Image Denoising Using Block-Rotation-Based SVD Filtering in Wavelet Domain

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SUMMARY This paper proposes an image denoising method using singular value decomposition (SVD) with block-rotation-based operations in wavelet domain. First, we decompose a noisy image to some sub-blocks, and use the single-level discrete 2-D wavelet transform to decompose each sub-block into the low-frequency image part and the high-frequency parts. Then, we use SVD and rotation-based SVD with the rank-1 approximation to filter the noise of the different high-frequency parts, and get the denoised sub-blocks. Finally, we reconstruct the sub-block from the low-frequency part and the filtered the high-frequency parts by the inverse wavelet transform, and reorganize each denoised sub-blocks to obtain the final denoised image. Experiments show the effectiveness of this method, compared with relevant methods.

key words: image denoising, threshold denoising, singular value decomposition, peak signal-to-noise ratio, structural similarity index

1. Introduction

The classical image denoising methods include the spatial domain algorithms and the transform domain algorithms [1]. There are many spatial domain methods, such as median filtering and Wiener filtering. Transform domain methods include Fourier transform and wavelet transform [2]–[8].

Because of the multi-resolution properties, once an image is decomposed by wavelet transform, the noise of the image will mainly retain in the high-frequency parts, a common approach to denoising is to set the threshold with wavelet coefficients.

Singular value decomposition (SVD) can be used for noise filtering and has been widely used in image denoising and feature extraction, with good numerical robustness [9]. A denoised image can be reconstructed through the larger singular values and their corresponding eigenvectors, while the high-frequency parts of an image need to rotate to horizontal or vertical orientation with directional features of SVD. Recently, there are some image denoising methods based on the SVD, for example, block-based SVD (BSVD), SVD in wavelet domain [10], and K-means and SVD (K-SVD) [11]. A K-SVD-based image denoising method is proposed [12], which to find a few atoms with the best linear combination from the dictionary to represent each sub-blocks. This method is robust as well as computationally expensive. A denoising algorithm using the adaptive SVD-based (ASVD) is proposed [13]. A patch-based weighted-SVD denoising method proposed [14] with preserving important features. In [15], an image denoising method is proposed using the higher order SVD (HOSVD). A denoising algorithm using SVD in wavelet domain is proposed [16], which enhance the directional features. A denoising algorithm using the low-rank approximation (LRA) and the nonlocal self-similarity is proposed [17]. A new wavelet threshold determination method considering inter-scale correlation in signal denoising is proposed [18]. A denoising method for blind image using SVD and local pixel grouping is proposed [19].

In BSVD and Wavelet transform filtering, the directional features in sub-blocks and the local stationarity of whole image are not considered. In this paper, a denoising algorithm via block-rotation-based SVD (BRSVD) in wavelet domain is proposed. First, we divide the noisy image to some sub-blocks, decompose each sub-block into the low-frequency and the high-frequency parts by the one-layer discrete 2-D wavelet transform. Then, use SVD with the rank-1 approximation to filter the noise of the horizontal and vertical high-frequency parts. In particular, use the rotation-based SVD (RSVD) with rotating 45 degrees to remove the noise of the diagonal high-frequency part, and rotate it back after filtering. Finally, reconstruct the sub-block from the low-frequency and denoised high-frequency parts by the inverse wavelet transform, and reorganize each denoised sub-blocks to obtain the final denoised image.

The next of this paper is organized as follows. In Sect. 2, we briefly review the principles of Wavelet Transform and SVD denoising. In Sect. 3, we present the proposed denoising method in detail. In Sect. 4, we give some numerical experiments and performance analysis. Finally, the Sect. 5 is the conclusion remarks.

2. Mathematical Preliminaries

2.1 Wavelet Transform and Threshold Denoising

Using the wavelet transform, an image can be decomposed into its low-frequency and high-frequency parts. The low-frequency part represents the approximate energy of an image, while the three high-frequency parts represent the detail information of an image, including the horizontal, vertical, and diagonal parts. New wavelet coefficients are estimated by thresholding the original wavelet coefficients. Common threshold functions include hard and soft threshold functions.
2.2 SVD Denoising

Every two-dimensional image $A$ of size $M \times N$ ($M \geq N$) can be decomposed into three matrices by SVD

$$A = U S V^T$$  \hspace{1cm} (1)

where $U$ and $V$ are the left and right singular matrices of $A$, with column vectors $\vec{u}_i$ and $\vec{v}_i$, respectively. The rank of $A$ is the number of the nonzero singular values. For $\text{rank}(A) = R$ ($R \leq N$), and when the diagonal singular values matrix $S = \text{diag}(s_1, s_2, \ldots, s_R)$ is a non-negative matrix, the nonzero singular values can be arranged as $s_1 \geq s_2 \geq \ldots \geq s_R > 0$. These singular values reflect the energy distribution of the image, and $s_i$ can be considered the representation coefficient. Therefore, image $A$ with $\text{rank}(A) = R$ can be expressed by ignoring singular values having value zero as follows:

$$A = \sum_{i=1}^{R} s_i u_i v_i^T$$  \hspace{1cm} (2)

Moreover, we can choose the best appropriate number $r$ or threshold $\lambda$ of the singular values to reconstruct $A$, describes as the rank-$r$ approximation $\tilde{A}$:

$$\tilde{A} = \sum_{i=1}^{r} s_i u_i v_i^T, \quad s_i \geq \lambda$$  \hspace{1cm} (3)

The threshold $\lambda$ is estimated as

$$\lambda \leq \sqrt{MN} \sigma$$  \hspace{1cm} (4)

where $M$ and $N$ represent the length and width of the noisy image, $\sigma$ represents the normalized variance of the noisy image.

2.3 Directional Features Using SVD

Based on the analysis of SVD theory, the directional features can be described as follows:

If an image is composed with mainly horizontal or vertical lines, then the energy of the image after the SVD filtering is mainly represented by a few large singular values, and the several leading singular values are far greater than other singular values. Otherwise, for the image with a non-horizontal/non-vertical line, the larger leading singular values do not have more energy compared to the smaller trailing singular values, and the energy is distributed over multiple singular values and their corresponding singular vectors. The use of only a small number of large singular values and the corresponding singular vectors to reconstruct the original image cannot achieve good denoising results.

For the image of noisy horizontal/vertical line shown in
Fig. 1 (a), which is added the white Gaussian noise (normalized variance $\sigma^2 = 0.01$). We use lower rank-1, rank-2, and rank-50 approximation to remove the noise, the denoised images are shown in Fig. 1 (b)–(d). It is clear that the rank-1 approximation also has a good denoising effect.

For the image of noisy 45° line shown in Fig. 2 (a), we use the conventional SVD with rank-1, rank-2, rank-3, rank-50 and rank-100 approximation to remove the noise, the denoised images are shown in Fig. 2 (b)–(f). As we can see that the lower rank approximation will lead to the loss of a great amount of original information, while the higher rank approximation retains more information but also hasn’t decreased noise prominently. An idea to overcome this problem is to rotate the 45° line to the horizontal or vertical direction, where apart of rotated image uses the noisy image to fill, as shown in Fig. 3 (a). Figure 3 (b) shows the denoised image with rank-1 approximation of Fig. 3 (a), following which Fig. 3 (c)–(d) are re-rotated and intercepted to the original angle and size of Fig. 2(a). As a comparison, the denoised image with rank-2, rank-3, rank-50 and rank-100 approximation after rotation as shown in Fig. 3 (e)–(h), the denoised PSNR results with different rank approximation using the conventional SVD and rotation-based SVD show in Fig. 3 (i). It is clear that the denoised PSNR with rank-1 approximation using the rotation-based SVD is best, that exceeds the maximum with rank-54 approximation using the conventional SVD. Therefore, we can use rotation-based SVD with rank-1 approximation to remove the noise.

Because the three high-frequency parts after wavelet transform show the horizontal, vertical, and diagonal information of image, therefore, we can use the directional features of SVD, that is, use the rotation-based SVD to rotate the diagonal part 45 degrees, and then remove the noise with rank-1 approximation. For horizontal and vertical parts, we use the conventional SVD filtering to get the denoised part.

2.4 Block-Rotation SVD Denoising

Considering the local stationarity and directionality of an image, we proposed the Block-Rotation SVD (BRSVD) method. The method can described as follows: the noisy image of $M \times N$ size is divided into several non-overlapping sub-blocks of $b \times b$ size, with $M = Kb$, $N = Lb$. $A_{kl}$ is the sub-block image after the rotation. By applying the SVD, we obtain

$$A_{kl} = U_{kl}S_{kl}V_{kl}^T$$

(5)

where $U_{kl}$ and $V_{kl}$ are the left and singular matrices of $A_{kl}$, respectively, and $S_{kl}$ is the diagonal singular value matrix. The reconstructed image $\tilde{A}_{kl}$ with rank-$r_{kl}$ is given by

$$\tilde{A}_{kl} = \sum_{i=1}^{r_{kl}} \lambda_{kl}^i u_{kl}^i v_{kl}^i T$$

(6)

3. Image Denoising in This Paper

The steps followed in the image denoising procedure using BRSVD filtering and wavelet transform are given below (Fig. 4):

1. Divide the noised image into sub-blocks with fixed size, and use two-dimensional extension to the edge of the sub-blocks.
2. Apply the single-level discrete 2-D wavelet transform to decompose each sub-blocks into one low-frequency and three high-frequency parts.
3. Use the conventional SVD to remove the noise of the horizontal and vertical high-frequency parts, and use the rotation-based SVD (RSVD) with rotating 45 degrees to remove the noise of the diagonal high-frequency parts.
Fig. 6 (a) The original image; (b) the noised image ($\sigma^2 = 0.01$); (c) the denoised image based on the conventional wavelet hard threshold; (d) the denoised image based on the conventional wavelet soft threshold; (e) the denoised image using the method in this paper (with 1 sub-blocks); (f) the denoised image using the method in this paper (with 4 sub-blocks); (g) the denoised image using the method in this paper (with 16 sub-blocks).

Fig. 7 (a) The original image; (b) the noised image ($\sigma^2 = 0.1$); (c) the denoised image based on the conventional wavelet hard threshold; (d) the denoised image based on the conventional wavelet soft threshold; (e) the denoised image based on the conventional SVD; (f) the denoised image using the method in this paper (with 1 sub-blocks); (g) the denoised image using the method in this paper (with 4 sub-blocks); (h) the denoised image using the method in this paper (with 16 sub-blocks).

Fig. 8 (a) The PSNR results for different methods of Figs. 5–7 ($\sigma^2 \in [0.001, 0.01]$); (b) The PSNR results for different methods of Figs. 5–7 ($\sigma^2 \in [0.01, 0.1]$); (c) The SSIM results for different methods of Figs. 5–7 ($\sigma^2 \in [0.001, 0.01]$); (d) The SSIM results for different methods of Figs. 5–7 ($\sigma^2 \in [0.01, 0.1]$).
4. Reconstruct the image from the low-frequency and denoised high-frequency parts by the inverse wavelet transform with rank-1 approximation, to get the de-

5. Finally, reorganizes each denoised sub-block to obtain the final denoised image.

4. Simulations

To validate the effect of this paper, in this section, we apply a number of experiments on three test images with added different white Gaussian noise. The random noise mean is 0, and the variance $\sigma^2$ is from 65 to 6500, that is, the normalized variance $\sigma^2$ used for MATLAB simulation is from 0.001 to 0.1,

$$\sigma^2 = \frac{\sigma_0^2}{(255)^2}$$  \hspace{1cm} (7)

The size of the test images are both $512 \times 512$, as shown in Fig. 5 (a), 9 (a), and 11 (a). Figs. 5–7 (b), 9–10 (b) and 13–14 (b) show the noisy images with different added variances.

4.1 Performance Metrics

We used two metrics to evaluate the denoising effectiveness:

1) Peak signal to noise ratio (PSNR)

PSNR represents the ratio between the maximum possible power of an original image and the power of denoised image, where a higher PSNR indicated a better denoising effect. PSNR can be defined as
Fig. 12 The PSNR results with four denoising methods of Figs. 9–10: (a) $\sigma^2 = 0.01$; (b) $\sigma^2 = 0.04$; (c) $\sigma^2 = 0.07$; (d) $\sigma^2 = 0.1$.

Fig. 13 (a) The original image; (b) the noised image ($\sigma^2 = 0.001$); (c) the denoised image based on the conventional wavelet hard threshold; (d) the denoised image based on the conventional wavelet soft threshold; (e) the denoised image based on the conventional SVD; (f) the denoised image using the method in this paper (with 1 sub-blocks); (g) the denoised image using the method in this paper (with 4 sub-blocks); (h) the denoised image using the method in this paper (with 16 sub-blocks).

Fig. 14 (a) The original image; (b) the noised image ($\sigma^2 = 0.01$); (c) the denoised image based on the conventional wavelet hard threshold; (d) the denoised image based on the conventional wavelet soft threshold; (e) the denoised image based on the conventional SVD; (f) the denoised image using the method in this paper (with 1 sub-blocks); (g) the denoised image using the method in this paper (with 4 sub-blocks); (h) the denoised image using the method in this paper (with 16 sub-blocks).

PSNR

$$PSNR = 10 \log_{10} \left( \frac{255^2}{MSE} \right)$$

$$MSE = \frac{\sum_{i=1}^{M} \sum_{j=1}^{N} (f(i, j) - f_0(i, j))^2}{MN}$$

where $f(i, j)$ and $f_0(i, j)$ are the pixel gray values of the denoised and original image, respectively.

2) Structural Similarity Index (SSIM)

SSIM are used to measure the luminance, contrast and structure between the original and denoised images by using mean value, where a higher SSIM indicated more similarity degree of images X and Y. SSIM calculated as [20]

$$SSIM(X, Y) = \frac{(2\mu_X\mu_Y + C_1)(2\sigma_{XY} + C_2)}{(\mu_X^2 + \mu_Y^2 + C_1)(\sigma_X^2 + \sigma_Y^2 + C_2)}$$

where $\mu_X$, $\mu_Y$, $\sigma_X^2$ and $\sigma_Y^2$ are the averages and variances of
two images, \( \sigma_{XY} \) is the covariance between \( X \) and \( Y \), \( C_1 \) and \( C_2 \) are constants to maintain the stability.

\[
\mu_X = \sum_{i=1}^{M} \sum_{j=1}^{N} X(i, j) \\
\sigma_X^2 = \sum_{i=1}^{M} \sum_{j=1}^{N} (X(i, j) - \mu_X)^2 \\
\sigma_{XY} = \sum_{i=1}^{M} \sum_{j=1}^{N} (X(i, j) - \mu_X)(Y(i, j) - \mu_Y) \\
C_1 = (K_1 \times L)^2 \\
C_2 = (K_2 \times L)^2
\]

where \( L \) is the dynamic range of the pixel values (255 for 8-bit grayscale images), and \( K_1 \leq 1 \), \( K_2 \leq 1 \) are small constants. Throughout this paper, the SSIM measure uses the following parameter settings: \( K_1 = 0.01 \), \( K_2 = 0.03 \).

### 4.2 Denoising Performance

For the single-level discrete 2-D wavelet transform with db3 wavelets, Figs. 5–7 (c)–(h), 9–10 (c)–(h), and 13–14 (c)–(h), respectively, illustrate the results of:

- the denoising method proposed in this paper (with 1, 4 and 16 sub-blocks, respectively),
- image denoising based on RSVD and Wavelet transform,
- the conventional wavelet hard threshold method,
- the conventional wavelet soft threshold method,
- the conventional SVD filtering method.

The PSNR and the SSIM results for which are shown in Figs. 8, 11 and 15. As a comparison, Fig. 12 are the PSNR results using the conventional wavelet threshold methods, the method proposed in the literature [18], the conventional SVD method and the method proposed in this paper, and the denoised images are shown in Figs. 9–10 (i)–(j).

From Figs. 5–15, it is clear that this denoising method significantly improves the PSNR and SSIM, and provides a better denoising effect with greater detail preservation compared to the other methods. For example, for 5 (a) image with different normalized variance \( \sigma^2 \) (0.001–0.1), on average, the PSNR of our method (16 sub-blocks) is superior to wavelet hard thresholding by 5.0901 dB, to wavelet soft thresholding by 4.9510 dB, to conventional SVD by 4.7500 dB, to our method (1 sub-block) by 2.5572 dB, respectively. Moreover, the SSIM of our method (16 sub-blocks) is superior to wavelet hard thresholding by 0.2055, to wavelet soft thresholding by 0.2045, to conventional SVD by 0.2041, to our method (1 sub-block) by 0.1120, respectively. The effect of our method becomes more evident with stronger noise. Moreover, the PSNR and SSIM of our method with 4 and 16 sub-blocks are very close and far more than other methods. In order to reduce the amount of computation, we only need to divide the original noisy image into 4 sub-blocks.

In essence, conventional SVD method don’t consider the directional features of original image, so it’s denoising effect and PSNR is not good. Moreover, in conventional BSVD and wavelet transform filtering, the directional features in sub-blocks and the local stationarity of whole image are not considered, respectively. Therefore, this method consider the directional features in sub-blocks and the local stationarity of whole image, which propose an image denoising algorithm based on the block-rotation SVD filtering in wavelet domain. We consider the local stationarity so that divide the noisy image into several sub-block images. Because of the directional features of SVD, we uses the rotation-based SVD to remove the noise in the diagonal part after wavelet transform which represent 45° features. Therefore, these block division and rotation operations can lead to the pixel gray values of the denoised image probably closest to original image, thus the PSNR results obtained by the proposed method is superior to other relevant methods.

The denoising performance of this proposed method is not affected by other factors, except for the size and texture of the image. Because of wavelet transform, this method is suitable for any square and gray images, that is, the length and width of the gray image are equal. And this method consider the directional features, so the original image need contains the distinct textures.

### 5. Conclusions

In this paper, we have proposed an efficient denoising algorithm, using the block-rotation singular value decompo-
sition (BRSVD) in wavelet domain. Experimental results show that this proposed method is effective in denoising and has a better denoising performance compared with existing methods. This method can be used to process the remote sensing and medical images denoising.

Although the method has the advantages, it has more computational costs (e.g., than the traditional SVD method) and the usage of this method is mainly for square gray images with distinct directional textures.

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References


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