**LETTER**

Finding the Minimum Number of Open-Edge Guards in an Orthogonal Polygon is NP-Hard

Chuzo IWAMOTO

SUMMARY We study the problem of determining the minimum number of open-edge guards which guard the interior of a given orthogonal polygon with holes. Here, an open-edge guard is a guard which is allowed to be placed along open edges of a polygon, that is, the endpoints of the edge are not taken into account for visibility purpose. It is shown that finding the minimum number of open-edge guards for a given orthogonal polygon with holes is NP-hard.

key words: open-edge guards, orthogonal polygons, art gallery problem

1. Introduction

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We study the problem of determining the minimum number of open-edge guards for a given orthogonal polygon with holes. Here, an open-edge guard is a guard which is allowed to be placed along open edges of a polygon, that is, the endpoints of the edge are not taken into account for visibility purpose. It is shown that finding the minimum number of open-edge guards for a given orthogonal polygon with holes is NP-hard.

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2. Definitions and Results

The definitions of a polygon and a polygon with holes are as follows.

A **polygon** is defined by a finite set of segments such that every segment is extreme and shared by exactly two edges and no subset of edges has the same property. The segments are the edges and their endpoints are the vertices of the polygon. If each edge of a polygon is perpendicular to one of the coordinate axes, then the polygon is called **orthogonal**. Thus, an orthogonal polygon is a polygon all of whose edge intersections are at right angles. An orthogonal polygon is sometimes called a **rectilinear polygon**.

A **polygon with holes** is a polygon P enclosing several other polygons $H_1, H_2, \ldots, H_t$, the holes. None of the boundaries of $P$, $H_1, H_2, \ldots, H_t$ may intersect, and each of the holes is empty. P is said to bound a *multiply-connected region* with t holes: the region of the plane interior to or on the boundary of $P$, but exterior to or on the boundary of $H_1, H_2, \ldots, H_t$. (Such a region is called a "polygon P with holes" or simply "polygon P" in the rest of this paper.) Similarly, we define an **orthogonal polygon with holes** to be an orthogonal polygon with orthogonal holes, with all edges aligned with the same pair of orthogonal axes (see Fig. 1(a)).

Given a polygon P, two points $x$ and $y$ are said to be visible if the line segment $xy$ does not contain any points outside the polygon P. A point $x$ in the polygon P is said to be visible from an edge if there exists an interior point $y$ of the edge such that $x$ and $y$ are visible. Here, an interior point of an edge is a point which is on the edge but is not on the
endpoints of the edge. A set of edges is said to guard the polygon $P$ if every point in $P$ is visible from at least one of these edges.

An instance of the open-edge guarding problem in this paper is an orthogonal polygon with holes and a positive integer $g$. The problem asks whether there exists a set of $g$ edges which guards the region of the orthogonal polygon with holes.

**Theorem 1:** The open-edge guarding problem for orthogonal polygons with holes is $NP$-hard.

### 3. Proof of Theorem 1

In Sect. 3.2, we will show a polynomial-time transformation from an arbitrary instance $C$ of the 3SAT problem to a polygon with holes and integer $g$ such that $C$ is satisfiable if and only if there exists a set of $g$ edges which guards the region of the polygon with holes.

#### 3.1 3SAT Problem

The definition of 3SAT is mostly from [LO1] of [3]. Let $U = \{x_1, x_2, \ldots, x_n\}$ be a set of Boolean variables. Boolean variables take on values 0 (false) and 1 (true). If $x$ is a variable in $U$, then $x$ and $\overline{x}$ are literals over $U$. The value of $\overline{x}$ is 1 (true) if and only if $x$ is 0 (false). A clause over $U$ is a set of literals over $U$, such as $\{\overline{x}_1, x_3, x_4\}$. It represents the disjunction of those literals and is satisfied by a truth assignment if and only if at least one of its members is true under that assignment.

An instance of 3SAT is a collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over $U$ such that $|c_j| \leq 3$ for each $c_j \in C$. The 3SAT problem asks whether there exists some truth assignment for $U$ that simultaneously satisfies all the clauses in $C$. (An example of $C$ is given in the caption of Fig. 8.) It is known that the 3SAT problem is $NP$-complete even if each variable occurs exactly once positively and exactly twice negatively in $C$ [1].

#### 3.2 Transformation from 3SAT-Instance to Polygon

Figure 2 is an outline drawing of orthogonal polygon $P$ of width $3w$. Each of the left and right parts of width $w$ has a comb-like structure. The top side of the center part of width $w$ has $m$ projections labeled with $c_1, c_2, \ldots, c_m$. (See also the detailed drawing of $c_1$ and $c_j$ in Fig. 3.)

Each comb has $2m$ teeth, and thick and thin teeth appear alternately. One can see that the red area of Fig. 2 can be guarded by $2m$ edges, indicated by red nodes of thick teeth. The width $w$ of the comb-like part is sufficiently large so that (i) at least $2m$ edges are required for guarding thin teeth, and (ii) those $2m$ edges cannot guard any projection $c_j$.

Let $d$ be the thickness of thin teeth (see Fig. 2). In the figure, the thickness of thick teeth is $4d$, the depth of each projection is $d$, and the width of the comb-like part is $w = 32d$. If the value of $d$ is determined so that the width $p$ of projections $c_1$ and $c_m$ is less than the value of $4d$, then conditions (i) and (ii) are satisfied.

Figure 3 is a gadget for variable $x_h$, which appears in clause $c_i$ positively and in $c_j$ and $c_k$ negatively in $C$. The detailed drawing of the variable gadget for $x_h$ is shown in Figs. 4, 5, and 6.

In order to see the red area of Fig. 6, at least one guard must be placed on an edge which touches that red area (see the edge indicated by blue node $s$ of Fig. 4). By the same reason, at least nine guards must be placed in the variable gadget $x_h$ in Fig. 4 (see the nine blue edges indicated by blue nodes in that figure). Those nine edges see the left and right rectangular halls $A$ and $B$ excepting two small white triangular areas near the green nodes $a$ and $b$.

The red area of Fig. 4 is a connecting corridor between two halls $A$ and $B$. It is not difficult to show that at least two edges touching the red area are required to guard all points of the connecting corridor. If there is a guard on the edge indicated by green node $a$ (resp. green circle $b$) of Fig. 4, then small white triangular area near $b$ (resp. $a$) is guarded by one of the two green edges in the corridor indicated by $c_i$.

![Fig. 2](image-url)  An outline drawing of orthogonal polygon $P$ of width $3w$. The center part of width $w$ has $m$ projections labeled with $c_1, c_2, \ldots, c_m$. 


two green nodes (resp. two green circles). Three edges indicated by green nodes (resp. green circles) correspond to the assignment \( x_h = 1 \) (resp. \( x_h = 1 \)).

In Fig. 3, there are four pairs of narrow rectangular portions of size \( l_1 \times d_1 \) and \( 2l_1 \times d_1 \). Those portions can be guarded by four edges (see the four red nodes of rectangular portions of size \( 2l_1 \times d_1 \)). Here, we assume that width \( d_1 \) is sufficiently small as compared with length \( l_1 \) so that the
value $d_1 + 2\delta$ in Fig. 3 is almost equivalent to $d_1$. Namely, for an arbitrary small constant $\delta > 0$, the value $d_1$ is determined so that \( \frac{d_1}{2l_1} \leq \frac{\delta}{8dm} \) holds (i.e., $d_1 \leq \frac{\delta}{8dm}$), where $8dm$ is the height of the red area of Figs. 2 and 3. One can see four edges are sufficient and necessary to guard the four pairs of rectangular portions.

Similar rectangular portions of size $l_2 \times d_2$ and $2l_2 \times d_2$ can be found in Fig. 4; four edges are sufficient and necessary for guarding such portions (see the four edges indicated by four red nodes of Fig. 4). Here, width $d_2$ is sufficiently small as compared with $l_2$. (The exact value of $d_2$ can be computed by a manner similar to the previous paragraph.)

Consider a “hammer-shaped” gadget labeled with $v_h$ of Fig. 3. This gadget is for guarding the grey area $V$ of Fig. 4. Obviously, one blue edge is sufficient and necessary for guarding both grey area $V$ and the inside of the hammer.

Consider projections labeled with $c_i$, $j$, and $k$ of Figs. 3 and 4. Edges $a$ and $b$ of Fig. 4 can see edges labeled with $c_i$, $j$, and $k$, respectively. (Recall that variable $x_h$ appears in clauses $j$ and $k$ negatively and in $i$ positively.)

The position of blue dotted segment $c_i$ of Fig. 4 can be controlled by moving all the 26 vertices placed on the right side of dotted line $r$, to the right or to the left. The length of the blue dotted segment $c_i$ can be controlled by changing the gap $d_3$. The position and length of segments $j$ and $k$ can be controlled similarly. If the movement of those 26 vertices causes a collision with a $2l_1 \times d_1$ rectangular portion in Fig. 3, then we reconstruct a variable gadget so that halls $A$ and $B$ have a free space for the movement as shown in Fig. 7. Figure 7 (a) is a simplified illustration of hall $B$ of Fig. 4. We reconstruct the variable gadget so that the size of halls is reduced as shown in Fig. 7 (b). Note that two green visibility lines in Fig. 7 (a) are coincident with those in Fig. 7 (b). Furthermore, in Fig. 7 (a), a hammer-shaped portion (illustrated by dotted lines) can be replaced with smaller one such that both of them have the same visibility line.

Figure 8 is an orthogonal polygon $P$ (with holes) transformed from a 3SAT-instance $C$, where $C$ is given in the caption of Fig. 8. In this figure, the detailed drawing of Fig. 4 is illustrated as a simplified figure of Fig. 9. Figure 10 is the variable gadget for $x_3$. The variable gadget for $x_4$ is a mirror image of this figure. From Fig. 8, one can see that there is a truth assignment $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfying all clauses. (In this figure, the detailed drawing of Fig. 4 is illustrated as a simplified figure of Fig. 9.)
Fig. 10 Variable gadget for $x_3$ in Fig. 8. Variable gadget for $x_4$ is a mirror image of this figure.

are $2m$ red nodes. Hence, let $g = 2m + (10 + 4 + 3)n + 2m = 4m + 17n$. From this construction of polygon $P$ with holes, one can see that $C$ is satisfiable if and only if there exists a set of $g$ edges which guards polygon $P$ with holes.

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References


