s' \) has first-order stochastic dominance over \( S_k | s \) for \( s' > s \).

**Proof:**

Conditional cumulative distribution function \( F_{(k+1)\tau | k\tau}(u | s) \) has the form
\[ F_{(k+1)\tau | k\tau}(u | s) = \int_0^u f_{(k+1)\tau | k\tau}(x | s) \, dx, \]
where \( f_{(k+1)\tau | k\tau}(x | s) \) is the corresponding conditional density probability function of \( S_{k+1} \) given \( S_k = s \). Since \( S_k \) follows a geometric Brownian motion, \( f_{(k+1)\tau | k\tau}(x | s) \) can be derived as
\[ f_{(k+1)\tau | k\tau}(x | s) = \frac{1}{\sqrt{2\pi \sigma^2 \tau x}} \exp \left\{ -\frac{\ln \left( \frac{x}{s} \right) - \left( \beta - \sigma^2 \right) \tau}{2\sigma^2 \tau} \right\}. \]

We take the difference between \( F_{(k+1)\tau | k\tau}(u | s') \) and \( F_{(k+1)\tau | k\tau}(u | s) \) for \( s' > s \) and obtain
\[ F_{(k+1)\tau | k\tau}(u | s') - F_{(k+1)\tau | k\tau}(u | s) = \int_0^u f_{(k+1)\tau | k\tau}(x | s) \, dx - \int_0^u f_{(k+1)\tau | k\tau}(x | s') \, dx = \Phi(z_s) - \Phi(z_{s'}), \]
where \( \Phi \) is the cumulative distribution function of the standard normal distribution and

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\[ z_s = \left( \ln \left( \frac{t}{\tau} \right) - \left( \frac{\beta - \frac{\sigma^2}{2}}{\tau} \right) \right) \sigma \sqrt{t} . \]

Since \( z_s > z_{s'} \), we have

\[ F_{(k+1)\tau|kr}(u|s) - F_{(k+1)\tau|kr}(u|s') > 0 \quad \text{for} \quad s' > s. \quad \text{(9)} \]

As \( u \) approaches infinity,

\[ \lim_{u \to \infty} \left\{ F_{(k+1)\tau|kr}(u|s) - F_{(k+1)\tau|kr}(u|s') \right\} = 0. \]

This completes the proof of lemma 1.

**Lemma 2**

For a non-decreasing function \( g(x) \) that is differentiable for \( x \),

\[ \int_0^\infty g(x)f_{(k+1)\tau|kr}(x|s')dx \geq \int_0^\infty g(x)f_{(k+1)\tau|kr}(x|s)dx \]

holds for \( s' > s \).

**Proof:**

Let \( S_k = s' \) be the amount of deterioration at the \( k \)-th time epoch. We define indicator function \( I \) as

\[ I_x(u) = \begin{cases} 1 & u \leq x \\ 0 & u > x \end{cases} \]

Since \( g(x) \geq 0 \) is a bounded non-decreasing function that is differentiable for \( x \), let

\[ \lim_{x \to 0} g(x) = 0 \]

and

\[ \lim_{x \to \infty} g(x) = K. \]

Moreover, \( g(x) \) can be written as

\[ g(x) = \int_0^x g'(u)du = \int_0^\infty I_x(u) \cdot g'(u)du. \quad \text{(10)} \]

From equation (10), we have

\[ \int_0^\infty g(x)f_{(k+1)\tau|kr}(x|s')dx = \int_0^\infty f_{(k+1)\tau|kr}(x|s') \int_0^\infty I_x(u) \cdot g'(u)du dx \]

\[ = \int_0^\infty \int_u^\infty f_{(k+1)\tau|kr}(x|s')dx \int_0^\infty g'(u)du \]

\[ = \int_0^\infty \left( 1 - F_{(k+1)\tau|kr}(u|s') \right) g'(u)du. \]

Similarly,

\[ \int_0^\infty g(x)f_{(k+1)\tau|kr}(x|s)dx = \int_0^\infty \left[ 1 - F_{(k+1)\tau|kr}(u|s) \right] g'(u)du \]

for \( S_k = s(< s') \). Since \( g(u) \) is a non-decreasing function, \( g'(u) \geq 0 \), and

\[ 1 - F_{(k+1)\tau|kr}(u|s') \geq 1 - F_{(k+1)\tau|kr}(u|s), \]

\[ \int_0^\infty g(x)f_{(k+1)\tau|kr}(x|s')dx \geq \int_0^\infty g(x)f_{(k+1)\tau|kr}(x|s)dx \]

holds for any \( s' > s \). This completes the proof of lemma 2.

**Lemma 3**

The expected operating cost \( W(k,s) \) for the \( k \)-th time period is non-decreasing in \( k \) and \( s \).

**Proof:**

Expected operating cost \( W(k,s) \) has the form

[DOI : 10.17929/ tqs.7.113]
Since deterioration amount follows a geometric Brownian motion, $f_{t\mid k\tau}(u \mid s) = \frac{1}{\sqrt{2\pi\sigma^2(t-k\tau)u}} \exp\left\{-\frac{(\ln(u/s) - (\beta - \frac{\sigma^2}{2})(t-k\tau))^2}{2\sigma^2(t-k\tau)}\right\},$ \hspace{1cm} (11)

Calculating the expectation of deterioration amount within time interval $k\tau \leq t < (k+1)\tau$ given $S_k = s$ we get

$$\int_0^\infty f_{t\mid k\tau}(u \mid s)\,du = s \cdot e^{\beta(t-k\tau)},$$ \hspace{1cm} (12)

Hence,

$$W(k, s) = s \int_{k\tau}^{(k+1)\tau} h(t)e^{\beta(t-k\tau)}\,dt = s \cdot \Gamma(k),$$

where $\beta$ is the mean. It is easily to show that $W(k, s)$ is increasing in $s$ since $\Gamma(k) > 0$ due to the fact $h(t) > 0$ and $e^{\beta(t-k\tau)} > 0$. To show that $W(k, s)$ is non-decreasing for all $k$, it is sufficient to show that $\Gamma(k)$ is non-decreasing in $k$. Let $q(t) = h(t)e^{\beta(t-k\tau)}$, where $q(t)$ is non-negative and non-decreasing for all $t$.

$$Q(t) = \int_0^t q(u)\,du,$$

then $Q'(t) = q(t) > 0$, and $Q''(t) = q'(t) \geq 0$ since $h(t) > 0$ and $h'(t) \geq 0$. Hence, $Q(t)$ is a convex function for all $t$. We rewrite $\Gamma(k)$ as follows,

$$\Gamma(k) = \int_{k\tau}^{(k+1)\tau} h(t)e^{\beta(t-k\tau)}\,dt = Q((k+1)\tau) - Q(k\tau).$$

Since $Q(t)$ is a convex function, we have

$$\Gamma(k) - \Gamma(k-1) = Q((k+1)\tau) + Q((k-1)\tau) - 2Q(k\tau)$$

$$= 2\left[\frac{1}{2}Q((k+1)\tau) + Q((k-1)\tau)\right] - Q\left[\frac{1}{2}((k+1)\tau + (k-1)\tau)\right] \geq 0,$$

which is known as Jensen’s inequality. This completes the proof of Lemma 3.

### 3.2 Proposition of optimal replacement policy

Using the lemmas proved in Section 3.1, we derive the properties of the optimal replacement policy.

**Proposition 1**

The total expected discounted cost $V(k, s)$ is monotonically non-decreasing in $s$ for any $k$.

**Proof:**

First, we prove the finite case of $V_i(k, s)$ ($i = 1, \ldots, n$) inductively.

**I.** For $i = n$, $V_n(k, s') = V_n(k, s) = 0$ for any $k$.

**II.** For $i = n - 1$, the difference between two measured values $s < s' \leq l$ for $V_{n-1}(k, s)$ is taken. When continue operating is selected, $V_{n-1}^C(k, s') - V_{n-1}^C(k, s) = W(k, s') - W(k, s) \geq 0$ for $s < s'$ from Lemma 3. When preventive replacement is selected, $V_{n-1}^P(k, s') - V_{n-1}^P(k, s) = e^{-\gamma\tau}(c_p - c_p) = 0$.

The same holds when corrective replacement is selected.

**III.** For $i = m$, we assume

$$V_m(k, s') - V_m(k, s) \geq 0$$ \hspace{1cm} (13)

holds.

**IV.** For $i = m - 1$, the difference between two measured values $s < s' \leq l$ is taken. When continue operating is selected,
On the basis of the standard argument of contraction mapping theory [8], the control limit deterioration amount if it satisfies

Moreover, from the standard argument of contraction mapping theory [8], the relation

holds when \( n \) is expanded to infinity. This completes the proof of proposition 1.

Proposition 2
The total expected discounted cost \( V(k, s) \) is monotonically non-decreasing in \( k \) for any \( s \).

Proof:
We prove the monotonicity of \( V(k, s) \) in \( k \) for any given deterioration amount \( s \in [0, \infty) \). First, we prove the finite case of \( V(k, s) \) (\( i = 1, ..., n \)) inductively.

I. For \( i = n \), \( V_n(k + 1, S_{k+1} = s) = V_n(k, S_k = s) = 0 \) for any \( k \).

II. For \( i = n - 1 \), the difference between two ages \( k \) and \( k + 1 \) is taken.

When continue operating is selected,

When preventive replacement is selected,

The same holds when corrective replacement is selected.

III. For \( i = m \), we assume

holds.

IV. For \( i = m - 1 \), the difference between two ages \( k \) and \( k + 1 \) is taken.

A) When continue operating is selected,

which holds from inductive hypothesis condition (14).

B) When preventive replacement is selected,

The same holds when corrective replacement is selected.

From the above four steps,

for \( i = 1, ..., n \) can be proved inductively.

On the basis of the standard argument of contraction mapping theory [8],

holds when \( n \) approaches infinity. This completes the proof of proposition 2.

We investigate the control limit where the optimal action changes in deterioration amount and age. Let \( s_k^* \) be the control limit deterioration amount if it satisfies \( V^f(k, s_k^*) = V^c(k, s_k^*) \) for a given age \( k \). Let
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\[ k^*_s = \arg\min_{k^*_s \in \{1, 2, \ldots \}} \{ V^C(k^*_s, s) \geq V^P(k^*_s, s) \} \]

be the control limit age \( k^* \) for a given \( s \). On the basis of propositions 1 and 2, we obtain a theorem for the replacement policy.

**Theorem**

Preventive replacement should be performed when deterioration amount \( s (\leq l) \) exceeds control limit \( s^*_k \) for age \( k \). Moreover, control limit \( s^*_k \) is non-increasing in \( k \).

**Proof:**

Preventive replacement is optimal when the following condition holds:

\[ c_P + e^{-\gamma t} V(0, 0) < W(k, s) + e^{-\gamma t} E[V(k + 1, S_{k+1}) | k, s]. \]  \hspace{1cm} (15)

The left side of condition (15) is a constant function of \( k \) and \( s \). From proposition 1, the right side is monotonically non-decreasing in \( s \). Hence, there exists at most one control limit \( s^*_k \) for any given \( k \). On the other hand,

\[ V^C(k, s^*_k) \leq V^C(k + 1, s^*_k) \]  \hspace{1cm} (16)

from proposition 2. Since

\[ V^C(k, s^*_k) = V^P(k, s^*_k) \]  \hspace{1cm} (17)

from the definition of \( s^*_k \) and

\[ V^P(k, s^*_k) = V^P(k + 1, s^*_k+1), \]  \hspace{1cm} (18)

from conditions (16), (17), and (18). Therefore, we have \( s^*_k \geq s^*_k+1 \). This implies that \( s^*_k \) is non-increasing in \( k \), and the theorem is proved.

The theorem also implies that \( k^*_s \), which is the control limit with respect to age \( k \) for a given deterioration amount \( s \), is non-increasing in \( s \).

**Fig. 1. Properties of control limit policy in preventive replacement**

Fig. 1 illustrates the properties of the control limit policy in preventive replacement. The horizontal axis shows the time epoch and system age, and the vertical axis shows the deterioration amount. The theorem obtained above can be explained using this figure. At time \( i \), the amount of system deterioration exceeds control limit \( s^*_i \), so the optimal action changes from continue operating to preventive replacement. Preventive replacement is thus selected, and both age and deterioration amount are reset to 0 from the next time period. At time \( j \), the deterioration amount exceeds preset safety limit \( l \), so corrective replacement is selected, and the age and deterioration amount are again reset to 0. Furthermore, the figure shows that control limit \( s^*_k (k = 0, 1, \ldots) \) and the corresponding \( k^*_s \) are monotonically non-increasing in \( k \) and \( s \), respectively.

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The following corollary summarizes these properties.

**Corollary**
The optimal replacement policy for a deteriorating system subjected to a geometric Brownian motion is given as

\[
V(k, s) = \begin{cases} 
V^c(k, s) & 0 \leq s < s_k^* \\
V^p(k, s) & s_k^* \leq s < l \\
V^f(k, s) & l \leq s
\end{cases}
\]

for \(0 \leq k < k_s^*\), and

\[
V(k, s) = \begin{cases} 
V^p(k, s) & 0 \leq s < l \\
V^f(k, s) & l \leq s
\end{cases}
\]

at the age \(k_s^*\). Note that \(k_s^*\) and \(s_k^*\) depend on the cost functions and parameters of the geometric Brownian motion. Therefore, control limits \(k_s^*\) and \(s_k^*\) may not exist in some cases.

4. Numerical example

A numerical example is provided to illustrate the results obtained in this research. Let the deterioration process follow a geometric Brownian motion model with drift coefficient \(\beta = 0.2\) and diffusion coefficient \(\sigma = \sqrt{0.1}\). The system is periodically inspected at intervals of \(\tau = 1\). The deterioration amount is classified into seven levels: \(s = 0\) if the deterioration amount is in the interval \([0,1)\) and \(s = 1, 2, 3, 4, 5, 6\) if it is in the intervals \([1,2), [2,5), [5,10), [10,50), [50,100), [100, +\infty)\), respectively. The safety limit is 100, which means the system is replaced if deterioration level \(l\) reaches 6. The costs for preventive and corrective replacements are \(c_p = 1000\) and \(c_f = 3300\). The incurred cost is discounted by \(e^{-0.02}\). For simplicity, we set \(h(t) = t\) and derive \(W(k, s)\) by taking the integral over each time interval. Given these parameters, we can calculate the total expected discounted cost for each \((k, s)\) and determine the optimal replacement policy, which is shown in Fig. 2.

Fig. 2 clearly shows that the optimal replacement policy has three action regions: continue operating (CO), preventive replacement (PR), and corrective replacement (CR). It also shows that the optimal replacement policy depends on the age of the system. For example, PR is not necessary for systems aged 0 or 1 while it is preferable to replace the system preventively before the deterioration level exceeds safety limit \(l\) for systems aged more than 2. Furthermore, Fig. 2, shows that the control limit, where the optimal action changes from CO to PR or CR, decreases with an increase in the deterioration level. For example, it is \(k_s^* = 5\) for deterioration level \(s = 0, 1, 2\); it is \(k_s^* = 4\) and \(2, 1\) for \(s = 3\) and \(s = 4, 5\), respectively. Similarly, the figure illustrates the monotonicity of \(s_k^*\) in age \(k\). To investigate how PR cost \(c_p\) and CR cost \(c_f\) affect the optimal replacement policy, we performed a sensitivity analysis. The results are shown in Figs. 3 and 4.
In Figs. 3 and 4, the horizontal axis is cumulative operating time, which is the age of the system, and the vertical axis is the control limit with respect to the cumulative deterioration amount. Sensitivity analysis was performed by varying PR cost \( c_P \) and CR cost \( c_F \) each among three different values. As shown in Fig. 3, the optimal control limit \( s_k^* \) tends to increase as \( c_P \) increases. This means that, as \( c_P \) increases, the effectiveness of implementing PR tends to decrease. On the other hand, as shown in Fig. 4, as \( c_F \) increases, the control limit policy becomes more conservative. These findings indicate that the effect of varying \( c_F \) is not as large as that of varying \( c_P \). Compared with PR, CR is seldom carried out because safety limit \( l \) in this example is set to 6, which corresponds to serious deterioration. The seldom use of CR can be explained by the low frequency of sudden failures. Hence, the CR cost has little effect on the control limit.

We also investigated the effect of different values of the geometric Brownian motion parameters on the control limits. In Figs. 5 and 6, the horizontal axis and vertical axis represent age and control limit. Sensitivity analysis was performed for three values each of mean \( \beta \) and standard deviation \( \sigma \). As shown in Fig. 5, optimal control limit \( s_k^* \) decreases as \( \beta \) increases. This means it is preferable to implement PR at an earlier stage if the system deteriorates rapidly since a larger mean of geometric Brownian motion indicates a higher deterioration rate. This can be explained as follows. A larger \( \beta \) increases the future cost if CO is selected. On the other hand, the total expected cost for PR remains constant. This means that control limit \( s_k^* \), where the optimal action changes from CO to PR, appears sooner with respect to the deterioration amount. Fig. 6 shows the relationship between control limit \( s_k^* \) and standard deviation \( \sigma \). Compared with \( \beta \), \( \sigma \) has little effect on the control limit since \( \sigma \) does not affect deterioration as much as \( \beta \).
5. Conclusion

The optimal replacement policy for systems with an increasing deterioration rate was investigated by modeling the deterioration process using a geometric Brownian motion and formulating the optimization problem using a discrete-time Markov decision process. The total expected discounted cost $V(k,s)$ was shown to be monotonically non-decreasing in both age $k$ and deterioration amount $s$. These results indicate that there is at most one switching point for corrective replacement, preventive replacement, and continue operating for deterioration amount. Furthermore, it was shown that the control limit with respect to the deterioration amount is monotonically non-increasing in age.

This study did not consider corrective replacement in the optimization process. The optimization was performed with a fixed safety limit. The effect of varying the safety limit should be investigated in future work. In addition, two possible actions, continue operating and replacement, were discussed in this research. However, sometimes it is possible to carry out an imperfect repair to restore the state to an interim level with a lower cost than full replacement. Future work includes examining a maintenance policy including imperfect repair.

References:


Acknowledgments:

This work was supported by JSPS KAKENHI Grant Number 17K01290.

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[DOI: 10.17929/tqs.7.113]