An Efficient Approximate Algorithm for the 1-Median Problem on a Graph*

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**SUMMARY** We propose a heuristic approximation algorithm for the 1-median problem. The 1-median problem is the problem of finding a vertex with the highest closeness centrality. Starting from a randomly selected vertex, our algorithm repeats to find a vertex with higher closeness centrality by approximately calculating closeness centrality of each vertex using simpler spanning subgraphs, which are called k-neighbor dense shortest path graphs with shortcuts. According to our experimental results using real networks with more than 10,000 vertices, our algorithm is more than 100 times faster than the exhaustive search and more than 20 times faster than the state-of-the-art approximation algorithm using annotated information to the vertices while the solutions output by our algorithm have higher approximation ratio.

**key words:** 1-median problem, closeness centrality, graph mining

1. Introduction

Since various networks in society can be represented as weighted graphs, methods for finding important vertices in a graph have a lot of applications. What is important depends on applications, and various importance measures have been proposed so far.

One of the most popular such measures is closeness centrality, which measures centrality in terms of distance to all the vertices. The closeness centrality of a vertex \(v\) is calculated using the (weighted) sum of the lengths of the shortest paths from \(v\) to all the vertices, and smaller length sum means higher closeness centrality. The problem of finding the highest closeness centrality is known as the 1-median problem, which is a special case of the k-median problem: the problem of finding a \(k\)-sized set \(S\) that minimizes the (weighted) sum of the lengths of the shortest paths from \(v\) to all the vertices, where \(v\) is the vertex in \(S\) that is nearest to \(u\).

The k-median problem is known to be NP-hard for \(k > 1\) [8] but the 1-median problem can be solved in \(O(n^2 \log n + mn)\) time by Dijkstra’s algorithm using a Fibonacci heap [3], [7], where \(n\) is the number of vertices and \(m\) is the number of edges. Recently, however, we have to deal with huge networks, for which algorithms are required to run in time close to linear in the network size.

In this paper, we propose a heuristic approximation algorithm for the 1-median problem. Our algorithm is iterative: starting from a randomly selected initial vertex \(v_0\), the algorithm finds a vertex \(v_s\) with closeness centrality higher than that of \(v_0\), and repeats the same process replacing \(v_0\) with \(v_s\) until such vertex \(v_s\) is not found. In each iteration, the algorithm approximately calculates closeness centrality of each vertex using a simpler subgraph containing the shortest path tree from the vertex \(v_0\). The highest-closeness-centrality vertex \(v_s\) for such simpler subgraphs has centrality that is at least the centrality of \(v_0\) for the original graph because the centrality of \(v_0\) is the same for both the graphs. As a subgraph, the shortest path tree from \(v_0\) itself can be used and then we can obtain the exact solution for the subgraph in \(O(n)\) time [5]. The centrality of a vertex \(v\) for the spanning tree is an upper bound of its centrality for the original graph, but the gap between them is large unless \(v = v_0\) in most graphs, which results in few iterations and a bad approximate solution. To obtain a better centrality upper bound for each vertex \(v\), we propose to use a k-neighbor dense shortest path graph from \(v_0\) with shortcuts, which is composed of (1) all the edges between k-nearest vertices of \(v_0\) and (2) at most \(k - 1\) edges between \(v\)’s partition and the other \(k - 1\) partitions in addition to the shortest path tree from \(v_0\), where each partition is composed of all the vertices whose nearest vertex among the k-nearest vertices of \(v_0\) is the same. Our algorithm, which we call FAOM (Fast Approximation of One Median), runs in \(O((kn + m) \log n + k^3 \log k \ell)\) time and \(O(m + n \log n)\) space, where \(\ell\) is the number of iterations, which is at most 7 in our experiments using a graph with 17,903 vertices and 197,031 edges.

According to the results of our experiments using real and synthetic datasets, for scale-free networks, FAOM outperforms the exhaustive search and an algorithm using DTZ (Distance To Zone) [11] to calculate approximate distances; for two real networks and one synthetic scale-free network, FAOM with \(k \leq 128\) is faster than the exhaustive search and the algorithm using DTZ while the approximation ratio of FAOM is always better than that of the algorithm using DTZ. Especially for the real networks, which have more than 10,000 vertices, FAOM with \(k \leq 128\) is at least 21 times faster than the algorithm using DTZ and at least 143 times faster than the exhaustive search.

2. Related Work

The graph median problem has been studied more than half
a century. Hakimi wrote a paper on the 1-median problem in 1964 [6]. The 1-median problem is related to the problem of constructing a shortest path tree from a given vertex because the 1-median problem can be solved by constructing a shortest path tree from each vertex, which we call the exhaustive search in this paper. For the shortest path tree construction problem, $O(n \log n + m)$-time algorithm using Fibonacci heap [3], [7] has been proposed, where $n$ is the number of nodes and $m$ is the number of edges.

Since the exhaustive search is too slow for a huge network, faster exact algorithms for a simpler graph or faster approximation algorithms have been developed. Burkard et al. proposed the exact algorithm for cactus graphs [2]. Rattigan et al. proposed an approximation algorithm for the centrality measure using annotated information to the vertices [11].

The 1-median problem is, in other words, the problem of finding a vertex with the highest closeness centrality [4]. In terms of closeness centrality, approximation algorithms for the ranking problems have been also developed [10], [15].

As for more general $k$-median problem, its NP-hardness was proved [8], and approximation algorithms have been proposed [1], [9], [12].

### 3. Problem Setting

Let $G = (V, E)$ be an undirected connected graph, where $V$ and $E$ are the sets of vertices and edges, respectively. Each vertex $v \in V$ has a weight $w(v) > 0$, and each edge $(u, v) \in E$ has a length $l(u, v) > 0$, where $(u, v)$ for $u, v \in V$ represents an edge between two vertices $u, v \in V$. The number of vertices and edges are denoted as $n$ and $m$, respectively.

The distance between any two vertices $u$ and $v$ in $G = (V, E)$ is defined as the shortest path length between $u$ and $v$ and denoted as $d_G(u, v)$. The distance $d_G(u, v)$ is also written as $d(u, v)$ by omitting $G$ when $G$ is clear from context.

Let $d_G(v)$ denote the $w(u)$-weighted sum of distances $d_G(u, v)$ from $v$ to all the vertices $u$, that is,

$$d_G(v) = \sum_{u \in V} d_G(u, v)w(u).$$

In this paper, we consider the problem of finding a vertex $v$ with the minimum $d_G(v)$ among all vertices $v \in V$.

#### Problem 3.1 (1-median problem): For a given undirected connected graph $G = (V, E)$ with a vertex weight function $w : V \to (0, \infty)$ and an edge length function $d : E \to (0, \infty)$, find a vertex $v$ with the minimum $d_G(v)$ among all vertices $v \in V$.

This problem can be solved exactly by constructing a shortest path tree from each vertex, which takes $O(n^2 \log n + mn)$ time using Dijkstra’s algorithm, and $O(nm)$ time using the Thorup’s algorithm [14] in the case with positive integer length function.

The followings are notions and notations related to shortest path trees that are used in this paper. The shortest path tree $T(v)$ of $G$ from $v$ is the spanning tree of $G$ that contains a shortest path from $v$ to all $u \in V \setminus \{v\}$. The shortest path tree $T(v)$ can be regarded as a rooted tree with root $v$. We let $D_T(v)$ denote the set of the descendants of $v$ in the rooted tree $T(v)$. Note that $u$ is a descendant of $v$. We let $p_T(v)$ denote the parent of $u$ in $T(v)$, and let $p_T^{-1}(u)$ denote the parent of $p_T(v)$. Define $W_T(v)$ and $d_T(v)$ as

$$W_T(v) = \sum_{u \in D_T(v)} w(u)$$

and

$$d_T(v) = \sum_{u \in D_T(v)} d_T(v, u)w(u),$$

respectively. We sometimes omit the subscript $T(v)$ when it is clear from context.

### 4. Iterative Algorithm Framework

To obtain an approximate solution for 1-median problem, we propose an iterative algorithm framework shown in Algorithm 1. Starting from a randomly selected vertex, an algorithm in our framework finds a better vertex (a vertex $v$ with smaller $d_G(v)$) repeatedly until failing to find such a vertex. The point is how we can find a vertex $v$ with $d_G(v)$ smaller than $d_G(v_0)$ of a given vertex $v_0$ efficiently.

In our framework, an algorithm calculates $d_G(v_0)$ by constructing the shortest path tree $T(v_0)$ from $v_0$. Then, in order to efficiently find vertex $v$, with $d_G(v_0) \leq d_G(v)$, the algorithm calculates $d_G(v)$ for each $v \in V$, where $G_n$ is a subgraph of $G$ that contains $T(v_0)$. The algorithm in this framework can be implemented efficiently if $G$ has a simple structure; in the case with $G_n = T(v_0)$, $d_G(v)$ for all $v \in V$ can be calculated in $O(n)$ time [5]. Furthermore, $d_G(v) \leq d_G(v_0)$ if $d_G(v) \leq d_G(v_0)$ because $d_G(v) \leq d_G(v_0)$ and $d_G(v_0) = d_G(v_0)$, where the last equality holds because $G_n$ contains the shortest path tree $T(v_0)$ of $G$ from $v_0$. Thus, $d_G(v) \leq d_G(v_0)$ is guaranteed for $v = arg \min d_G(v)$, which means that no worse vertex is obtained by any iteration. Note that $d_G(v_0) \leq d_G(v_0)$ is guaranteed even using an upper bound $\hat{d}_G(v)$ of $d_G(v)$ if $\hat{d}_G(v_0) = d_G(v_0)$ holds.

#### Algorithm 1 Iterative algorithm framework for 1-median problem

1. $v_0 \leftarrow$ a randomly selected vertex from $V$;
2. repeat
3. $v_0 \leftarrow v_0$
4. Calculate $d_G(v_0)$ constructing the shortest path tree $T(v_0)$ from $v_0$.
5. Calculate $d_G(v)$ for each $v \in V$, where $G_n$ is a subgraph of $G$ that contains $T(v_0)$.
6. Set $v_0 = arg \min_{v \in V} d_G(v)$.
7. until $d_G(v_0) \geq d_G(v_0)$
8. output $v_0$.
5. Subgraphs for Efficient Approximation

In order to efficiently obtain a tighter upper bound $d_{G_v}(v)$ of $d_G(v)$, what subgraph $G_v$ of $G$ should be used? Under the constraint that $G_v$ must contain the shortest path tree $T(v_0)$ from $v_0$, it is ranged from $T(v_0)$ to $G$ itself. A simple subgraph $G_v$ enables fast calculation of $d_{G_v}(v)$ but brings a loose upper bound of $d_G(v)$. Conversely, a tight upper bound of $d_{G_v}(v)$ can be obtained if $G_v$ is close to $G$ though slow calculation of $d_{G_v}(v)$ is inevitable. So, we should choose a subgraph balancing this trade-off.

5.1 $k$-Neighbor Dense Shortest Path Graph

If the shortest path tree $T(v_0)$ is used as $G_v$, then the difference between $d_{G_v}(v)$ and $d_G(v)$ is 0 at $v = v_0$ and expected to increase as the distance between $v$ and $v_0$ increases. The difference is expected to become close to 0 for neighbors of $v_0$ if the edges between the neighbors are added.

From this consideration, as a simple subgraph of $G$ that is an extension of the shortest path tree $T(v_0)$, we propose a $k$-neighbor dense shortest path graph of $G$ from $v_0$, $k$NSPG for short. Let $N_k(v_0)$ denote the set of $k$ nearest vertices of $v_0$ in $V$, that is, the set of vertices with the smallest distance $d_{G_v}(v, v_0)$ from $v_0$. Note that $v_0 \in N_k(v_0)$. The $k$NSPG of $G$ from $v_0$, which is denoted as $T(v_0; k)$, is defined as a subgraph of $G$ that is constructed from the shortest path tree $T(v_0)$ of $G$ from $v_0$ by adding all the edges in $E$ between the vertices in $N_k(v_0)$. The $k$NSPG $T(v_0; k)$ of $G$ from $v_0$ is an extension of the shortest path tree $T(v_0)$ of $G$ from $v_0$ because $T(v_0)$ is just the 1NSPG $T(v_0; 1)$.

Example 5.1: Let $G$ be the leftmost graph of Fig. 1. Then, the center graph in the figure is a 4NSPG $T(v_0; 4)$ of $G$ from $v_0$. The four vertices in the gray box are members of $N_4(v_0)$. The edge between $v_1$ and $v_2$ is not contained in the shortest path tree $T(v_0)$ but it is contained in $T(v_0; 4)$.

The construction of $T(v_0; k)$ and the calculation of $d_{T(v_0; k)}(v)$ for all the vertices $v \in V$ can be done efficiently for small $k$.

Theorem 5.2: For a given graph $G = (V, E)$, $d_{T(v_0; k)}(v)$ for all the vertices $v \in V$ can be calculated in $O(n \log n + m + k^3)$ time and $O(m)$ space.

(proof) See Appendix A. □

5.2 $k$NSPG with Shortcuts

By using the $k$NSPG $T(v_0; k)$ as $G_v$, the upper bound $d_{G_v}(v)$ of $d_G(v)$ is expected to become tight for neighbors of $v$ of $v_0$, but it might be still loose for the vertices $v$ that are far from $v_0$ if $k$ is small. In order to improve the upper bound for such vertices, we consider a further extension of $k$NSPG $T(v_0; k)$ for each $v$ by adding at most $k - 1$ edges depending on $v$.

Before describing the extension, we introduce some notions and notations. For each vertex $v \in V$, define the closest $v_0$-neighbor $C(v, v_0; k)$ of $v \in V$ in $T(v_0; k)$ as the nearest vertex in $N_k(v_0)$ from $v$, that is, $C(v, v_0; k) = \arg \min_{w \in N_k(v_0)} d_{T(v_0; k)}(v, u)$. If $v$ is in $N_k(v_0)$, $C(v, v_0; k)$ is $v$ itself.

For each $v \in N_k(v_0)$, we define the $v$-subtree $ST(v, v_0; k)$ of $T(v_0; k)$ as the vertex-induced subgraph of $T(v_0; k)$ that is composed of all the vertices $u \in V$ with $C(u, v_0; k) = v$. A $k$NSPG $T(v_0; k)$ is partitioned into $k$ disjoint $v$-subtrees $ST(v, v_0; k)$ for $v \in N_k(v_0)$. In the following, $C(v, v_0; k)$ and $ST(v, v_0; k)$ are also written as $C(v)$ and $ST(v)$ by omitting “$v_0; k$” when it is clear from context.

Example 5.3: In the center graph 4NSPG $T(v_0; 4)$ of Fig. 1,

\[
C(v_0) = C(v_4) = C(v_8) = C(v_9) = v_0, \\
C(v_1) = v_1, \\
C(v_2) = C(v_3) = C(v_6) = v_2, \\
C(v_5) = C(v_7) = v_3.
\]

Thus, $T(v_0; 4)$ are partitioned into $ST(v_0), ST(v_1), ST(v_2)$ and $ST(v_3)$, which are the vertex-induced subgraphs of vertex-sets \{v_0, v_4, v_8, v_9\}, \{v_1\}, \{v_2, v_3, v_6\} and \{v_5, v_7\}, respectively.

Now, we describe our extension of $k$NSPG $T(v_0; k)$ of $G$ from $v_0$. A $k$NSPG of $G$ from $v_0$ with shortcuts for $v$ is defined as a subgraph of $G$ that is constructed from $T(v_0; k)$ by adding at most one edge $(s, t) \in E$ between $V_{ST}(v_0)$ and $V_{ST}(u)$ for each $u \in N_k(v_0) \setminus \{C(v)\}$, where $V_{ST}(v_0)$ and $V_{ST}(u)$ are the set of vertices in $ST(C(v))$ and $ST(u)$, respectively.

Example 5.4: The rightmost graph in Fig. 1 is a $k$NSPG of the leftmost graph $G$ from $v_0$ with shortcuts for $v_0$.

Let $T^S(v_0; k)$ denote the $k$NSPG of $G$ with shortcuts for $v$ that is constructed from $T(v_0; k)$ by adding edges in $S$. Now, the problem is how to efficiently find $S$ that is good for $v$. To address the problem, we first consider effect of...
adding an edge.

For $v \in V$, $s \in D_{T(v_0)}(v)$, $e = (s, t)$ and $u \in D_{T(v_0)}(C(t)) (C(t) \neq C(v))$, define

$$a_{T(v_0,k),e}(v,u) = d_{T(v_0,k)}(s,u) - d_{T(v_0,k)}(e,v)$$

$$= d_{T(v_0,k)}(e,v) + d_{T(v_0,k)}(C(t),v) + d_{T(v_0,k)}(C(t),u) - (d_{T(v_0,k)}(v,s) + \text{len}(s,t) + d_{T(v_0,k)}(u,t)).$$

(1)

We call $u \in \text{ST}(C(t))$ an affected vertex of $e = (s,t)$ for any vertex $v$ if $a_{T(v_0,k),e}(v,u) > 0$. For a descendant $s$ of $v$ in the tree $\text{ST}(C(v))$ rooted by $C(v)$ and $u \in N_k(v_0) \setminus \{C(v)\}$, define the nearest affected vertex $v_{op} \in \text{ST}(u)$ of an edge $(s,t) \in (\text{VST}(C(v)) \times \text{VST}(u)) \cap E$ for $v$ as

$$v_{op} = \min_{u' \in \text{VST}(v), u' \neq u} a_{T(v_0,k),e}(v,u').$$

Note that $v_{op}$ is the vertex that is nearest to $v_0$ among the vertices $u'$ in $\text{VST}(u)$ for which the distance from $v$ to $u'$ is shortened by the path using edge $(s,t)$. The effect $\delta_{T(v_0,k),e}(v)$ of edge $e$ for $v$ is defined as $d_{T(v_0,k)}(v) - d_{T(v_0,k)}(e,v)$ and the restricted effect $\tilde{\delta}_{T(v_0,k),v}(e)$ of $e$ for $v$ is defined as the effect of $e = (s,t)$ restricted to the set of vertices $\text{ST}(C(t))$, that is, $\tilde{\delta}_{T(v_0,k),v}(e) = \sum_{u \in \text{VST}(C(t))} a_{T(v_0,k),e}(v,u)u(u)$.

Then,

$$\tilde{\delta}_{T(v_0,k),v}(e) = a_{T(v_0,k),e}(v,v_{op})W_T(v_{op})$$

$$+ \sum_{i=1}^{\text{deg}} d_{T(v_0)}(p^i(t), p^{i-1}(t))W_T(p^{i-1}(t))$$

(2)

holds, where $p^{i,v}(t) = v_{op}$.

**Theorem 5.5:** Given a $k$NSPG $T(v_0; k)$ of $G$ from $v_0$ and $d_{T(v_0,k)}(v,u)$ for all $v, u \in N_k(v_0)$ ($v \neq u$), after $O(n \log n)$-time $O(n \log n)$-space preparation, the restricted effect $\tilde{\delta}_{T(v_0,k),v}(e)$ of any edge $e$ for any vertex $v$ can be calculated in $O(n \log n)$ time.

(proof) See Appendix B.

The restricted effect $\tilde{\delta}_{T(v_0,k),v}(e)$ is a lower bound of effect $\delta_{T(v_0,k),v}(e)$, and the both coincide if $v_{op} \neq C(t)$ for $e = (s,t)$. The merits of calculating $\tilde{\delta}_{T(v_0,k),v}(e)$ are not only computational efficiency but also disjointness from the effect of other edge between $\text{ST}(C(v))$ and $u$-subtree for any $u \in N_k(v_0)$. Let $S$ be the set of at most $k - 1$ edges in $E$ that satisfies

$$(s_1, t_1), (s_2, t_2) \in S \Rightarrow s_1, s_2 \in D_{T(v_0)}(v), C(t_1), C(t_2) \neq C(v), C(t_1) \neq C(t_2).$$

Then

$$d_{T(v_0,k)}(v) \leq d_{T(v_0,k)}(v) - \sum_{e \in S} \tilde{\delta}_{T(v_0,k),v}(e)$$

is derived from disjointness property of $\tilde{\delta}_{T(v_0,k),v}(e)$. Since our purpose is to obtain a tighter upper bound of $d_G(v)$ efficiently and $d_{T(v_0,k)}(v)$ is an upper bound of $d_G(v)$, it is no problem to use $d_{T(v_0,k)}(v) - \sum_{e \in S} \tilde{\delta}_{T(v_0,k),v}(e)$ as an upper bound of $d_G(v)$; it is tighter than $d_{T(v_0,k)}(v)$ by $\sum_{e \in S} \tilde{\delta}_{T(v_0,k),v}(e)$.

For each $v \in N_k(v_0) \setminus \{C(v)\}$, we want to know the maximum $\tilde{\delta}_{T(v_0,k),v}(e)$ among $e \in (D_{T(v_0)}(v) \times \text{VST}(u)) \cap E$, but it is computationally too heavy. Giving up to find the optimal value, we try to find a good edge with large $\tilde{\delta}_{T(v_0,k),v}(e)$. Let $E''_v = \{(v,t) \in E \mid t \in \text{VST}(u)\}$.

We define an edge $e_{u,v} \in (D_{T(v_0)}(v) \times \text{VST}(u)) \cap E$ for each $v \in V$ and $u \in N_k(v_0) \setminus \{C(v)\}$ as

$$e_{u,v} = \begin{cases} \arg \max_{e \in E''_v} \tilde{\delta}_{T(v_0,k),v}(e) & (D_{T(v_0)}(v) = \{v\}) \\ \arg \max_{e \in E''_v \cup \{e_{u,v}\} : p^{i,v}(t) = u} \tilde{\delta}_{T(v_0,k),v}(e) & (D_{T(v_0)}(v) \neq \{v\}). \end{cases}$$

Note that $e_{u,v}$ can be calculated in a bottom-up manner from the leaf nodes $v$ of $T(v_0)$. For leaf nodes $v$ of $T(v_0)$, edges $e_{u,v}$ are optimal but the optimality is not guaranteed for other vertices. Since $\tilde{\delta}_{T(v_0,k),v}(e)$ is expected to be close to $\delta_{T(v_0,k),v}(e)$, $\tilde{\delta}_{T(v_0,k),v}(e)$ is expected to be large if $\tilde{\delta}_{T(v_0,k),v}(e)$ is large. Thus, even for non-leaf nodes $v$, edges $e_{u,v}$ are expected to have large $\tilde{\delta}_{T(v_0,k),v}(e)$. Define $S_v$ as $S_v = \{e_{u,v} \mid u \in N_k(v) \setminus \{C(v)\}\}$. Then, we use $T^S(v_0; k)$ for a subgraph of $G$ to calculate an upper bound of $d_G(v)$.

**Theorem 5.6:** Given a $k$NSPG $T(v_0; k)$ of $G$ from $v_0$ and $d_{T(v_0,k)}(v,u)$ for all $v, u \in N_k(v_0)$ ($v \neq u$), after $O(n \log n)$-time $O(n \log n)$-space preparation, upper bounds

$$d_{T(v_0,k)}(v) - \sum_{e \in S} \tilde{\delta}_{T(v_0,k),v}(e)$$

of $d_{T^S(v_0,k)}(v)$ for all $v \in V \setminus N_k(v_0)$ can be calculated in $O((kn + m) \log n)$ time.

(proof) See Appendix C.

$\square$

6. Algorithm FAOM

We propose algorithm FAOM (Fast Approximation of One Median) of the iterative algorithm framework using $k$NSPGs $T^S(v_0; k)$ with the set $S_v$ of shortcuts for each vertex $v$ as a subgraph $G_v$ of a given graph $G$ to calculate an upper bound of $d_G(v)$.

A pseudocode of FAOM is shown in Algorithm 2. FAOM repeats the execution (Line 5) of procedure HigherCentralityVertex which returns $\nu$ and an upper bound $d_v$ of $d_G(v)$, with $d_v \leq d_{T(v_0)}(v)$ given an input vertex $v_0$. In HigherCentralityVertex, the $k$NSPG $T(v_0; k)$ of $G$ from $v_0$ is constructed (Line 12) and $d_{T(v_0,k)}(v)$ is calculated for all $v \in V$ (Line 13). After the preparation for fast calculation of $\tilde{\delta}_{T(v_0,k),v}(e)$ (Lines 14-15), explained in the proof of Theorem 5.5 (Line 14), upper bounds
Algorithm 2 FAOM

1: function FAOM(G = (V, E), k)
2: \( v_0 \leftarrow \) a randomly selected vertex from V
3: repeat
4: \( v_0 \leftarrow v_*; \)
5: \((v_*, d_*) \leftarrow \text{HigherCentralityVertex}(G, k, v_0)\)
6: until \( d_* \geq d_G(v_0)\)
7: output \( v_0; \)
8: end function
9:
10: function HigherCentralityVertex\((G = (V, E), k, v_0)\)
11: Calculate \( d_G(v_0)\) constructing the shortest path tree \( T(v_0)\) from \( v_0. \)
12: Construct \( kNSPG \) \( T(v_0; k)\) of \( G \) from \( v_0 \)
by adding edges between the vertices in \( N_k(v_0) \) to \( T(v_0)\)
13: Calculate \( d_T(v_0; k)(v)\) for all \( v \in V. \)
14: Do preparation for fast calculation of \( \overline{d}_{T(v_0k)}(v)\)
\((v \in V \setminus N_k(v_0), s \in D_T(v_0)(v), t \notin V_T\).
(See Appendix B.)
15: \( v_* \leftarrow v, \ d_* \leftarrow d_G(v_0)\)
16: for all \( v \in V \setminus N_k(v_0)\) do
17: \( \overline{d}_* \leftarrow \overline{d}_{T(v_0k)}(v) - \sum_{e \in E_T(v_0k)}(v)\)
18: if \( \overline{d}_* < d_*\) then
19: \( v_* \leftarrow v, \ d_* \leftarrow \overline{d}_*\)
20: end if
21: end for
22: return \((v_*, d_*)\);
23: end function

For the network with \( n \) vertices, \( \ell \) is the number of the main-loop iterations. Though we have not obtained any non-trivial upper bound of \( \ell \) yet, \( \ell \) was at most 7 in our experiment even for the network with \( n > 10,000 \) and \( m > 100,000. \)

Remark 6.2: Assume that the number of the main-loop iterations is \( O(1) \), and let us consider the time and space complexities of FAOM to those of other algorithms that are used as comparative methods in our experiments. The exhaustive search using Dijkstra’s algorithm runs in \( O((kn + m) \log n + k^3) \) time and \( O(m + n \log n) \) space.

Remark 6.3: Unfortunately, for \( n > 2k \), FAOM’s approximation ratio is not good in the worst case even when \( w(v) = 1 \) for all \( v \in V \) because we have

\[
\sup_{G, len} \frac{d_G(\hat{v})}{d_G(v_0)} = n - 1 \quad \text{for } v_* = \arg \max_{v \in V} d_G(v) \quad \text{and FAOM’s output } \hat{v}\]

for \( k \); then

\[
\text{Inequality } \frac{d_G(\hat{v})}{d_G(v_0)} \leq n - 1 \quad \text{for any } G \quad \text{and len holds because } d_G(\hat{v}) \leq d_G(v) \leq d_G(v_0) \leq d_G(v_0) \text{.}
\]

Thus, Equality \( \sup_{G, len} \frac{d_G(\hat{v})}{d_G(v_0)} = n - 1 \) holds for this \( G \). Approximation ratio of FAOM’s outputs is close to 1 in our experiments, so what condition makes FAOM output a vertex with good approximation ratio is an interesting issue to pursue.

7. Experiments

We conducted experiments to check the effectiveness of our method using synthetic and real datasets.

7.1 Experimental Setting

We used four datasets shown in Table 1.

As real datasets, we used two datasets of Stanford Large Network Dataset Collection\(^\text{11}\). Dataset ca-AstroPh is a collaboration network of Arxiv Astro Physics category, in which vertices represent authors of scientific papers and edges represent co-author relationship. Dataset Oregon-1(May26) is a social system peering information inferred from Oregon route-views, in which vertices represent autonomous systems and edges represent existence of communication between them.

As synthetic datasets, we generate a random and a scale-free graph. Dataset ER is a random graph generated using Erdős-Rényi model, in which all the pairs of vertices are connected randomly with a given probability \( p \). We set \( p = 0.0012 \) in our experiments. Dataset BA is a scale-free graph generated using Barabási-Albert model, which was generated starting from the complete graph of three vertices by repeatedly adding a vertex \( v \) and edges between \( v \) and (at most) three other existing vertices that were selected according to the probability distribution proportional to the current

\(^\text{11}\)https://snap.stanford.edu/data/index.html
vertex degree\footnote{Selection were done three times independently according to the same distribution, and distinct ones of the three selected vertices were chosen.}. In a graph of Barabási-Albert model, the distribution of vertex degrees is known to obey power-law distribution.

All the datasets but BA are disconnected, so the maximum connected components, whose numbers of vertices and edges are shown in Table 1, were used in those datasets.

In ER and BA, we generated the vertex weights according to this distribution, so \( \text{len}(v) \) is improved by using \( \frac{d_G(v)}{d_G(v)} \) for each of the three. The result for the ca-AstroPh is shown in Fig. 2. We made histograms of the values with range width 0.01 for \( d_G \). The distribution of vertex weights is known to obey power-law distribution, so \( \text{len}(v) \) are generated according to this distribution, so \( \text{len}(v) \) holds for any vertices \( u, v \) of the generated graphs \( G \).

In algorithm FAOM, we use an upper bound of \( d_T(v_0;k)(v) \) to obtain an upper bound of \( d_G(v) \) for \( v \in V \). As a subgraph of \( G \) containing \( T(v_0) \), we checked effectiveness of using \( T^S(v_0;k) \) comparing with simpler subgraphs \( T(v_0) \) and \( T(v_0;k) \).

For each of the randomly selected 100 initial vertices \( v_0 \), we calculated \( d_T(v_0)(v)/d_T(v_0)(v) \) and \( d_T(v_0;k)(v)/d_T(v_0;k)(v) \) and \( (d_T(v_0;k)(v) - \sum_{e \in E} \varpi_{T(v_0;k)\varepsilon} T(v_0;k)(v)/d_T(v_0;k)(v) \) is an upper bound of \( d_T(v_0;k)(v)/d_T(v_0;k)(v) \) for all the vertices \( v \). Then, we made histograms of the values with range width 0.01 for each of the three. The result for the ca-AstroPh is shown in Fig. 2. As we can see on this graph, the approximation ratio is improved by using \( T(v_0;k) \) and furthermore by using \( T^S(v_0;k) \).

7.3 Number of Main-Loop Iterations

Efficiency of algorithm FAOM depends on the number of main-loop iterations, that is, the number of executions of HigherCentralityVertex. So, we checked the distribution of the number of the iterations using randomly selected 100 initial vertices \( v_0 \) for each graph. The result is shown in Table 2. As compared with the number of vertices, the number of iterations is very small (at most 7) on any graph in our experiments. As a result, our algorithm runs fast for the datasets.

7.4 Effect of Using Larger \( k \)

FAOM has parameter \( k \) which controls the complexity of subgraphs \( T^S(v_0;k) \) of \( G \). Larger \( k \) is expected to improve approximation ratio \( d_G(v)/d_G(v) \) of FAOM’s output \( \hat{v} \) for the optimal vertex \( v_* \). We compared FAOM’s performance to those of exact method and one state-of-the-art approximation method. Exact method (Exact) is the exhaustive search for the optimal vertex by constructing shortest path trees from all the vertices using Dijkstra’s algorithm. As an approxima-

\begin{table}[h]
\centering
\begin{tabular}{|c|cccccccc|}
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
ca-AstroPh & 14 & 38 & 27 & 15 & 6 &  &  & \\
Oregon1(May26) & 22 & 41 & 28 & 9 &  &  &  & \\
BA & 10 & 48 & 36 & 5 & 1 &  &  & \\
ER & 2 & 24 & 40 & 21 & 6 & 5 & 2 & \\
\hline
\end{tabular}
\caption{The number of main-loop iterations}
\end{table}


\footnote{DTZ and Exact are too slow to execute 1000 times.}
Table 3 Approximation ratio \(d_G(\hat{v})/d_G(v_0)\) and running time [sec] of three methods, FAOM, DTZ and Exact, where \(\hat{v}\) is the vertex found by a method and \(v_0\) is the optimal vertex. The results are averaged over 1000 runs for FAOM, 100 runs for DTZ and 1 run for Exact. The width of 95% confidence interval is shown in parentheses for approximation ratio and omitted for running time because it is ignobly small.

<table>
<thead>
<tr>
<th>method</th>
<th>parameters</th>
<th>ca-AstroPh</th>
<th>Oregon1(May26)</th>
<th>ER</th>
<th>BA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>app. ratio</td>
<td>app. ratio</td>
<td>time</td>
<td>app. ratio</td>
</tr>
<tr>
<td>FAOM</td>
<td>(k = 1)</td>
<td>1.023(±0.002)</td>
<td>1.951</td>
<td>1.000(±0.000)</td>
<td>0.207</td>
</tr>
<tr>
<td></td>
<td>(k = 2)</td>
<td>1.024(±0.002)</td>
<td>3.588</td>
<td>1.000(±0.000)</td>
<td>0.337</td>
</tr>
<tr>
<td></td>
<td>(k = 4)</td>
<td>1.030(±0.003)</td>
<td>5.143</td>
<td>1.000(±0.000)</td>
<td>0.459</td>
</tr>
<tr>
<td></td>
<td>(k = 8)</td>
<td>1.026(±0.002)</td>
<td>7.154</td>
<td>1.000(±0.000)</td>
<td>0.506</td>
</tr>
<tr>
<td></td>
<td>(k = 16)</td>
<td>1.025(±0.003)</td>
<td>9.602</td>
<td>1.000(±0.000)</td>
<td>0.567</td>
</tr>
<tr>
<td></td>
<td>(k = 32)</td>
<td>1.018(±0.002)</td>
<td>11.764</td>
<td>1.000(±0.000)</td>
<td>0.738</td>
</tr>
<tr>
<td></td>
<td>(k = 64)</td>
<td>1.025(±0.003)</td>
<td>13.290</td>
<td>1.000(±0.000)</td>
<td>1.160</td>
</tr>
<tr>
<td></td>
<td>(k = 128)</td>
<td>1.021(±0.002)</td>
<td>16.172</td>
<td>1.000(±0.000)</td>
<td>3.434</td>
</tr>
<tr>
<td></td>
<td>(k = 256)</td>
<td>1.014(±0.001)</td>
<td>41.303</td>
<td>1.000(±0.000)</td>
<td>22.111</td>
</tr>
<tr>
<td></td>
<td>(k = 512)</td>
<td>1.007(±0.001)</td>
<td>197.273</td>
<td>1.000(±0.000)</td>
<td>159.656</td>
</tr>
<tr>
<td>DTZ</td>
<td>(k = 2, d = 10)</td>
<td>1.167(±0.001)</td>
<td>1677.122</td>
<td>1.232(±0.001)</td>
<td>713.922</td>
</tr>
<tr>
<td></td>
<td>(k = 5, d = 4)</td>
<td>1.111(±0.001)</td>
<td>883.119</td>
<td>1.181(±0.001)</td>
<td>349.941</td>
</tr>
<tr>
<td></td>
<td>(k = 10, d = 2)</td>
<td>1.084(±0.000)</td>
<td>525.265</td>
<td>1.145(±0.000)</td>
<td>198.245</td>
</tr>
<tr>
<td></td>
<td>(k = 20, d = 1)</td>
<td>1.064(±0.001)</td>
<td>344.416</td>
<td>1.111(±0.003)</td>
<td>123.802</td>
</tr>
<tr>
<td>Exact</td>
<td>-</td>
<td>1.000(±0.000)</td>
<td>3037.151</td>
<td>1.000(±0.000)</td>
<td>493.967</td>
</tr>
</tbody>
</table>

Fig. 3 Upper graph: Relation between parameter \(k\) and approximation ratio \(d_G(\hat{v})/d_G(v_0)\), where \(v_0\) is the optimal vertex and \(\hat{v}\) is the output of FAOM. The error bars show 95% confidence interval. Lower graph: Relation between parameter \(k\) and running time of FAOM.

In this paper, we proposed an approximation algorithm for the 1-median problem that repeats to find a vertex with higher closeness centrality starting a randomly selected initial vertex. The key of the success of our iterative approach is what subgraph is used to efficiently obtain a tight upper

8. Conclusion and Future Work

In this paper, we proposed an approximation algorithm for the 1-median problem that repeats to find a vertex with higher closeness centrality starting a randomly selected initial vertex. The key of the success of our iterative approach is what subgraph is used to efficiently obtain a tight upper...

DTZ stores \(kd\) values per vertex in memory for fast calculation.
bound of closeness centrality of each vertex. FAOM uses
$k$-neighbor dense shortest path graphs with shortcuts, which
results in empirical efficiency and approximation ratio close to
$1$. It is an important remaining issue to theoretically clarify
input graph conditions under which those subgraphs are
effective. Furthermore, there may be better subgraphs for this
approach. It is an interesting research direction to study
what subgraph is appropriate for the approach. Another
interesting research direction is extension of our algorithm to
the $k$-median problem.

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Appendix A: Proof of Theorem 5.5

(proof) As preparation, for all $v \in V$, we calculate
(a) $W(T_{(0)}(v), C(v))$ and $d_{T_{(0)}(v)}(v, C(v))$,
(b) $d_{T_{(0)}(v)}(v, p^j)$ for $j = 0, \ldots, \log(\text{depth}_{ST(C(v))}(v))$ and
(c) $\sum_{i=1}^{2^j} d_{T_{(0)}(p^{i}(v), p^{i-1}(v))} W_{T_{(0)}(p^{i-1}(v))}$
for $j = 0, \ldots, \log(\text{depth}_{ST(C(v))}(v))$, where
$\text{depth}_{ST(C(v))}(v)$ is the depth of $v$ in the rooted tree
$ST(C(v))$.

(a) can be calculated by recursive call starting from $u_0$ and
traversing $T_{(0)}(v)$ in $O(n)$-time and $O(n)$-space. (b) and
(c) can also be calculated by similar recursive call stacking
$d_{T_{(0)}(v)}(v)$ and $\sum_{i=1}^{\text{depth}_{v_0}(v)} d_{T_{(0)}(p^{i}(v), p^{i-1}(v))} W_{T_{(0)}(p^{i-1}(v))}$
to a stack array, from which (b) and (c) for $v$ can be calculated
in $O(\log n)$-time and $O(n)$-space. Totally, (a), (b), and
(c) are calculated in $O(n \log n)$-time and $O(n \log n)$-space.

We show that $d_{T_{(0)}(v)}(v)$ can be calculated in $O(\log n)$
time using above (a), (b), and (c). Let $e = (s, t)$. First,
$v_{op}$ can be calculated by a kind of binary search using
(b) starting from the comparison between $d_{T_{(0)}(e)}(v, t) =
d_{T_{(0)}(v)}(v, C(v)) + d_{T_{(0)}(C(v))}(C(v), f(t)) + d_{T_{(0)}(f(t))}(f(t), t)$
and $d_{T_{(0)}(v)}(v, t) = d_{T_{(0)}(v)}(v, s) + d(s, t)$. This can be done in
$O(\log n)$ time. During the binary search for the calculation of
$v_{op}$, $\sum_{i=1}^{\text{depth}_{v_0}(v)} d_{T_{(0)}(p^{i}(v), p^{i-1}(v))} W_{T_{(0)}(p^{i-1}(v))}$
can be calculated in $O(\log n)$ time using (c). Thus, $d_{T_{(0)}(v)}(v)$ can be calculated in $O(\log n)$ time.

Appendix C: Proof of Theorem 5.6

(proof) By Theorem 5.5 after $O(n \log n)$-time $O(n \log n)$-
space preparation, the restricted effect $\delta_{T(v_0;k,e)}(v)$ of any edge $e$ for any vertex $v$ can be calculated in $O(\log n)$ time. So, we show that total number of pairs $(v,e)$ to calculate $T(v_0;k,e)$ for any vertex $v$ can be calculated in $O(\log n)$ time. For each $v \in V \setminus N_k(t_0)$ and each $u \in N_k(t_0) \setminus \{C(v)\}$, $\delta_{T(v_0;k,e)}(v)$ for edges $e$ in $E^u \cup \{e_{v',u} \mid p(v') = v\}$ are calculated. So, the total number of edges is

$$\sum_{u \in V \setminus \{N_k(t_0) \cup \{C(v)\}\}} \sum_{v \in V \setminus \{N_k(t_0) \cup \{C(v)\}\}} (|E^u_v| + |e_{v',u} \mid p(v') = v|)).$$

Since $\sum_{u \in V \setminus \{N_k(t_0) \cup \{C(v)\}\}} |E^u_v|$ is upper bounded by $2m$ and $\sum_{v \in V \setminus \{N_k(t_0) \cup \{C(v)\}\}} |e_{v',u} \mid p(v') = v|$ is upper bounded by $n - k$, the total number of edges is $(k - 1)(n - k) + 2m = O(kn + m)$.

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