The Panpositionable Pancyclicity of Locally Twisted Cubes

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SUMMARY In a multiprocessor system, processors are connected based on various types of network topologies. A network topology is usually represented by a graph. Let G be a graph and u, v be any two distinct vertices of G . We say that G is pancyclic if G has a cycle C of every length $l(C)$ satisfying $3 \leq l(C) \leq |V(G)|$, where $|V(G)|$ denotes the total number of vertices in G . Moreover, G is panpositionably pancyclic from r if for any integer m satisfying $r \leq m \leq \frac{|V(G)|}{2}$, G has a cycle C containing u and v such that $d_C(u, v) = m$ and $2m \leq l(C) \leq |V(G)|$, where $d_C(u, v)$ denotes the distance of u and v in C . In this paper, we investigate the panpositionable pancyclicity problem with respect to the n -dimensional locally twisted cube LTQ_n , which is a popular topology derived from the hypercube. Let $D(LTQ_n)$ denote the diameter of LTQ_n . We show that for $n \geq 4$ and for any integer m satisfying $D(LTQ_n) + 2 \leq m \leq \frac{|V(LTQ_n)|}{2}$, there exists a cycle C of LTQ_n such that $d_C(u, v) = m$, where (i) $2m + 1 \leq l(C) \leq |V(LTQ_n)|$ if $m = D(LTQ_n) + 2$ and n is odd, and (ii) $2m \leq l(C) \leq |V(LTQ_n)|$ otherwise. This improves on the recent result that u and v can be positioned with a given distance on C only under the condition that $l(C) = |V(LTQ_n)|$. In parallel and distributed computing, if cycles of different lengths can be embedded, we can adjust the number of simulated processors and increase the flexibility of demand. This paper demonstrates that in LTQ_n , the cycle embedding containing any two distinct vertices with a feasible distance is extremely flexible.

key words: cycle embedding, pancyclicity, locally twisted cube, interconnection network

1. Introduction

In a multiprocessor system, processors are connected based on various types of network topologies. A network topology is usually represented by a graph, in which the vertices and edges of the graph represent the processors and communication links of the multiprocessor system. In this paper, we use an undirected graph to represent a network topology, and we follow the graph-theory definitions and notations given by Golumbic [6].

Let $G = (V, E)$ be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. We use $|V(G)|$ to denote the number of vertices in G , and we use uv to denote the edge incident with vertices u and v . Two vertices u and v of G are adjacent if $uv \in E(G)$. A path P of G is a sequence of distinct vertices such that any two consecutive vertices of P are adjacent in G . The length of P , denoted by $l(P)$, is the number of edges in P . For a path P of length k , $k \geq 1$, we write

$P = [v_1, v_2, \dots, v_k, v_{k+1}]$. The i th vertex of P is denoted by $P(i)$. If $P^* = [v_i, \dots, v_j]$, $i \leq j$, is a segment of P , we can also write P as $[v_1, v_2, \dots, v_i, P^*, v_j, \dots, v_{k+1}]$. A cycle is a close path with at least three vertices, where the last vertex is adjacent to the first one.

The distance between vertices u and v in G , denoted by $d_G(u, v)$, is the length of the shortest path between u and v . The diameter of G , denoted by $D(G)$, is the maximum of all distances for all pairs of vertices in G . A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, H is a spanning subgraph of G , or we can say that H spans G . A Hamiltonian path (respectively, Hamiltonian cycle) of G is a path (respectively, cycle) which spans G . A graph G is Hamiltonian if it has a Hamiltonian cycle, and G is Hamiltonian connected if it has a Hamiltonian path joining any pair of vertices of G .

Let u and v be any two distinct vertices of G . We say that G is pancyclic if it has a cycle of every length m satisfying $3 \leq m \leq |V(G)|$ [2]. More specifically, G is vertex-pancyclic (respectively, edge-pancyclic) if its any vertex (respectively, edge) lies on a cycle of every length m , $3 \leq m \leq |V(G)|$. On the other hand, G is panconnected if it has a path of every length m joining u and v with $d_G(u, v) \leq m \leq |V(G)| - 1$ for any two vertices u and v of G [1]. If there exists a Hamiltonian cycle C of G such that $d_C(u, v) = m$ for any integer m with $d_G(u, v) \leq m \leq \frac{|V(G)|}{2}$, then G is panpositionably Hamiltonian [14]. From the above definitions, we can further define that a graph G is panpositionably pancyclic if it has a cycle C containing u and v such that (i) $d_C(u, v) = m$ for any integer m with $d_G(u, v) \leq m \leq \frac{|V(G)|}{2}$, and (ii) $2m \leq l(C) \leq |V(G)|$ for every $l(C)$; i.e., vertices u and v can be positioned on various cycles C of feasible lengths in G such that u and v keep any given distance from each other on C . In this paper, we will investigate the panpositionable pancyclicity problem with respect to locally twisted cubes.

The locally twisted cube was introduced by Yang *et al.* [16]. It can be derived from the hypercube by changing half part of links. An n -dimensional locally twisted cube, denoted by LTQ_n , has 2^n vertices, and each vertex corresponds to an n -bit binary string. The locally twisted cube is recursively constructed as follows:

- (i) LTQ_2 is a graph with four vertices $w = 00$, $x = 01$, $y = 10$, $z = 11$, and the edge set of LTQ_2 is $\{wx, wy, xz, yz\}$.
- (ii) For $n \geq 3$, let LTQ_{n-1}^0 and LTQ_{n-1}^1 be two copies of LTQ_{n-1} with $V(LTQ_{n-1}^0) = \{0u_{n-1}u_{n-2} \dots u_1 | u_i =$

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Fig. 1 LTQ_3 and LTQ_4 .

0 or 1 for $1 \leq i \leq n-1$ and $V(LTQ_{n-1}^1) = \{1u_{n-1}u_{n-2} \dots u_1 | u_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq n-1\}$. Then, LTQ_n is formed by connecting LTQ_{n-1}^0 and LTQ_{n-1}^1 with 2^{n-1} edges such that a vertex $u = 0u_{n-1}u_{n-2} \dots u_1$ in LTQ_{n-1}^0 is adjacent to a vertex $v = 1v_{n-1}v_{n-2} \dots v_1$ in LTQ_{n-1}^1 if and only if $v_{n-1} = (u_{n-1} + u_1) \bmod 2$ and $v_i = u_i$ for every $i, 1 \leq i \leq n-2$.

Figure 1 shows the LTQ_3 and LTQ_4 . It is proven that the diameter $D(LTQ_n)$ is $n-1$ if $n \in \{3, 4\}$ and $D(LTQ_n)$ is $\lceil \frac{n+3}{2} \rceil$ if $n \geq 5$, which is about half that of the hypercube [16].

According to its definition and structure, a locally twisted cube has no cycle of length 3. That is, for any edge uv of LTQ_n , there exists no path of length 2 joining u and v in LTQ_n . Therefore, locally twisted cubes are not fully pancyclic or panconnected. This property also can be found in several hypercube variants. Many researchers then proposed loosed versions of pancyclicity and panconnectedness for various cubes [3], [5], [10], [12]. Generally, a graph G is defined to be r -pancyclic if it has a cycle of every length m satisfying $r \leq m \leq |V(G)|$, and G is defined to be r -panconnected if it has a path of every length m , $r \leq m \leq |V(G)| - 1$, joining u and v for any two distinct vertices u and v of G .

In [17], Yang *et al.* showed that locally twisted cubes are 4-pancyclic and Hamiltonian connected. Hu *et al.* further showed that locally twisted cubes are 4-node-pancyclic and 4-edge-pancyclic [7]. In [12], Ma and Xu proposed the panconnectedness of locally twisted cubes, where for any two distinct vertices u and v of LTQ_n , $n \geq 3$, there exists a path of every length m joining u and v with $d_{LTQ_n}(u, v) \leq m \leq 2^n - 1$ and $m \neq d_{LTQ_n}(u, v) + 1$. Kung and Chen in [9] combined panconnectedness with Hamiltonian connectedness to show that the n -dimensional locally twisted cube is Hamiltonian n -panconnected for $n \geq 5$. More precisely, for any three distinct vertices x , y , and z of LTQ_n , there exists a Hamiltonian path P of LTQ_n such that $P(1) = x$, $P(m+1) = y$, and $P(2^n) = z$ for any integer m satisfying $n \leq m \leq 2^n - n - 1$. In [8], Kung discussed the panpositionable Hamiltonicity of locally twisted cubes. He showed that for any two distinct vertices u and v of LTQ_n , $n \geq 4$, and for any integer m with $d_{LTQ_n}(u, v) \leq m \leq 2^{n-1}$ and $m \neq d_{LTQ_n}(u, v) + 1$, there exists a Hamiltonian cycle C of LTQ_n such that $d_C(u, v) = m$. In this paper, we will improve the above result of panpositionable Hamiltonicity so that the

cycle C of LTQ_n can be of various feasible lengths instead of a Hamiltonian cycle.

The rest of this paper is organized as follows. In Sect. 2, we introduce some properties of locally twisted cubes. In Sect. 3, we propose the main result of this paper and prove its correctness. Finally, some concluding remarks are given in Sect. 4.

2. Preliminaries

Let $u = u_n u_{n-1} \dots u_1$ and $v = v_n v_{n-1} \dots v_1$ be any two distinct vertices of LTQ_n . Then, u and v are adjacent along the i th dimension, $1 \leq i \leq n$, if and only if one of the following conditions is satisfied [17]:

- (i) For $i \in \{1, 2\}$, vertices u and v differ only with $u_i \neq v_i$.
- (ii) For $3 \leq i \leq n$, vertices u and v are with (a) $u_i \neq v_i$, (b) $u_{i-1} = (v_{i-1} + u_1) \bmod 2$, and (c) $u_j = v_j$ for all $j \in \{1, 2, \dots, n\} - \{i-1, i\}$.

For convenience, we say that v is the i -neighbor of u , and it is denoted by $v = (u)^i$. We also call that uv is an i -dimensional edge. The following lemmas describe some properties of cycle embedding and path embedding of locally twisted cubes.

Lemma 1: [9] Let uv be any n -dimensional edge of LTQ_n , $n \geq 3$. Then, vertices u , v , $(u)^i$, and $(v)^i$ form a cycle of length 4 if $2 \leq i \leq n-1$.

Lemma 2: [8] Let u and v be any two distinct vertices of LTQ_n , $n \geq 2$, where u is in LTQ_{n-1}^0 and v is in LTQ_{n-1}^1 . Then, $d_{LTQ_n}(u, (v)^n) = d_{LTQ_n}(u, v) - 1$ or $d_{LTQ_n}((u)^n, v) = d_{LTQ_n}(u, v) - 1$.

Lemma 3: [12] Let u and v be any two distinct vertices of LTQ_n , $n \geq 3$. For any integer m satisfying $d_{LTQ_n}(u, v) \leq m \leq 2^n - 1$ and $m \neq d_{LTQ_n}(u, v) + 1$, there exists a path P of LTQ_n joining u and v such that $l(P) = m$.

Lemma 4: [8] Let u and v be any two distinct vertices of LTQ_n , $n \geq 4$. For any integer m satisfying $d_{LTQ_n}(u, v) \leq m \leq 2^n - 1$ and $m \neq d_{LTQ_n}(u, v) + 1$, there exists a Hamiltonian path P of LTQ_n such that $P(1) = u$ and $P(m+1) = v$; i.e., $d_P(u, v) = m$.

Let G be a graph and F be a set of vertices and/or edges of G . We use $G-F$ to denote the resulting graph after removing the elements of F from G . Then, a Hamiltonian graph G is f -fault-tolerant Hamiltonian (or f -Hamiltonian) if $G-F$ remains Hamiltonian for every F with $|F| \leq f$. Similarly, a Hamiltonian connected graph G is f -fault-tolerant Hamiltonian connected (or f -Hamiltonian connected) if $G-F$ remains Hamiltonian connected for every F with $|F| \leq f$. The following lemmas give some properties of fault-tolerance for locally twisted cubes.

Lemma 5: [13] Locally twisted cubes LTQ_n , $n \geq 3$, are $(n-2)$ -Hamiltonian and $(n-3)$ -Hamiltonian connected.

Lemma 6: [15] Let F be a set of faulty vertices and/or edges in LTQ_n , where $n \geq 3$ and $|F| \leq n-3$. Moreover, let uv be any edge of $LTQ_n - F$. Then, there exists a path

Fig. 2 Illustration of the proof of Lemma 7. (A straight line represents an edge, and a curve represents a path.)

P of $LTQ_n - F$ joining u and v such that $l(P) = m$ for every integer m satisfying $5 \leq m \leq 2^n - |F| - 1$.

Lemma 7: Let w be any vertex of LTQ_n , $n \geq 4$. Moreover, let u and v be any two distinct vertices of $LTQ_n - \{w\}$. Then, there exists a path P of $LTQ_n - \{w\}$ joining u and v such that $l(P) = m$ for every integer m satisfying $\lceil \frac{n+3}{2} \rceil + 3 \leq m \leq 2^n - 2$.

Proof. We prove this lemma by induction. The correctness of the induction base LTQ_4 can be verified by our computer program [4]. Assume, as the inductive hypothesis, that this lemma holds for any LTQ_k , $4 \leq k \leq n-1$. We will show that this lemma also holds for LTQ_n . Without loss of generality, assume w is in LTQ_{n-1}^0 . Moreover, assume u is in LTQ_{n-1}^0 if u and v are respectively in different subcubes LTQ_{n-1}^i and LTQ_{n-1}^{1-i} of LTQ_n , $i \in \{0, 1\}$. According to the possibilities of u and v , we distinguish the following cases.

Case 1. u and v are in LTQ_{n-1}^0 . In this case, we also need to consider the following subcases.

Subcase 1.1. $\lceil \frac{n+3}{2} \rceil + 3 \leq m \leq 2^{n-1} - 2$. By the inductive hypothesis, there exists a path R of $LTQ_{n-1}^0 - \{w\}$ joining u and v with $\lceil \frac{(n-1)+3}{2} \rceil + 3 \leq l(R) \leq 2^{n-1} - 2$. Certainly, this path is a path of $LTQ_n - \{w\}$ and it also satisfies $\lceil \frac{n+3}{2} \rceil + 3 \leq l(R) \leq 2^{n-1} - 2$. We let $P = R$, and P is our required path.

Subcase 1.2. $2^{n-1} - 1 \leq m \leq 2^n - 2$. By the inductive hypothesis, $LTQ_{n-1}^0 - \{w\}$ has a path R joining u and v with $2^{n-2} - 1 \leq l(R) \leq 2^{n-1} - 2$. We write R as $[u, R_1, p, q, R_2, v]$ for some adjacent vertices p and q . By Lemma 3, LTQ_{n-1}^1 has a path S joining $(p)^n$ and $(q)^n$ with $2^{n-2} - 1 \leq l(S) \leq 2^{n-1} - 1$. We set $P = [u, R_1, p, (p)^n, S, (q)^n, q, R_2, v]$, then P is a path of $LTQ_n - \{w\}$ joining u and v with $l(P) = l(R) + l(S) + 1$. It is easy to obtain that $2^{n-1} - 1 \leq l(P) \leq 2^n - 2$, and $l(P) = m$. See Fig. 2 (a) for illustration.

Case 2. u and v are in LTQ_{n-1}^1 . In this case, two subcases are distinguished.

Subcase 2.1. $\lceil \frac{n+3}{2} \rceil + 3 \leq m \leq 2^{n-1} - 1$. By Lemma 3, we have a path S of LTQ_{n-1}^1 joining u and v with $\lceil \frac{(n-1)+3}{2} \rceil + 2 \leq l(S) \leq 2^{n-1} - 1$, which also satisfies $\lceil \frac{n+3}{2} \rceil + 3 \leq l(S) \leq 2^{n-1} - 1$. We can let $P = S$.

Subcase 2.2. $2^{n-1} \leq m \leq 2^n - 2$. In LTQ_{n-1}^1 , by Lemma 3, there exists a path S joining u and v with $2^{n-2} - 1 \leq l(S) \leq 2^{n-1} - 1$. We can write S as $[u, S_1, p, q, S_2, v]$ for some adjacent vertices p and q with $(w)^n \notin \{p, q\}$. By the inductive hypothesis, $LTQ_{n-1}^0 - \{w\}$ has a path R join-

ing $(p)^n$ and $(q)^n$ with $2^{n-2} \leq l(R) \leq 2^{n-1} - 2$. We set $P = [u, S_1, p, (p)^n, R, (q)^n, q, S_2, v]$, and $l(P) = l(R) + l(S) + 1$. Then, we have that $2^{n-1} \leq l(P) \leq 2^n - 2$, and $l(P) = m$.

Case 3. u is in LTQ_{n-1}^0 and v is in LTQ_{n-1}^1 and uv is an edge of LTQ_n . By Lemma 1, we can find a cycle of length 4 that contains u and v and does not contain w . We write this cycle as $[u, p, (p)^n, v, u]$. By the inductive hypothesis, there exists a path R in $LTQ_{n-1}^0 - \{w\}$ joining u and p with $l(R) \in \{1, \lceil \frac{n+2}{2} \rceil + 3, \lceil \frac{n+2}{2} \rceil + 4, \dots, 2^{n-1} - 2\}$. In LTQ_{n-1}^1 , Lemma 4 ensures that there exists a Hamiltonian path S such that $S(1) = (p)^n$ and $d_S((p)^n, v) \in \{1, 3, 4, \dots, 2^{n-1} - 1\}$. We can write $S = [(p)^n, S_1, v, S_2, z]$ for some vertex z of LTQ_{n-1}^1 . Note that $v = z$ if $l(S_2) = 0$. We set $P = [u, R, p, (p)^n, S_1, v]$, and $l(P) = l(R) + l(S_1) + 1$. It is obvious that $l(P) = 3$ or $5 \leq l(P) \leq 2^n - 2$. Since $\lceil \frac{n+3}{2} \rceil + 3 > 5$ for $n \geq 5$, $l(P)$ satisfies $\lceil \frac{n+3}{2} \rceil + 3 \leq l(P) \leq 2^n - 2$, and P can be our required path. See Fig. 2 (b) for illustration.

Case 4. u is in LTQ_{n-1}^0 and v is in LTQ_{n-1}^1 and u is not adjacent to v . In this case, we need to consider the following two subcases.

Subcase 4.1. $\lceil \frac{n+3}{2} \rceil + 3 \leq m \leq 2^{n-1}$. By Lemma 3, LTQ_{n-1}^1 has a path S joining $(u)^n$ and v with $\lceil \frac{n+2}{2} \rceil + 2 \leq l(S) \leq 2^{n-1} - 1$. We set $P = [u, (u)^n, S, v]$, and $l(P) = l(S) + 1$, which satisfies $\lceil \frac{n+3}{2} \rceil + 3 \leq l(P) \leq 2^{n-1}$. Thus, P can be our required path.

Subcase 4.2. $2^{n-1} + 1 \leq m \leq 2^n - 2$. Let p be a vertex of $LTQ_{n-1}^0 - \{w\}$ with $(p)^n \neq v$. By the inductive hypothesis, $LTQ_{n-1}^0 - \{w\}$ has a path R joining u and p with $2^{n-2} \leq l(R) \leq 2^{n-1} - 2$. By Lemma 3, LTQ_{n-1}^1 has a path S joining $(p)^n$ and v with $2^{n-2} \leq l(S) \leq 2^{n-1} - 1$. We set $P = [u, R, p, (p)^n, S, v]$, and $l(P) = l(R) + l(S) + 1$. Then, we have that $2^{n-1} + 1 \leq l(P) \leq 2^n - 2$, and $l(P) = m$. See Fig. 2 (c) for illustration.

The above argument completes the proof. $\square$

3. The Panpositionable Pancyclicity Problem

In this section, we propose the panpositionable pancyclicity problem of locally twisted cubes LTQ_n for $n \geq 4$. Let u and v be any two distinct vertices of LTQ_n . Lemma 3 ensures that there exists a path of LTQ_n joining u and v with every length from $d_{LTQ_n}(u, v) + 2$ to $2^n - 1$. Since the diameter of LTQ_n is $n - 1$ for $n = 4$ and $\lceil \frac{n+3}{2} \rceil$ for $n \geq 5$, we discuss this problem based on the assumption that u and v are contained in cycles C of various feasible lengths in LTQ_n and

$d_C(u, v)$ satisfies any integer from $D(LTQ_n) + 2$ to 2^{n-1} . The following theorems describe the main result of this paper.

Theorem 1: Let u and v be any two distinct vertices of LTQ_4 . Then, for any integer m satisfying $D(LTQ_4) + 2 \leq m \leq 8$, there exists a cycle C of LTQ_4 containing u and v such that $d_C(u, v) = m$ and $2m \leq l(C) \leq 16$.

Proof. Since $D(LTQ_4) = 3$, we have that $5 \leq m \leq 8$. The correctness of this theorem is verified by our computer program [4]. $\square$

Theorem 2: Let u and v be any two distinct vertices of LTQ_n , $n \geq 5$. Then, for any integer m satisfying $D(LTQ_n) + 2 \leq m \leq 2^{n-1}$, there exists a cycle C of LTQ_n containing u and v such that $d_C(u, v) = m$, where (i) $2m + 1 \leq l(C) \leq 2^n$ if $m = D(LTQ_n) + 2$ and n is odd, and (ii) $2m \leq l(C) \leq 2^n$ otherwise.

Proof. Since $n \geq 5$, we have that $D(LTQ_n) = \lceil \frac{n+3}{2} \rceil$. We prove this theorem by induction. By our computer program [4], the correctness of the induction base LTQ_5 is verified. Assuming that this theorem is true for any LTQ_k , $5 \leq k \leq n-1$, we show that this theorem is also true for LTQ_n . Without loss of generality, assume u is in LTQ_{n-1}^0 . Consider the following cases.

Case 1. v is in LTQ_{n-1}^1 and u is not adjacent to v . By Lemma 2, we have that $d_{LTQ_n}(u, (v)^n) = d_{LTQ_n}(u, v) - 1$ or $d_{LTQ_n}((u)^n, v) = d_{LTQ_n}(u, v) - 1$. Without loss of generality, assume $d_{LTQ_n}((u)^n, v) = d_{LTQ_n}(u, v) - 1$. Since $D(LTQ_n) = \lceil \frac{n+3}{2} \rceil$, $d_{LTQ_n}((u)^n, v) \leq \lceil \frac{n+3}{2} \rceil - 1$. In LTQ_{n-1}^1 , by Lemma 4, there exists a Hamiltonian path S such that $S(1) = (u)^n$ and $d_S((u)^n, v) = m - 1$, where $\lceil \frac{n+3}{2} \rceil + 2 \leq m \leq 2^{n-1}$. We can write S as $[(u)^n, S_1, v, S_2, z]$ for some vertex z with $l(S_1) = m - 1$, and S_2 can be written as $[v, S_2^1, x, S_2^2, z]$ for some vertex x . Note that $x = v$ if $l(S_2^1) = 0$, $x = z$ if $l(S_2^1) = 2^{n-1} - m$, and $x = v = z$ if both $l(S_1) = 2^{n-1} - 1$ and $l(S_2^1) = 0$. For LTQ_{n-1}^0 , distinguish the following subcases.

Subcase 1.1. n is even. Then, $\lceil \frac{(n-1)+3}{2} \rceil + 2 = \lceil \frac{n+3}{2} \rceil + 1$. By Lemma 3, LTQ_{n-1}^0 has a path R joining u and $(x)^n$ with $m - 1 \leq l(R) \leq 2^{n-1} - 1$. We set $C = [u, (u)^n, S_1, v, S_2^1, x, (x)^n, R, u]$, and $l(C) = l(R) + l(S_1) + l(S_2^1) + 2$. It is easy to obtain that $d_C(u, v) = d_S((u)^n, v) + 1 = m$ and $2m \leq l(C) \leq 2^n$. See Fig. 3 for illustration.

Subcase 1.2. n is odd. Then, $\lceil \frac{(n-1)+3}{2} \rceil + 2 = \lceil \frac{n+3}{2} \rceil + 2$. Consider the following conditions.

Condition 1.2.1. $m = \lceil \frac{n+3}{2} \rceil + 2$. In LTQ_{n-1}^0 , Lemma 3 ensures that there exists a path R joining u and $(x)^n$ with $m \leq l(R) \leq 2^{n-1} - 1$. By setting $C = [u, (u)^n, S_1, v, S_2^1, x, (x)^n, R, u]$, we have that $l(C) = l(R) + l(S_1) + l(S_2^1) + 2$. Then, $d_C(u, v) = m$ and $2m + 1 \leq l(C) \leq 2^n$. Figure 3 also illustrates this subcase.

Condition 1.2.2. $\lceil \frac{n+3}{2} \rceil + 3 \leq m \leq 2^{n-1}$. By Lemma 3, there exists a path R of LTQ_{n-1}^0 joining u and $(x)^n$ with $m - 1 \leq l(R) \leq 2^{n-1} - 1$. We set $C = [u, (u)^n, S_1, v, S_2^1, x, (x)^n, R, u]$, then $d_C(u, v) = m$ and $2m \leq l(C) \leq 2^n$.

Case 2. v is in LTQ_{n-1}^1 and uv is an edge of LTQ_n . Consider the following subcases.

Fig. 3 Case 1 in the proof of Theorem 2. (A straight line represents an edge, and a curve represents a path.)

Subcase 2.1. $\lceil \frac{n+3}{2} \rceil + 2 \leq m \leq 2^{n-2}$ and $2m \leq l(C) \leq 2^{n-1}$. Let $x = (u)^2$ and $y = (u)^3$. By Lemma 1, vertices $u, x, (x)^n, v$ form a cycle of length 4, and so do vertices $u, y, (y)^n$, and v . In $LTQ_{n-1}^1 - \{(y)^n\}$, by Lemma 6, there exists a path S joining v and $(x)^n$ with $5 \leq l(S) \leq 2^{n-1} - 2$. Since $\lceil \frac{n+3}{2} \rceil \geq 5$ for $n \geq 6$, path S also satisfies $\lceil \frac{n+3}{2} \rceil \leq l(S) \leq 2^{n-2} - 2$, and we can let $l(S) = m - 2$. Also by Lemma 6, $LTQ_{n-1}^0 - \{x\}$ has a path R joining u and y with $5 \leq l(R) \leq 2^{n-1} - 2$ which also satisfies $m - 2 \leq l(R) \leq 2^{n-1} - m - 2$. We set $C = [u, x, (x)^n, S, v, (y)^n, y, R, u]$, then $d_C(u, v) = l(S) + 2$ and $l(C) = l(R) + l(S) + 4$. Since $l(S) = m - 2$ and $m - 2 \leq l(R) \leq 2^{n-1} - m - 2$, we obtain that $d_C(u, v) = m$ and $2m \leq l(C) \leq 2^{n-1}$ no matter n is odd or even. This result certainly satisfies $2m + 1 \leq l(C) \leq 2^{n-1}$ for $m = \lceil \frac{n+3}{2} \rceil + 2$ with n odd. The illustration of Subcase 2.1 is shown in Fig. 4 (a).

Subcase 2.2. $\lceil \frac{n+3}{2} \rceil + 2 \leq m \leq 2^{n-2}$ and $2^{n-1} + 1 \leq l(C) \leq 2^n$. Let $x = (u)^2$. By Lemma 1, vertices $u, x, (x)^n, v$ form a cycle of length 4. In LTQ_{n-1}^1 , $d_{LTQ_{n-1}^1}((x)^n, v) = 1$, and there exists a Hamiltonian path S by Lemma 4 such that $S(1) = (x)^n$ and $d_S((x)^n, v) = m - 2$. We write S as $[(x)^n, S_1, v, S_2, z]$ for some vertex z , and S_2 can be rewritten as $[v, S_2^1, y, S_2^2, z]$ for some vertex $y \neq v$. Note that $y = z$ if $l(S_2^1) = 2^{n-1} - m + 1$. In LTQ_{n-1}^0 , by Lemma 7, there exists a path R of $LTQ_{n-1}^0 - \{x\}$ joining u and $(y)^n$ with $2^{n-1} - m - 1 \leq l(R) \leq 2^{n-1} - 2$. We set $C = [u, x, (x)^n, S_1, v, S_2^1, y, (y)^n, R, u]$, then $d_C(u, v) = l(S_1) + 2$ and $l(C) = l(R) + l(S_1) + l(S_2^1) + 3$. Since $l(S_1) = m - 2$ and $1 \leq l(S_2^1) \leq 2^{n-1} - m + 1$, we have that $d_C(u, v) = m$ and $2^{n-1} + 1 \leq l(C) \leq 2^n$. The illustration of Subcase 2.2 is shown in Fig. 4 (b).

Subcase 2.3. $2^{n-2} + 1 \leq m \leq 2^{n-1}$ and $2m \leq l(C) \leq 2^n$. As in Subcase 2.2, we can find a Hamiltonian path $S = [(x)^n, S_1, v, S_2^1, y, S_2^2, z]$ in LTQ_{n-1}^1 such that $d_S((x)^n, v) = l(S_1) = m - 2$ and $1 \leq l(S_2^1) \leq 2^{n-1} - m + 1$. In LTQ_{n-1}^0 , by Lemma 7, there exists a path R of $LTQ_{n-1}^0 - \{x\}$ joining u and $(y)^n$ with $m - 2 \leq l(R) \leq 2^{n-1} - 2$. By setting $C = [u, x, (x)^n, S_1, v, S_2^1, y, (y)^n, R, u]$, we have that $d_C(u, v) = l(S_1) + 2$ and $l(C) = l(R) + l(S_1) + l(S_2^1) + 3$. It is obvious that $d_C(u, v) = m$ and $2m \leq l(C) \leq 2^n$. Figure 4 (b) also illustrates this subcase.

Case 3. v is in LTQ_{n-1}^0 . The following subcases are distinguished.

Subcase 3.1. $\lceil \frac{n+3}{2} \rceil + 2 \leq m \leq 2^{n-2}$ and $2m \leq l(C) \leq 2^{n-1}$. If n is odd, then $n - 1$ is even, and $\lceil \frac{(n-1)+3}{2} \rceil + 2 = \lceil \frac{n+3}{2} \rceil + 2$. By the inductive hypothesis, there exists a cycle C of LTQ_{n-1}^0

Fig. 4 Case 2 in the proof of Theorem 2. (A straight line represents an edge, and a curve represents a path.)

Fig. 5 Case 3 in the proof of Theorem 2. (A straight line represents an edge, and a curve represents a path.)

containing u and v such that $d_C(u, v) = m$ and $2m \leq l(C) \leq 2^{n-1}$. If n is even, then $n-1$ is odd, and $\lceil \frac{(n-1)+3}{2} \rceil + 2 = \lceil \frac{n+3}{2} \rceil + 1$. Also by the inductive hypothesis, there exists a cycle C of LTQ_{n-1}^0 containing u and v such that $d_C(u, v) = m$ and $2m \leq l(C) \leq 2^{n-1}$. Thus, we can find our required cycles in LTQ_n . Since $\lceil \frac{n+3}{2} \rceil + 2 \leq m \leq 2^{n-2}$ and $2m \leq l(C) \leq 2^{n-1}$ no matter n is odd or even, this result satisfies $2m + 1 \leq l(C) \leq 2^{n-1}$ for $m = \lceil \frac{n+3}{2} \rceil + 2$ with n odd.

Subcase 3.2. $\lceil \frac{n+3}{2} \rceil + 2 \leq m \leq 2^{n-2}$ and $2^{n-1} + 1 \leq l(C) \leq 2^n$. By Lemma 4, we have a Hamiltonian path R of LTQ_{n-1}^0 such that $R(1) = u$ and $d_R(u, v) = m$. We write R as $[u, R_1, v, R_2, z]$ for some vertex z , where $l(R_1) = m$. For convenience, R_2 can be written as $[v, R_2^1, x, R_2^2, z]$ for some vertex x . Note that $x = v$ if $l(R_2^1) = 0$ and $x = z$ if $l(R_2^1) = 2^{n-1} - m - 1$. In LTQ_{n-1}^1 , by Lemma 3, there exists a path S joining $(u)^n$ and $(x)^n$ with $2^{n-1} - m - 1 \leq l(S) \leq 2^{n-1} - 1$. We set $C = [u, R_1, v, R_2^1, x, (x)^n, S, (u)^n, u]$, and $l(C) = l(R_1) + l(R_2^1) + l(S) + 2$. Since $0 \leq l(R_2^1) \leq 2^{n-1} - m - 1$ and $2^{n-1} - m - 1 \leq l(S) \leq 2^{n-1} - 1$, we can obtain that $2^{n-1} + 1 \leq l(C) \leq 2^n$. Thus, C is our required cycle. Figure 5(a) illustrates this subcase.

Subcase 3.3. $2^{n-2} + 1 \leq m \leq 2^{n-1} - 1$ and $2m \leq l(C) \leq 2^n$. As in Subcase 3.2, we can find a Hamiltonian path $R = [u, R_1, v, R_2^1, x, R_2^2, z]$ of LTQ_{n-1}^0 such that $d_R(u, v) = l(R_1) = m$ and $0 \leq l(R_2^1) \leq 2^{n-1} - m - 1$. In LTQ_{n-1}^1 , Lemma 3 ensures that there exists a path S joining $(x)^n$ and $(u)^n$ with $m - 2 \leq l(S) \leq 2^{n-1} - 1$. We set $C = [u, R_1, v, R_2^1, x, (x)^n, S, (u)^n, u]$, and $l(C) = l(R_1) + l(R_2^1) + l(S) + 2$. Then, we can obtain that $2m \leq l(C) \leq 2^n$. Figure 5(a) also illustrates this subcase.

Subcase 3.4. $m = 2^{n-1}$ and $l(C) = 2^n$. Let x and y be two distinct neighbors of u in $LTQ_{n-1}^0 - \{v\}$. By Lemma 5, there exists a Hamiltonian path R in $LTQ_{n-1}^0 - \{u, y\}$ joining x

and v with $l(R) = 2^{n-1} - 3$. Also by Lemma 5, there exists a Hamiltonian path S in $LTQ_{n-1}^1 - \{(u)^n, (x)^n\}$ joining $(y)^n$ and $(v)^n$ with $l(S) = 2^{n-1} - 3$. We set $C = [u, (u)^n, (x)^n, x, R, v, (v)^n, S, (y)^n, y, u]$, and we obtain that $d_C(u, v) = l(R) + 3 = 2^{n-1} = m$ and $l(C) = l(R) + l(S) + 6 = 2^n$. This subcase is illustrated in Fig. 5(b).

The above completes the proof. $\square$

4. Concluding Remarks

Cycles and paths are two fundamental network topologies for parallel and distributed computing. If cycles (paths) of different lengths can be embedded, we can adjust the number of simulated processors and increase the flexibility of demand [11]. Since the panpositionable pancyclicity problem combines cycle embedding with path embedding of various lengths, this problem supports extremely flexible cycle embedding. In this paper, we demonstrate that locally twisted cubes LTQ_n , $n \geq 4$, are panpositionably pancyclic. Precisely, for any two distinct vertices u and v of LTQ_n and for any integer m with $D(LTQ_n) + 2 \leq m \leq 2^{n-1}$, we can find a cycle C containing u and v such that $d_C(u, v) = m$, and the length $l(C)$ of cycle C satisfies (i) $2m + 1 \leq l(C) \leq 2^n$ if $m = D(LTQ_n) + 2$ and n is odd, and (ii) $2m \leq l(C) \leq 2^n$ otherwise. This result improves on the previous one that locally twisted cubes are panpositionably Hamiltonian, where any two distinct vertices can be positioned with a given distance only on a Hamiltonian cycle. This paper indicates that in locally twisted cubes, the cycle embedding containing any two vertices with a feasible distance is extremely flexible. For a future work, we are motivated to investigate if the panpositionable pancyclicity of locally twisted cubes can be $d_{LTQ_n}(u, v) + 2 \leq m \leq 2^{n-1}$ with $2m \leq l(C) \leq 2^n$ instead of $D(LTQ_n) + 2 \leq m \leq 2^{n-1}$.

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