PAPER

Fuzzy Levy-GJR-GARCH American Option Pricing Model Based on an Infinite Pure Jump Process

Huiming ZHANG^{†a)}, Member and Junzo WATADA^{††b)}, Nonmember

SUMMARY This paper focuses mainly on issues related to the pricing of American options under a fuzzy environment by taking into account the clustering of the underlying asset price volatility, leverage effect and stochastic jumps. By treating the volatility as a parabolic fuzzy number, we constructed a Levy-GJR-GARCH model based on an infinite pure jump process and combined the model with fuzzy simulation technology to perform numerical simulations based on the least squares Monte Carlo approach and the fuzzy binomial tree method. An empirical study was performed using American put option data from the Standard & Poor's 100 index. The findings are as follows: under a fuzzy environment, the result of the option valuation is more precise than the result under a clear environment, pricing simulations of short-term options have higher precision than those of medium- and long-term options, the least squares Monte Carlo approach yields more accurate valuation than the fuzzy binomial tree method, and the simulation effects of different Levy processes indicate that the NIG and CGMY models are superior to the VG model. Moreover, the option price increases as the time to expiration of options is extended and the exercise price increases, the membership function curve is asymmetric with an inclined left tendency, and the fuzzy interval narrows as the level set α and the exponent of membership function n increase. In addition, the results demonstrate that the quasi-random number and Brownian Bridge approaches can improve the convergence speed of the least squares Monte Carlo approach.

key words: American option, fuzzy set theory, fuzzy simulation technology, Levy process, GJR-GARCH model, least squares Monte Carlo approach, binomial tree method, quasi-random number, Brownian Bridge approach

1. Introduction

An option is a kind of fundamental financial derivative, it represents a contract which offer a right to the buyer who can buy (call) or sell (put) a security or other financial assets at a agreed-upon price (the exercise price) without the obligation during fixed period or on a specific date (exercise date). The buyer should pay the fee for the seller to obtain this right, this fee is called option price. An option includes call option and put option, call option offer a right to the buyer who can buy the underlying asset at a fixed price before or on a specific date, put option offer a right to the buyer who can sell the underlying asset at a fixed

price before or on a specific date. In 1973, E. Black and M. Scholes wrote "The Pricing of Options and Corporate Liabilities" and proposed a comprehensive option pricing model that resolves the challenges in option pricing, which has contributed tremendously to the study of option pricing theory. However, the said pricing model primarily addresses pricing problems in European options; hence, it is unsuitable for the pricing of American options, which allow early exercise. Compared with European options, the pricing problems of American options are far more complicated because the holder of an option determines the best time to exercise the option by comparing the value of continuously holding the option at various points of time with the value of immediate option exercise. When the value of immediate option exercise is greater than the value of continuously holding the option, the option holder will choose to exercise the option immediately because that time is the best exercise time; this choice influences the pricing of the American option. A rational investor will choose the best time to exercise the option, which is known as the optimal stopping time problem in mathematics. From the perspective of partial differential equations, the problem is also considered a free-boundary problem. Therefore, the key challenge in the pricing of American options is to determine the best time for investor to exercise an option. This is one of the reasons why American option pricing theory has become a frontier and very active topic in the field of financial research.

The pricing problem of American options is usually solved with either analytical or numerical methods. Earlier studies mainly used analytical methods to determine the price of American options: Johnson (1983) [1] used approximate analysis to determine the value of an American option under the assumption of no dividend; Geske et al. (1984) [2] constructed a model to analyses an American option with a dividend pay-out, but no closed-form solution was obtained. Therefore, numerical methods to solve American option pricing began to emerge; these include the commonly used binomial tree, finite difference, and least squares Monte Carlo methods, among others. Cox et al. (1979) [3] proposed the binomial tree method, which offers simple and effective solutions, and it provides an accurate numerical solution by continuously shrinking the time step; therefore, it is often used as a reference to evaluate the accuracy of other numerical approaches. However, when the model includes multiple random influencing factors, the number of values that must be calculated in the binomial tree method increases exponentially, which often leads to the curse of

Manuscript received July 21, 2017.

Manuscript revised March 13, 2018.

Manuscript publicized April 16, 2018.

[†]The author is with the Graduate School of Information, Production and Systems, Waseda University, Kitakyushu-shi, 808-0135 Japan.

^{††}The author is with the Computer and Information Sciences Department, PETRONAS University of Technology, 32610 Seri Iskandar, Perak Darul Ridzuan, Malaysia.

a) E-mail: h.zhang@kurenai.waseda.jp

b) E-mail: junzo.watada@gmail.com

DOI: 10.1587/transinf.2017EDP7236

dimensionality. The finite difference method mainly converts the asset pricing differential equation into a difference equation, and by obtaining solutions through an iterative method, it avoids the difficulty of directly solving the differential equation. In 1978, Brennan et al. [4] applied this calculation method in the pricing of American options, but the curse of dimensionality persists when this method is used to solve high-dimensional problems. The Monte Carlo method has the characteristic of forward simulation, so it cannot be applied directly for the pricing of American options, which have a backward iterative search characteristic. Longstaff et al. (2001) [5] modified the Monte Carlo method using the least squares approach and proposed the least squares Monte Carlo approach, which solves the application difficulty of the said method in the pricing of American options; the authors also provided empirical evidence of the method effectiveness. This method uses the least squares approach to estimate the expected value of continuous holding for each path. By comparing the values to the value associated with immediate exercise, the exercise point of each path is determined. Finally, the value of the American option is obtained by computing the discounted average value of each path's exercise point.

The above calculation methods are effective in pricing American options; however, these studies use the Black-Scholes (B-S) model as their theoretical basis, in which the asset price random process is treated as a geometric Brownian motion, which is unfit for real-life financial markets. Empirical studies have demonstrated that fluctuations in asset price and rate of return are often characterized by non-continuity, clustering and leverage effects; consequently, there is a need to construct a more flexible asset pricing model to accurately reflect how asset prices change in reality. There are usually jumps in asset price movements, and by adding a Levy process in the pricing model, we can construct a jump model with random jumps of different strengths. Moreover, generalized autoregressive conditional heteroskedasticity (GARCH) models are most frequently used to express the volatility in asset price fluctuations and leverage effects, and such models are highly expandable and more capable of providing accurate descriptions of volatility; therefore, by combining the two models to form the Levy-GARCH model, we can better capture the characteristics of the volatility of the underlying asset. The Levy-GARCH model is widely used in the pricing of European options, but due to the complexity of American options, the model is less frequently applied as a theoretical model for American options. Based on the background described above, jump measure, time-varying volatility and leverage effects are incorporated in this study to construct the Levy-GARCH pricing model for American options proposed by Glosten, Jagannathan and Rundle (Levy-GJR-GARCH) on the basis of an infinite pure jump Levy process and an asymmetric GARCH model. In addition, in real-life financial markets, many subjective and objective uncertainty factors lead to randomness and fuzziness in the price of the option; therefore, it is necessary to incorporate fuzzy the-

ory in the pricing model to improve the classic pricing theory. Hence, this study analysed the American option pricing model under a fuzzy environment, incorporated fuzzy simulation technology, and used the least squares Monte Carlo approach with higher operational efficiency to analyse the model and compare the operating results with the results computed using the binomial tree method. Lastly, through empirical analysis, we compared the option pricing simulation results of the three Levy processes (variance gamma (VG), normal inverse Gaussian (NIG), Carr-Geman-Madan-Yor (CGMY)) combined with the GJR-GARCH model, and we verified the convergence efficiency of the modified least squares Monte Carlo approach using the quasi-random number and Brownian Bridge approaches.

2. Literature Review

Jin-Chuan Duan (1995) [6] was the first to apply the GARCH model in European option pricing theory. He also performed a comparative analysis of pricing results obtained using the B-S model under risk conditions. His findings demonstrated that the GARCH model is effective in reducing the systematic error in pricing. Saez Marc (1997) [7] studied the effect of stochastic volatility on option pricing using symmetric and asymmetric GARCH models based on data regarding Spanish options. He discovered that the IEGARCH (1,2)-M-S model is the most effective model for capturing the stochastic volatility of the rate of return of IBEX-35 stocks. Lars Stentoft (2012) [8] applied a GARCH model to the pricing of American options; he used a Monte Carlo approach to form simulation analysis. The results indicate that asset prices with GARCH effect can better reflect the actual conditions. Therefore, we can conclude that by incorporating a GARCH model in an option pricing model, we can improve the pricing accuracy. In addition, many researchers have incorporated Levy processes in option pricing model and studied the non-normality and jumping characteristics of the underlying asset. Levy processes include finite jumps and infinite pure jumps, the earliest research on finite jumps was proposed by Merton in 1976, the subsequently scholars proposed the single exponential and the double exponential jump diffusion models for option pricing, for example, Levendorskiĭ (2004) [9] introduced the exponential jump diffusion process to American option price problem and provided a effective pricing solution; Maller et al. (2006) [10] based on multinomial approximation and exponential jump diffusion process studied American option pricing problem, he consider that this scheme is relatively applicable to path-dependent options pricing problem. However, finite jumps can not better characterize high-frequency small jumps, thus Madan et al. (1990) [11] used the Gamma variable to describe the Levy process with a VG distribution; although only one parameter was added, the model fits high-order moment characteristics of asset prices well. The NIG process proposed by Barndorff Nielsen (1997) [12] is one of the most commonly used Levy processes, and this process offers high operating effi-

ciency and the ability to provide accurate characterisation of the tail behaviour of asset prices. And some scholars consider that infinite pure jumps can substitute jump diffusion, such as Carr et al. (2002) [13] and Daal et al. (2005) [14]. Thus infinite pure jumps offering wider application scope in option pricing, for example, Avramidis et al. (2006) [15] based on VG model studied the Monte-Carlo algorithm for path-dependent options. Song et al. (2011) [16] based on asymptotic expansion and nonlinear regression method to obtain the approximate option price for the infinite pure jump levy process option pricing problem. As theoretical research advances, Peter Christoffersen et al. (2010) [17] have combined a GARCH model and Levy processes, resulting in the GARCH-Levy option pricing model, which better suits the financial environment. Byun et al. (2013) [18] studied the dynamic volatility and non-normality of underlying asset using the Levy-GARCH model, and based on an empirical analysis of the S&P500, they demonstrated that their model has higher precision in option pricing than previous models.

Regarding the application of fuzzy theory in option pricing, Cheng-Few Lee et al. (2005) [19] were among the first to incorporate fuzzy decision space in investor decision making, deriving a B-S model under a fuzzy environment. Their research demonstrates that models that fail to incorporate fuzzy numbers tend to underestimate the values of call options. However, empirical research indicates that volatility in asset prices is often time-varying and non normality; consequently, under the framework of a fuzzy system, Leandro Maciel et al. (2015) [20] took into account the time-varying volatility, clustering of volatility and other factors. Using empirical testing, they showed that under a fuzzy environment, the GJR-GARCH model provides better prediction than the traditional GARCH model. Liu Wen-Qiong et al. (2013) [21] used fuzzy set theory to study the European option pricing problem under the condition of jump-diffusion. They treated the interest rate and jump frequency as triangular fuzzy numbers and obtained the option fuzzy price range through empirical analysis. Feng Zhi-Yuan et al. (2015) [22] applied a time-varying Levy process with high-frequency jumps and stochastic volatility for fuzzy pricing of European call option; using option data for the S&P500 Index, they demonstrated that their model better fits the data than state-of-the-art models. Therefore, only by simultaneously studying the fuzziness, time-varying volatility and jump characteristic of the asset prices can we obtain option pricing that fits the actual conditions.

Regarding research about American options based on numerical approaches, Richard Breen (1991) [23] and Mark Broadie et al. (1994) [24] have each applied the convergence acceleration technology and extrapolation methods in the binomial tree option pricing model, which in turn improves the model's convergence speed. Clement et al. (2002) [25] verified the convergence of the least squares Monte Carlo approach; they demonstrated that the asymptotic error obeys an asymptotic Gaussian distribution. Lars Stentoft (2004) [26] demonstrated that among the high-dimensional

calculation methods, the least squares Monte Carlo approach is clearly superior to the binomial tree method and finite difference algorithm; the least squares Monte Carlo approach can easily perform ten-dimensional mathematical operation, whereas the finite difference method is no longer valid for calculations with more than five dimensions. Afterwards, in 2008, Lars Stentoft [27] compared the least squares Monte Carlo approach with the American option pricing method proposed by Carriere in 1996, and he found that the least squares Monte Carlo approach provides better results. From financial model research, Jorg Kienitz et al. (2012) [28] performed key analysis regarding how the quasi-random number and Brownian Bridge approach can be used to improve Monte Carlo method, and they applied the improved calculation method in option pricing. However, there are relatively few studies regarding American option pricing theory under a fuzzy environment based on numerical approaches, for example, Yoshida et al. (2006) [29] based on Black-Scholes model constructed an American option pricing model which set the underlying price as fuzzy variable and through numerical simulation to verify the proposed model effectiveness. Rather, most of existing literature provides analysis based on the binomial tree method, such as Silvia Muzzioli et al. (2008) [30] treat the volatility as a fuzzy number and used the multiple-period binomial tree method to obtain risk-neutral valuations of American options.

Through review of the existing literature, we observe that there are abundant studies regarding European option, studies about American option pricing are still limited. Furthermore, the existing studies mainly focus on numerical method improvements, and there is insufficient research intended to improve the theoretical model. Therefore, we constructed the Levy-GJR-GARCH American option pricing model, which is more consistent with reality, and evaluate the model's simulation accuracy using empirical analysis. The rest of this paper is structured as follows: Sect. 3 introduces some related preliminaries on infinite pure jump levy processes and parabolic fuzzy variable; Sect. 4 deduces the Levy-GJR-GARCH American option pricing model under a fuzzy environment; Sect. 5 provides a brief introduction of fuzzy simulation technology, then based on it design the algorithms for fuzzy American option pricing model, such as fuzzy binomial tree method, fuzzy least squares Monte Carlo method and especially using quasi-random numbers and Brownian Bridge approach to improve the convergence speed of Monte Carlo method. Section 6 combines the Standard & Poor's 100 index (S&P 100 Index) American put option prices to perform empirical testing, followed by a comparative analysis of the fitting precision of different models under fuzzy environments and clear environments and an examination of the convergence efficiency of the quasi-Monte Carlo and Brown Bridge method. Section 7 summarizes the findings of this study and provides direction for future research.

3. Preliminaries

As the notations used in the remainder of this paper are listed as follows:

Acronyms	Description
VG	Variance gamma process
NIG	Normal inverse Gaussian process
CGMY	Carr-Geman-Madan-Yor process
GARCH	Generalized autoregressive conditional heteroskedasticity model
EGARCH	Exponential GARCH model
TGARCH	Threshold GARCH model
GJR-GARCH	Glosten, Jagannathan and Rundle-GARCH model

Variable	Description
S_0	The underlying asset price at initial time
S_t	The underlying asset price at time t
T	Time to expiration
K	Exercise price
r	Risk-free interest rate
$V(S_t, t)$	Option price at time t
R_t	Logarithmic return rate of the underlying asset price at time t
σ_t	Volatility at time t
X_t	Levy process
θ	Drift rate
v	Jump rate
F_t	Information set at time t
I_t	Indicator Function at time t
w	Intercept
α	The influence coefficient of the variance of previous period to the variance of current period
β	The influence coefficient of the residual of previous period to the residual of current period
δ	Asymmetric effect coefficient
z_t	The innovation of the mean equation at time t
g	Gamma function
Γ	The optimal boundary for exercise
$\tilde{A}$	Fuzzy set
$\tilde{A}_\alpha$	α -level set of fuzzy set $\tilde{A}$
$\mu_{\tilde{A}}(x)$	The membership function of the fuzzy set $\tilde{A}$
$V(\tilde{S}_t, t)$	Fuzzy option price at time t

3.1 Infinite Pure Jump Levy Process

A Levy process is a self-adapting process with independent and stable random increments. Its characteristic function is $\Phi_{x_t} = (u|F_t) = E\{\exp(iux_t)\} = \exp(\phi(u))$, where $\phi(u)$ is the characteristic exponent of the characteristic function. $\phi(u)$ is formulated as follows:

$$\phi(u) = i\theta u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux1_{|x|\leq 1})v(dx) \quad (1)$$

The whole expression consists of drift, diffusion and jump elements; θ and σ represent the measure of drift and the measure of diffusion, whereas v is the measure of jump. Hence, (θ, σ, v) represent all information in the Levy process, and it is also known as the three elements of Levy.

Levy processes include two major types of processes: jump-diffusion processes and pure jump processes. The jump-diffusion processes include the Merton model, and double exponential jump model, for example, whereas examples of pure jump processes include the VG model, NIG model, and CGMY model. Compared with jump-diffusion processes, there are fewer parameters in the pure jump process, it contains more high-order moment characteristics and offers simpler calculations, and it has broader applications in current research; therefore, this study chose to use a pure jump process for our model analysis.

This study used more mature Levy processes VG, NIG and CGMY simulation technology to analyse the option pricing and performed comparative analysis of the simulation effect of these three jump processes. The VG process is a random process driven by Gamma process; it has good mathematical properties and is considered to be a limited-variation process. Moreover, the incremental part of the VG process exhibits leptokurtic and fat-tailed characteristics, which can resolve the issue of the “volatility smile”. The NIG process is a random process formed when the IG process performs time-varying on Wiener process; this process can provide excellent description of fat-tailed data, and it also possess other good properties, such as the additive property, which makes formula deduction and application easier. Simultaneously, it is easier to achieve measure conversion for this process, which increases its application in option pricing. The CGMY process is a tempered stable process based on the VG process with the addition of the Y parameter. The existence of the Y parameter increases the structural complexity of the model, but it also enriches the model’s data presentation capability and allows better description of financial data characteristics, such as infinite jumps and infinite variance. The characteristic function of these three Levy processes are expressed as follows:

(1) VG process:

$$E(e^{iuX_t}) = \varphi(u; \sigma, v, \theta) = \left(1 - iu\theta v + \frac{1}{2}\sigma^2 v u^2\right)^{-\frac{1}{v}} \quad (2)$$

of which,

$$C = \frac{1}{v} > 0$$

$$M = \left(\sqrt{\frac{1}{4}\theta^2 v^2 + \frac{1}{2}\sigma^2 v} + \frac{1}{2}v\theta\right)^{-1} > 0$$

$$G = \left(\sqrt{\frac{1}{4}\theta^2 v^2 + \frac{1}{2}\sigma^2 v} - \frac{1}{2}v\theta\right)^{-1} > 0$$

(2) NIG process:

$$E(e^{iuX_t}) = \varphi(u; \lambda, \eta, \kappa) = \exp\left(\eta\sqrt{\lambda^2 - \eta^2} - \kappa\sqrt{\lambda^2 - (\eta + iu)^2}\right) \quad (3)$$

of which, $\lambda > 0, \kappa > 0, -\lambda < \eta < \lambda$.

(3) CGMY process:

$$\begin{aligned} E(e^{iuX_t}) &= \varphi(u; C, G, M, Y) \\ &= \exp(Cg(-Y)(M - iu)^Y \\ &\quad + (G + iu)^Y - M^Y - G^Y) \end{aligned} \quad (4)$$

of which, $C > 0, G > 0, M > 0, Y < 2$, g represents gamma function.

3.2 Parabolic Fuzzy Variable

In terms of fuzzy theory development, the possibility measure and necessity measure were proposed by Zadeh in 1978 [31]; afterwards, Liu et al. proposed the credibility measure in 2002 [32], and an uncertainty theory with an axiomatic basis was established in 2004 [33].

If ξ is the function of probability space $\xi(\Theta, P(\Theta), Pos)$ to real data set R , ξ is a fuzzy variable, and $\xi_\alpha = \{\xi(\theta) | \theta \in \Theta, Pos(\theta) \geq \alpha\}$ is the α level set of ξ . The function $u(x) = Pos\{\theta \in \Theta | \xi(\theta) = x, x \in R\}$ derived from the probability measure is known as the membership function of ξ [33].

The most commonly used fuzzy numbers include triangular, trapezoidal and parabolic fuzzy numbers, of which triangular and trapezoidal fuzzy numbers are special forms of parabolic fuzzy numbers.

Assuming the membership function of the σ is set as follows:

$$\mu_{\tilde{\sigma}}(\sigma) = \begin{cases} \left(\frac{\sigma - a_1}{a_2 - a_1}\right)^n, & a_1 \leq \sigma \leq a_2 \\ 1, & a_2 \leq \sigma \leq a_3 \\ \left(\frac{\sigma - a_4}{a_3 - a_4}\right)^n, & a_3 \leq \sigma \leq a_4 \\ 0, & \text{others} \end{cases} \quad (5)$$

$\tilde{\sigma}$ is viewed as a parabolic fuzzy number, where $\tilde{\sigma} = (a_1, a_2, a_3, a_4)_n$. When $n = 1$, this fuzzy number is referred to as a trapezoidal fuzzy number; when $n = 1$ and $a_2 = a_3$, this fuzzy number is called a triangular fuzzy number. At this point, the α level set of $\tilde{\sigma}$ can be expressed as $\tilde{\sigma}_\alpha = [\tilde{\sigma}_\alpha^L, \tilde{\sigma}_\alpha^U] = [a + \sqrt[n]{\alpha}(b - a), d - \sqrt[n]{\alpha}(d - c)]$, where $\tilde{\sigma}_\alpha^L$ is the α pessimistic value of fuzzy variable $\tilde{\sigma}$ and $\tilde{\sigma}_\alpha^U$ is the α optimistic value of $\tilde{\sigma}$. The membership function is shown in Fig. 1.

4. Fuzzy Levy-GJR-GARCH American Option Pricing Model

4.1 The Process of the Underlying Asset Price

In this paper, we assumed the fluctuation of the underlying asset price has the characteristics of time-varying, jump

Fig. 1 Plot of membership function of a parabolic fuzzy number.

and leverage effect, thus the sequence of the rate of return of the underlying asset is described using an asymmetric conditional heteroskedasticity model. Among GARCH-type models, models that can express the conditional heteroskedasticity “leverage effect” include the exponential GARCH (EGARCH), threshold GARCH (TGARCH) and GJR (Glosten, Jagannathan and Rundle)-GARCH models, of which the TGARCH and GJR-GARCH models have similar structures and pricing effects. Compared with the EGARCH model, the GJR-GARCH model has better simulation accuracy; therefore, we chose to use GJR-GARCH model proposed by Glosten et al. (1993) [34] as the specific form of the asset’s return rate model, specifically as follows:

$$\begin{cases} R_t = \ln\left(\frac{S_t}{S_{t-1}}\right) = u_t - \gamma_t + \sigma_t z_t \\ \sigma_t^2 = w + \alpha \sigma_{t-1}^2 + \beta \sigma_{t-1}^2 z_{t-1}^2 + \delta I_{t-1} \sigma_{t-1}^2 z_{t-1}^2 \\ I_t = \begin{cases} 1, & z_t < 0 \\ 0, & z_t \geq 0 \end{cases} \\ z_t | F_{t-1} \sim D(0, 1; \theta_D) \end{cases} \quad (6)$$

In the asset’s return rate model (6), R_t is asset’s logarithmic return rate, u_t is the expected rate of return under the condition of information set F_{t-1} , γ_t is the mean correction factor, and σ_t^2 is the time-varying variance sequence, I_t represents the indicator function. w represents intercept, α is the influence coefficient of the variance of previous period to the variance of current period, β is the influence coefficient of the residual of previous period to the residual of current period, δ represents asymmetric effect coefficient. z_t represents the innovation of the mean equation, and it follows distribution $D(\bullet)$ with mean value of 0, variance of 1, and parameter θ_D , for which this study will establish several different infinite pure jump Levy processes, such as VG, NIG and CMGY process, which were introduced at Sect. 3.

Compared with a standard normal distribution, an infinite pure jump Levy process can describe better high-order moment characteristic of financial data, such as skewness or fat tails. Therefore, it is necessary to incorporate an infinite pure jump Levy process in the GARCH model because the

normal random number replaced by the Levy process random number will improve the model's pricing accuracy.

4.2 The Risk-Neutral Conversion of the Underlying Asset Pricing

In theory, there should be no arbitrage in the option value; therefore, the asset's return rate model (see Eq. (6)) requires risk-neutral conversion to ensure the validity of the no-arbitrage assumption. Under risk-neutral measure Q , $E^Q(S_t|S_{t-1}) = S_{t-1}e^{r_t}$, where r_t represents the risk-free rate of return. Here, the risk-neutral model is,

$$S_t = S_{t-1}e^{r_t - \varphi^Q(\sigma_t) + \sigma_t \varsigma_t^Q} \quad (7)$$

Above, $\varphi^Q(\sigma_t) = E^Q(e^{\sigma_t \varsigma_t^Q})$ is the mean correction factor, where ς_t^Q is white noise with mean of 0 and variance of 1. Using the Christoffersen et al. (2010) [35] method to construct the pricing kernel $\{\varsigma_t\}$, we establish a Radon-Nikodym derivative sequence that can materialise real measurement of risk-neutral measure conversion:

$$\frac{dQ}{dP}|_{F_{t-1}} = \exp\left(-\sum_{i=1}^t (\varsigma_i \sigma_i z_i + \psi(\varsigma_i))\right) \quad (8)$$

Under a non-normal environment, the kernel sequence $\{\varsigma_t\}$ is not the only one that fulfils the following formula:

$$\psi_t(\varsigma_t - 1) - \psi_t(\varsigma_t) + u_t - r_t - \gamma_t = 0 \quad (9)$$

Here, $\psi(\bullet)$ represents the exponential part of the moment-generating function. Based on the characteristics of the moment-generating function $\psi'_t(0) = E_{t-1}[\sigma_t z_t]$, $\psi''_t(0) = \text{Var}_{t-1}[\sigma_t z_t] = \sigma_t^2$, we obtain the following analytical expression for the kernel sequence $\{\varsigma_t\}$:

$$\varsigma_t \approx \frac{1}{2} + \frac{u_t - r_t - \gamma_t - \psi'_t(0)}{\psi''_t(0)} = \frac{1}{2} + \frac{u_t - r_t - \gamma_t}{\sigma_t^2} \quad (10)$$

After obtaining the kernel sequence $\{\varsigma_t\}$, we can perform risk-neutral adjustment on the stochastic item $\varepsilon_t = \sigma_t z_t$ and obtain the following formula:

$$\varepsilon_t^Q = \varepsilon_t - E_{t-1}^Q[\varepsilon_t] = \varepsilon_t - \psi'_t(\varsigma_t) \quad (11)$$

Therefore, under the risk-neutral measure, the mean equation can be expressed as follows:

$$R_t^Q = r_t - \psi_{\varepsilon_t^Q}^Q(1) + \varepsilon_t^Q = r_t - \psi_{\varepsilon_t^Q}^Q(\sigma_t^Q) + \sigma_t^Q z_t^Q \quad (12)$$

The conditional variance formula for the risk-neutral asset's return rate model can be expressed as

$$(\sigma_t^Q)^2 = w^Q + \alpha^Q(\sigma_{t-1}^Q)^2 + \beta^Q(\varepsilon_{t-1}^Q + \psi'(\varsigma_{t-1}))^2 + \delta^Q I_{t-1}(\varepsilon_{t-1}^Q + \psi'(\varsigma_{t-1}))^2 \quad (13)$$

At this point, we can see that there is some discrepancy between the risk-neutral measure and the real measure of sequence ε_t^Q and $(\sigma_t^Q)^2$; therefore, it is necessary to perform

parameter adjustment using kernel sequence $\{\varsigma_t\}$.

4.3 American Option Pricing under a Fuzzy Environment

Although we can obtain the logarithmic return rate through a heteroskedastic model and obtain the asset price, this method does not provide complete control over future uncertainty factors; therefore, in this study, we assume the asset price volatility σ is a parabolic fuzzy variable and analyse the American option pricing model under random and fuzzy environments.

Unlike European options, American options allow early exercise; therefore, American option pricing is a free-boundary problem, in which there is an optimal boundary for exercise, and the region $\{0 \leq S \leq \infty, 0 \leq t \leq T\}$ can be segregated into two parts: a region corresponding to continuing option holding and a region corresponding to stopping holding. For a non-dividend-paying American put option, in the continuation holding region $\sum_1$, $V(S, t) > (K - S)^+$; in the stopping holding region $\sum_2$, $V(S, t) = (K - S)^+$; K represents exercise price; the optimal boundary for exercise $\Gamma : S = B(t)$ is located between the two regions. In the above equations, $V(S, t)$ and S each represents the option value and asset price at time t . Therefore, the following relationship can be obtained:

$$\sum_1 = \{(S, t) | B(t) \leq S < \infty, 0 \leq t \leq T\} \quad (14)$$

$$\sum_2 = \{(S, t) | 0 \leq S \leq B(t), 0 \leq t \leq T\} \quad (15)$$

The relationship for the optimal boundary for exercise, Γ , is

$$V(B(t), t) = K - B(t) \quad (16)$$

$$\frac{\partial V}{\partial S}(B(t), t) = -1 \quad (17)$$

Subsequently, when $S \rightarrow \infty$, $V \rightarrow 0$, and when $t = T$, $V(S, T) = (K - S)^+$. Because $B(t)$ is a free boundary, the problem of pricing American put options can be viewed as a parabolic free-boundary problem.

Because the asset price S at time t is a function of σ , when σ is a fuzzy number $\tilde{\sigma}$, S is also a fuzzy number $\tilde{S}$. Similarly, because the option value $V(S, t)$ is a function of S , when S is fuzzy number $\tilde{S}$, $V(S, t)$ is also a fuzzy number $\tilde{V}(\tilde{S}, t)$. At this point, the α level set of $\tilde{V}(\tilde{S}, t)$ can be expressed as

$$\begin{aligned} (\tilde{V})_\alpha &= [(\tilde{V})_\alpha^L, (\tilde{V})_\alpha^U] \\ &= [\min_{\tilde{S}_\alpha^L \leq S \leq \tilde{S}_\alpha^U} V(S, t), \max_{\tilde{S}_\alpha^L \leq S \leq \tilde{S}_\alpha^U} V(S, t)] \end{aligned} \quad (18)$$

Therefore, based on credibility theory, the expected value $E(\tilde{V}(\tilde{S}, t))$ of $\tilde{V}(\tilde{S}, t)$ can be expressed as

$$\begin{aligned} E(\tilde{V}(\tilde{S}, t)) &= \int_0^{+\infty} Cr\{(\tilde{V}(\tilde{S}, t)) \geq r\} dr \\ &= \frac{1}{2} \int_0^1 ((\tilde{V})_\alpha^L + (\tilde{V})_\alpha^U) d\alpha \end{aligned} \quad (19)$$

At this point, the optimal exercise boundary for fuzzy American options, the continuation holding region and the stopping holding region under fuzzy environment are expressed as follows:

(1) Optimal boundary for exercise: $V(B(t), t) = K - B(t), v - \Gamma : B(t) = E(\tilde{S})$.

(2) Continuation holding region: $\Sigma_1 = \{(\tilde{S}, t) | B(t) \leq E(\tilde{S}) < \infty, 0 \leq t \leq T\}$.

(3) Stopping holding region: $\Sigma_2 = \{(\tilde{S}, t) | 0 \leq E(\tilde{S}) \leq B(t), 0 \leq t \leq T\}$.

5. The Algorithms Design for Fuzzy American Option Pricing Model

Upon obtaining the Levy-GARCH model for an American option under a fuzzy environment, we combine fuzzy simulation technology [33] and calculation methods frequently used for American options to create a fuzzy pricing method for American options.

5.1 Fuzzy Simulation Technology

Fuzzy simulation technology is used for sampling test of fuzzy models based on probability distributions. This technology only provides a statistical estimate of the model, not the precise result, but it is the only effective method for complex problems for which analytical results are unattainable.

If ξ is a fuzzy variable with probability space $(\Theta, P(\Theta), Pos)$, the function $f(\xi)$ is also a fuzzy variable; at the same time, the membership function of $f(\xi)$ can be obtained using the following simulation method:

Step 1. Randomly and evenly extract a number ξ_k ($k = 1, 2, \dots, N$) from the level set of fuzzy variable ξ , calculate ξ_k membership from the membership function of ξ , and denote it as v_k .

Step 2. Based on the formula for function $f(\xi_k)$, calculate the function value $f(\xi_k)$.

Step 3. Repeat Steps 1 through 2 N times.

Step 4. Calculate the expected value $E(f(\xi)) = \frac{1}{N}f(\xi_k)$ of function $f(\xi)$ and draw the membership function of $f(\xi)$ based on $(f(\xi_k), v_k)$.

5.2 Fuzzy Binomial Tree Method

The binomial tree method assumes that asset prices obey a dispersed time process, where the time $[0, T]$ is divided into n equivalent time steps $\Delta t = t_i = T/n$, where $i = 1, 2, \dots, n$. There can only be two changes in the asset price S_i at time t_i , whereby the price either increases to u times its original price with probability p or decreases to d times its original price with probability $(1 - p)$, such that $0 < d < 1 < u, ud = 1$; therefore, the asset price at t_{i+1} can only be uS_i or dS_i . When the initial asset price is S_0 , there is $i + 1$ probability for asset price S_i at t_i : $S_0 u^j d^{i-j}$, where $j = 0, 1, 2, \dots, i$. The exact binomial tree is shown in Fig. 2.

When using a binomial tree to obtain the price of an American option at each node, pricing is mainly performed

Fig. 2 Multiple-period binomial tree.

using the backward inference method from back to front. Generally, u and d are set as functions of the volatility σ . When σ is a fuzzy number, u and d are also fuzzy numbers; consequently, the asset price S_i at time t_i is also a fuzzy variable. At this point, when the exercise price is K and the time to expiration is T , the value of the American put option is expressed as $V_n(S_{n,j}) = \max\{K - E(S_{n,j}), 0\}$, where $j = 0, 1, 2, \dots, n$. Using backward inference, we can obtain the option value at time t_i :

$$V_i(S_{i,j}) = \max\{K - E(S_{i,j}), \exp(-r\Delta t)(pV_{i+1,j}(uS_{i,j}) + (1 - p)V_{i+1,j}(dS_{i,j}))\} \quad (20)$$

The α level set of $V_i(S_{i,j})$ can be expressed as

$$\begin{aligned} \tilde{V}_{i,\alpha}(S_{i,j}) = & [\max\{K - \tilde{S}_{i,j}^U(\alpha), \exp(-r\Delta t)(p\tilde{V}_{i+1,j}^L(uS_{i,j}) \\ & + (1 - p)\tilde{V}_{i+1,j}^L(dS_{i,j}))\}, \\ & \max\{K - \tilde{S}_{i,j}^L(\alpha), \exp(-r\Delta t)(p\tilde{V}_{i+1,j}^U(uS_{i,j}) \\ & + (1 - p)\tilde{V}_{i+1,j}^U(dS_{i,j}))\}] \end{aligned} \quad (21)$$

Therefore, the calculation of option price using fuzzy binomial tree is as follows:

Step 1. Randomly and evenly extract a number σ_k ($k = 1, 2, \dots, N$) from the α level set of fuzzy variable $\tilde{\sigma}$, calculate σ_k membership from the $\tilde{\sigma}$ membership function, and denote it as v_k .

Step 2. Presume that the asset price upward factor u , downward factor d and probability p are $e^{\sigma_k \sqrt{\Delta t}}$, $e^{-\sigma_k \sqrt{\Delta t}}$ and $\frac{e^{r\Delta t} - d}{u - d}$, respectively; based on these, calculate the asset price $S_{i,j}^k$ at each node of the price tree.

Step 3. Based on option calculation formula, calculate the option value up to the expiry date $V_N^k(S_N^k)$ and use backward inference to obtain the option value at each node $V_i^k(S_i^k)$.

Step 4. Repeat Steps 1 through 3 N times.

Step 5. Calculate the expected value of the option price $E(V_0) = \frac{1}{N}V_0^k$ and draw the membership function diagram of the fuzzy option price according to (V_0^k, v_k) .

5.3 Fuzzy Least Squares Monte Carlo Approach

The least squares Monte Carlo approach mainly compares the exercise price of immediate option exercise and the conditional expected value of continuous option holding to determine the optimal exercise time of an American option.

When the value of immediate exercise is greater than or equal to the value of continuous holding, the investor will choose to exercise the option immediately.

Presuming that the number of Monte Carlo approach-simulated paths is N and that the time to expiration T is divided into M periods, at time t_i , the exercise price of path- j is $I_{i,j}(S_{i,j}) = \max(K - S_{i,j}, 0)$, where K is the exercise price and $S_{i,j}$ is the asset price on path j during t_i . The conditional expected price of continuous option holding can only be obtained using backward inference $E_{i,j}(S_{i,j}) = E[\exp(-r\Delta t)V_{i+1,j}(S_{i+1,j})|S_{i,j}]$. Therefore, the normal Monte Carlo method is not suitable for numerical simulation of the American option pricing model. The least squares Monte Carlo approach regards the present discounted value of the option price at time t_{i+1} , $\exp(-r\Delta t)V_{i+1,j}(S_{i+1,j})$, as the Y variable and $S_{i,j}$ and $S_{i,j}^2$ as X variable, constructing a least squares regression model for Y as a function of X and obtaining regression coefficients a_1 , a_2 and a_3 . The following formula can yield an approximation for $E_{i,j}(S_{i,j})$:

$$E_{i,j}(S_{i,j}) \approx a_1 + a_2 S_{i,j} + a_3 S_{i,j}^2 \quad (22)$$

Based on the above method, compare the value of continuous holding and the value of exercise at each node of N paths, thereby obtaining the optimal execution strategy for each path. Discount the option value of each path to the present period and obtain the average of each path's discounted option value; this said average value is the acquired option price.

If the asset price volatility σ is a fuzzy number, the asset price $S_{i,j}$ is also a fuzzy number, whereas the exercise price $I_{i,j}(S_{i,j})$ and value of continuous holding $E_{i,j}(S_{i,j})$ are both functions of $S_{i,j}$; therefore, $I_{i,j}(S_{i,j})$ and $E_{i,j}(S_{i,j})$ are also fuzzy numbers. Their α level set can be expressed as follows:

$$\tilde{I}_{i,j}^\alpha(S_{i,j}) = [\max\{K - \tilde{S}_{i,j}^U(\alpha)\}^+, \max\{K - \tilde{S}_{i,j}^L(\alpha)\}^+] \quad (23)$$

$$\tilde{E}_{i,j}^\alpha(S_{i,j}) = [a_1 + a_2 \tilde{S}_{i,j}^L + a_3 \tilde{S}_{i,j}^{L^2}, a_1 + a_2 \tilde{S}_{i,j}^U + a_3 \tilde{S}_{i,j}^{U^2}] \quad (24)$$

Because the asset price and option value are fuzzy variables, when comparing and solving the least squares regression equation, the expected value of fuzzy variable is used in the calculation. The calculation of the option price using the least squares Monte Carlo approach is as follows:

Step 1. Randomly and evenly extract a number σ_k ($k = 1, 2, \dots, N$) from the α level set of the fuzzy variable $\tilde{\sigma}$, calculate the membership degree of σ_k from the membership function of $\tilde{\sigma}$, and denote it as v_k .

Step 2. Based on the asset price formula, calculate the asset price $S_{k,j}$ ($j = 1, 2, \dots, M$) at each node of path j .

Step 3. Find the option exercise price $I_{k,j}(S_{k,j})$ at each node of path j , and calculate the value of continuous option holding $E_{k,j}(S_{k,j})$ at each node using the least squares method.

Step 4. Repeat Steps 1 through 3 N times.

Step 5. Calculate the expected option value $E(V_0) = V_{k,0}/N$, and based on $(V_{k,0}, v_k)$, draw the fuzzy option value membership function diagram.

5.4 The Improvement for Monte Carlo Approach

There are two key factors that affect the simulation effect of the Monte Carlo model: first, the sampling characteristic of the random sampling determines the skewness of the sample distribution; second, the manner in which the randomly simulated paths are constructed determines whether the simulated paths resemble real paths. Most studies regarding the simulation effectiveness of Monte Carlo method mainly focus on these two factors.

The random numbers generated from Monte Carlo method are pseudo-random numbers. These random points often present clustering or gap problems, causing relatively large deviations in the random number sequence. On the other hand, the quasi-random numbers generated from the quasi-Monte Carlo method incorporate the randomness and the evenness of the sequence distribution in random sequence; hence, using this method reduces the deviation in the random sequence. The Hatan, Sobol and Faure sequences are the most common quasi-random sequences; since the Sobol sequence has better evenness and is less time-consuming to generate, this study chose to use Sobol sequence to obtain the quasi-random numbers.

The Sobol sequence [36] is constructed based on a series of "direction numbers" v_i . When q_i is a positive odd number less than 2^i ,

$$v_i = \frac{q_i}{2^i} \quad (25)$$

The assisting coefficient for v_i and q_i is usually 0 or generated from a simple polynomial, and the form of the polynomial is as follows:

$$f(z) = z^p + a_1 z^{p-1} + \dots + a_{p-1} z + a_p \quad (26)$$

When $i > p$, the recursion formulas for v_i and q_i are

$$v_i = a_1 v_{i-1} \oplus a_2 v_{i-2} \oplus \dots \oplus a_p v_{i-p} \oplus \left\lfloor v_{i-p}/2^p \right\rfloor \quad (27)$$

$$q_i = 2a_1 q_{i-1} \oplus 2^2 a_2 q_{i-2} \oplus \dots \oplus 2^p a_p q_{i-p} \oplus q_{i-p} \quad (28)$$

where $\oplus$ indicates the binary bitwise exclusive-OR.

The Brownian Bridge approach [28] is a method for constructing a Monte Carlo simulation path. If $X(t)$ is a random process, let $t_1 < t_2$ and the density function of $x \sim F_{X(t_1)}$ and $y \sim F_{X(t_2)}$ be f_{t_1} and f_{t_2} respectively, where $f_{x,y}$ indicates the combined density function of x and y . When the density function of $z = x + y$ is f_z , the density function of $x|z$ can be obtained using the following formula:

$$f_{x|z} = \frac{f_{x,z}(x, z-x)}{f_z(z)} \quad (29)$$

For Brownian motion $W(t)$, presuming $t_i < t_j < t_k$ and

Table 1 Descriptive statistics of the data.

Indicator	Sample size	Maximum value	Minimum value	Average value	Standard deviation	Skewness	Kurtosis
Rate of return	1525	0.0188	-0.028	0.0002	0.004	-0.4512	7.6373
Option time to expiration	70	0.75	0.0833	0.231	0.184	1.7002	5.1386
Option exercise price	70	1095	990	1041.6	31.2	0.0000	1.8
Option price	70	65.8	0.6	23.4336	17.0523	0.4781	2.1542

that $W(t_i)$ and $W(t_k)$ are given, the average value and variance at time t_j satisfy the following Brown Bridge characteristic:

$$E[W(t_j)] = \left(\frac{t_k - t_j}{t_k - t_i} \right) W(t_i) + \left(\frac{t_j - t_i}{t_k - t_i} \right) W(t_k) \quad (30)$$

$$V[W(t_j)] = \frac{(t_j - t_i)(t_k - t_j)}{t_k - t_i} \quad (31)$$

Therefore, the following formula can be used to obtain the sample value of time sequence $\{t_0, t_1, \dots, t_n\}$, subsequently obtaining the diffused sample path:

$$f_{x|z}(x) = \frac{1}{\sqrt{2\pi \frac{(t_j - t_i)(t_k - t_j)}{t_k - t_i}}} \exp \left(-\frac{1}{2} \left(\frac{x - \frac{t_j - t_i}{t_k - t_i} z}{\sqrt{\frac{(t_j - t_i)(t_k - t_j)}{t_k - t_i}}} \right)^2 \right) \quad (32)$$

The Brownian Bridge approach mainly increases the low-order coordinate component in the random sequence and reduces the actual dimension of simulation problem to better illustrate the distribution characteristic of quasi-random numbers, hence improving its estimation effect.

6. Empirical Analysis

6.1 Source of Data and Descriptive Statistics

This study used the S&P 100 Index and American put options acquired from S&P 100 Index as the data for empirical analysis. The S&P 100 Index prices were selected from the closing prices of data of 1526 days dating from March 22, 2011 to March 23, 2017 (data source: Yahoo!Finance), and the American S&P 100 Index put option prices were selected to be the average prices of the final transacted prices for different expiry dates and different exercise prices for put options on March 23, 2017 (data source: Chicago Board of Options Exchange). The data used in this research excluded options with same month expiry, and we categorized options with durations of 1–3 months as short-term options, 4–6 months as medium-term options, and more than 6 months as long-term options. We only took into account American options with exercise prices within the range of 95%–105% of the index prices and eliminated contracts with option values close to 0. Consequently, we obtained 70 data points, of which 22 expire in April, 22 expire in May, 11 expire in June, 5 expire in July, 5 expire in September and 5 expire in December.

Fig. 3 Daily logarithmic return rate of the S&P 100 Index.

Figure 3 shows the logarithmic return rate data calculated from S&P 100 Index prices of 1,525 days; it reveals tremendous volatility in the rate of return. Descriptive statistics of the data are listed in Table 1. The sampling skewness of the rate of return is $-0.4512 < 0$, which indicates that the sample is skewed to the left. The kurtosis is $7.6373 > 3$, which indicates that the sample is leptokurtic and fat-tailed. Skewness represents the deviation degree of the sample data distribution relative to the symmetrical distribution, when skewness = 0, it represents the sample data distribution is symmetrical, when skewness < 0 , it represents the sample data is left-skewed distribution, when skewness > 0 , it represents the sample data is right-skewed distribution. Kurtosis represents the degree of the sample data distribution more or less peaked than a normal distribution, when kurtosis = 3, it represents the sample data is a normal distribution, when kurtosis > 3 , it represents the sample data is relatively peaked distribution (leptokurtic) and its tail is longer and fatter than a normal distribution, when kurtosis < 3 , it represents the sample data is flat-topped distribution (platykurtic) and its tail is shorter and thinner than a normal distribution.

6.2 Parameter Estimation

Compared with a Gaussian distribution, a Levy process can better characterize the jump behaviour of innovations; however, the form of the Levy process distribution function is complicated. After combining the Levy process distribution function with GJR-GARCH model, there are many parameters to be estimated. If estimation is performed using the maximum likelihood method, the calculation efficiency will

Table 2 Estimated results for the Levy-GJR-GARCH model parameters.

GJR-GARCH model parameters				VG process parameters			
ω	α	β	δ	θ	σ	ν	-
0.0001*** (2.9711)	0.0050 (0.490)	0.7364*** (40.3428)	0.4746*** (9.9941)	-0.2131 (-1.2262)	1.0676*** (6.1442)	0.3740*** (2.1527)	-
NIG process parameters				CGMY process parameters			
λ	η	κ	-	C	G	M	Y
1.7806*** (5.1771)	-0.2396 (-0.6968)	0.8883*** (2.5827)	-	6.6594*** (76.5830)	3.1636*** (36.3812)	2.7937*** (32.1268)	1.6961*** (19.5048)

Remark: the numerical values in parentheses correspond to the t-statistics of the parameter values, * indicates significant at the 10% significance level, ** indicates significant at the 5% significance level, and *** indicates significant at the 1% significance level.

be very low. Nevertheless, the form of the Levy process moment condition obtained from the characteristic function is relatively simple, and the parameters of a Levy process can be estimated using the generalised method of moments. Therefore, to reduce the complexity of parameter estimation, this study used a two-step method to estimate the parameters of GJR-GARCH model and Levy process: in step 1, set innovations as Gaussian distribution and use maximum likelihood estimation to estimate the parameters of the GJR-GARCH model; in step 2, based on the innovations data obtained in step 1, use the generalised method of moments to estimate the parameters of the VG, NIG and CGMY models. The results of the parameters estimation are presented in Table 2, from which we can see that the “leverage effect” parameter δ of GJR-GARCH model is greater than 0. At the 1% significance level, the significance is not 0, indicating that the changes in volatility are clearly asymmetric, with downward fluctuations stronger than upward fluctuations.

Figures 4 and 5 show the time-varying volatility sequence and the innovations sequence, and Table 3 presents the descriptive statistics of innovations. From the characteristics of time-varying volatility, we can observe the following phenomena: the volatility has a relatively strong clustering characteristic, i.e., major volatility is followed by major volatility, and minor volatility is followed by minor volatility. There is a clear “leverage effect” in volatility, i.e., increases in volatility are faster than decreases in volatility. Therefore, using a skewed GARCH model to describe the volatility data is more consistent with reality. From the characteristics of the innovation data, we can see that volatility in innovations is not white noise; the skewness and kurtosis are $-0.4951 < 0$ and $4.9872 > 3$, respectively, indicating a leptokurtic, fat-tailed distribution. Therefore, a Levy process can provide higher accuracy than a Gaussian distribution.

6.3 Empirical Result Analysis

We downloaded the data for option prices for 70 American options transacted on the S&P 100 Index from the official website of the Chicago Board Options Exchange and performed a comparative analysis of different exercise prices and different time to expiration. The closing price of S&P 100 Index on March 23, 2017 was USD 1,040, i.e.

Fig. 4 Time-varying volatility. **Fig. 5** Innovations sequence diagram.

$S_0 = 1,040$. We used the 10-year T-bond yield as of March 23, 2017 as the risk-free interest rate, $r = 2.4\%$ (data source: official website of US Treasury Department). To examine the pricing effect of the Levy-GJR-GARCH model under a fuzzy environment, we compared the pricing result with that of the Levy-GJR-GARCH model under a clear environment. Under fuzzy theory, the volatility σ of an asset price is set as a fuzzy variable, whereas in the GARCH model, the volatility σ is set as a time-varying variable. To reduce the complexity of the fuzzy calculation, the membership func-

Table 3 Descriptive statistics regarding volatility and innovations.

Indicator	Sample size	Maximum value	Minimum value	Average value	Standard deviation	Skewness	Kurtosis
Time-varying volatility	1525	0.0005	0.0000	0.0000	0.0000	7.7168	88.3883
Innovation	1525	3.4999	-6.6548	0.0753	1.0338	-0.4951	4.9872

Table 4 Option pricing results.

Time to expiration	Exercise price	Market price	Least squares Monte Carlo approach						Binomial tree method					
			Fuzzy environment (expected value)			Clear environment			Fuzzy environment (expected value)			Clear environment		
			VG	NIG	CGMY	VG	NIG	CGMY	VG	NIG	CGMY	VG	NIG	CGMY
0.083	1000	2.78	4.19	2.87	3.21	1.98	1.75	3.59	3.50	1.87	2.89	2.81	1.08	2.72
0.083	1020	5.05	6.81	4.91	6.26	3.68	3.01	7.88	7.05	3.73	3.65	6.94	3.75	3.02
0.083	1040	10.14	10.83	11.63	11.75	10.33	10.69	13.38	15.90	10.87	12.77	14.79	9.92	13.32
0.083	1060	21.24	18.73	25.69	20.17	20.63	25.11	22.86	26.29	23.61	28.40	21.30	25.36	23.55
0.083	1080	39.00	32.26	45.85	32.59	26.65	44.81	30.63	50.63	41.21	45.16	40.41	44.10	43.74
0.167	1000	7.50	4.56	3.02	4.32	5.33	2.62	7.03	9.32	2.52	6.18	5.84	1.55	5.46
0.167	1020	11.10	7.59	5.59	8.34	14.04	7.27	9.04	16.88	6.66	10.17	12.78	6.20	7.32
0.167	1040	17.55	10.45	14.65	13.62	21.79	17.16	16.64	26.10	14.35	17.70	18.48	15.59	15.64
0.167	1060	27.60	18.46	26.63	24.59	26.06	32.83	23.49	37.04	26.30	33.17	22.45	29.08	30.11
0.167	1080	42.50	26.76	46.60	39.76	30.47	51.52	39.68	53.62	42.47	49.47	41.41	45.60	46.10
0.250	1000	11.45	7.67	6.10	6.84	19.33	4.13	9.99	16.14	4.62	10.13	9.84	5.36	15.98
0.250	1020	16.50	11.33	15.59	14.62	21.91	10.52	15.76	24.15	9.59	16.84	17.03	14.95	16.31
0.250	1040	22.80	15.44	25.90	18.31	24.28	21.73	17.78	33.54	17.57	21.52	24.50	26.62	26.63
0.250	1060	32.60	23.76	44.85	28.44	28.24	37.05	26.60	44.37	29.00	36.93	27.05	35.07	38.92
0.250	1080	45.70	32.37	54.81	43.22	33.05	55.54	41.52	56.54	43.94	53.01	43.05	55.01	55.06
0.333	1000	16.55	12.81	14.51	9.38	20.42	6.86	15.98	22.19	6.06	14.58	11.56	8.44	20.98
0.333	1020	21.75	16.20	24.40	18.23	23.03	13.93	18.67	30.51	11.41	23.54	19.18	21.32	24.86
0.333	1040	28.60	24.39	34.73	22.59	25.74	25.76	22.68	35.03	19.48	26.26	25.82	31.26	32.96
0.333	1060	37.70	29.02	43.68	30.67	28.35	41.78	28.24	46.79	30.62	39.63	32.48	43.12	45.45
0.333	1080	49.90	36.02	56.62	44.32	32.73	60.30	43.72	62.73	45.07	55.60	53.03	56.93	63.08
0.500	1000	24.40	15.22	27.14	13.97	25.17	10.69	16.56	30.69	9.53	20.52	17.97	10.92	22.31
0.500	1020	30.10	18.35	37.05	23.26	27.30	19.24	20.68	39.26	15.63	26.81	21.42	24.40	32.63
0.500	1040	37.40	27.26	45.31	28.81	29.73	31.88	26.95	48.84	24.05	29.61	30.67	32.48	35.95
0.500	1060	46.20	33.21	56.29	35.44	32.86	47.85	33.48	54.46	35.07	44.91	41.64	47.62	54.65
0.500	1080	57.40	39.85	68.19	48.93	36.78	66.00	49.31	71.08	48.84	60.72	54.31	62.99	65.80
0.750	1000	35.30	18.24	35.61	17.98	22.86	12.54	19.32	33.78	13.82	26.20	21.29	17.62	27.13
0.750	1020	41.60	24.30	45.46	25.75	24.14	22.35	24.79	42.71	20.51	35.63	28.88	27.24	38.93
0.750	1040	49.20	30.79	53.72	31.66	27.33	35.29	30.99	62.43	29.15	45.47	33.88	38.86	47.85
0.750	1060	57.80	35.77	70.61	41.52	30.41	51.38	35.39	73.03	39.95	60.76	50.33	52.16	59.81
0.750	1080	68.30	41.26	84.45	53.12	33.29	69.45	51.87	84.37	52.94	66.43	61.33	66.83	68.05

tion of the time-varying volatility $\{\tilde{\sigma}_t\}$ was set as an equal form of parabolic membership function. Because estimation of four parameter values was required for parameter interval of parabolic fuzzy numbers, the historical rates of return of 1525, 1200, 800 and 400 trading days before March 23, 2017 were selected as observation samples based on different market information reflected by different sampling intervals. The parameter values required for parabolic fuzzy numbers and the results of option pricing are listed in Table 4.

The expected values under a fuzzy environment presented in Table 4 were obtained from the upper and lower weights of the $\alpha = 0.95$ level set of fuzzy number $\tilde{V}_t$, and the exact formula is as follows:

$$\begin{aligned}
 M(\tilde{V}) &= \frac{M(\tilde{V})^L + M(\tilde{V})^U}{2} \\
 &= \frac{\int_0^1 f(\alpha) \tilde{V}_\alpha^L d\alpha + \int_0^1 f(\alpha) \tilde{V}_\alpha^U d\alpha}{2} \\
 &= \int_0^1 \frac{f(\alpha)}{2} (\tilde{V}_\alpha^L + \tilde{V}_\alpha^U) d\alpha
 \end{aligned} \quad (33)$$

Based on the simulation results presented in Table 4,

the option prices corresponding to different time to expirations and different exercise prices are shown in Figs. 6 and 7. From Fig. 6, we can see that as the time to expiration of option lengthens, the option price increases gradually; this increase in option price is because the uncertainty increases as time increases. We can see that when the time to expiration is shorter, the model's simulation results are clustered around the market price, whereas the simulation results are more dispersed when the time to expiration is longer. This result indicates that the model has higher simulation accuracy for short-term options than for long-term options. From Fig. 7, we can see that the higher the exercise price, the higher the option price, which is consistent with the actual situation. Because for American put options, the option price is bearish, higher exercise prices correspond to higher option profit and thus higher option prices. Moreover, we can clearly observe that the model simulation results for shorter period cluster around the market price, and as the time frame lengthens, the simulated curve becomes more dispersed.

From the above analysis, we know that the theoretical model has greater simulation accuracy for short-term options compared with long-term options; therefore, we se-

Fig. 6 Comparison of option prices with different time to expiration.

Remark: "Fuzzy" denotes simulation under a fuzzy environment, "Clear" denotes simulation under a clear environment, "lsm" denotes the least squares Monte Carlo approach, and "bt" denotes the binomial tree method. VG, NIG and CGMY are Levy processes. The red broken line with arrow is the trend line, the blue line is the market price.

Fig. 7 Comparison of option prices with different exercise prices.

Fig. 8 Option pricing results for April 2017 expiry.

Table 5 The relationship between option pricing results and membership function/level set.

Level set α	$n = 1$		$n = 2$		$n = 3$	
	Fuzzy interval	Fuzzy expectation	Fuzzy interval	Fuzzy expectation	Fuzzy interval	Fuzzy expectation
0.95	[8.71, 24.8]	16.76	[8.81, 23.91]	16.36	[8.84, 23.61]	16.23
0.90	[8.51, 26.6]	17.56	[8.7, 24.85]	16.78	[8.77, 24.24]	16.51
0.85	[8.31, 28.39]	18.35	[8.6, 25.81]	17.21	[8.7, 24.9]	16.80
0.80	[8.09, 30.18]	19.14	[8.48, 26.8]	17.64	[8.63, 25.58]	17.11
0.75	[7.89, 31.97]	19.93	[8.37, 27.82]	18.10	[8.54, 26.29]	17.42

lected 22 short-term option pricing results with expiry in April 2017 to further analyse the differences in option pricing under fuzzy and clear environments. The result is shown in Fig. 8. From the simulation result, we can see that all market prices fall within the fuzzy interval of the VG, NIG and CGMY models under a fuzzy environment, which shows that the market prices of options are better covered when a fuzzy price interval is used. In contrast with the smaller fuzzy interval of the VG model and the greater fuzzy interval of the NIG model, the fuzzy interval of the CGMY model offers better simulation results. Simultaneously, we observe that under a clear environment, the simulation results of the VG and CGMY models are greater than the market price when the exercise price is lower and less than the market price when the exercise price is higher, whereas the simulation result of the NIG model is less than the market price when the exercise price is lower and greater than the market price when the exercise price is higher.

For exercise price $K = 1,050$, Table 5 provides the option pricing result using the NIG-GJR-GARCH model under a fuzzy environment, in addition to the relationship between its membership function and level set. When the exponent of membership function n remains unchanged, the fuzzy interval narrows as the level set α increases while the fuzzy expectation lowers; when the level set α remains unchanged, the fuzzy interval narrows as the membership function exponent n increases, and the fuzzy expectation

also decreases. The membership function diagram for different exponent n is shown in Fig. 9, from which we can see that when the rate of return of the membership function is parabolic, the option price membership function is also parabolic, and when $n = 1$, the parabolic membership function becomes the trapezoidal membership function. The left half interval of the function is monotonically increasing, whereas the right half interval is monotonically decreasing. The monotonically increasing part reflects that the seller's satisfaction increases as the price increases, whereas the monotonically decreasing part reflects that the buyer's satisfaction reduces as the price falls. Figure 10 shows the relationship between the option price and the level set, and combined with the membership function diagram, it clearly shows the changes in the option price interval under different level sets. The fuzzy expectation lowers as the level set increases, which shows that the membership function diagram of option prices is asymmetric with an inclined left tendency, which is consistent with the result shown in Fig. 9, where n represents the exponent of the parabolic fuzzy variable, when value of n varies, the shape of the membership function of the parabolic fuzzy variable will change accordingly; when $n = 1$, the parabolic fuzzy variable converts to a trapezoidal fuzzy variable; when $n = 2$, it represents the classical parabolic fuzzy variable.

Figure 11 shows the optimal option exercise boundaries when the time to expiration $T = 0.5$ year, exercise

Fig. 9 Diagrams of membership function.

Fig. 10 Relational chart between option price and level set.

price $K = 1,060$, and the exponent of membership function n takes on different values. The optimal option exercise boundary shifts higher as time goes. When $n = 1$, the position of the exercise boundary is at its highest, and when $n = 3$, the position of the exercise boundary is at its lowest. Since the price at the optimal exercise boundary satisfies $V(t) = K - S(t)$, as time progresses, $S(t)$ will decrease gradually. At the same time, we can see that as the expiry date approaches, the optimal exercise boundary tends to even out. At this point, if there is any sharp decrease in the option price, it may trigger early option exercise.

For the purpose of a comprehensive comparison of the accuracy of various option pricing models, we used four types of statistical methods – RMSE (root mean square error), RMSRE (root mean square relative error), AARE (average absolute relative error) and AAE (average absolute error) – to perform an error comparison of the pricing results

obtained from different models; the results are presented in Table 6. These four indicators quantify the deviation between the pricing result and the market price; the lower the value obtained, the higher the pricing accuracy. The calculation formulas for these indicators are as follows:

$$\text{RMSE} = \sqrt{\sum_{i=1}^N \frac{(C_i^{\text{Model}} - C_i^{\text{Market}})^2}{N}} \quad (34)$$

$$\text{RMSRE} = \sqrt{\sum_{i=1}^N \frac{(C_i^{\text{Model}} - C_i^{\text{Market}})^2}{N(C_i^{\text{Market}})^2}} \quad (35)$$

$$\text{AAE} = \frac{\sum_{i=1}^N |C_i^{\text{Model}} - C_i^{\text{Market}}|}{N} \quad (36)$$

$$\text{AARE} = \frac{\sum_{i=1}^N |C_i^{\text{Model}} - C_i^{\text{Market}}|}{NC_i^{\text{Market}}} \quad (37)$$

Fig. 11 The optimal exercise boundary.

Table 6 Pricing error of the models.

Model	RMSE	RMSRE	AAE	AARE
Fuzzy lsm-VG	9.622701	0.305166	7.231907	0.279272
Fuzzy lsm-NIG	5.876694	0.204024	4.453933	0.160488
Fuzzy lsm-CGMY	7.072131	0.230915	5.227281	0.198432
Clear lsm-VG	10.257905	0.321387	7.839648	0.287161
Clear lsm-NIG	7.177533	0.316647	5.297496	0.251002
Clear lsm-CGMY	8.001463	0.256294	5.800416	0.223266
Fuzzy bt-VG	9.817527	0.328137	8.210713	0.304872
Fuzzy bt-NIG	10.408733	0.362997	8.196607	0.304096
Fuzzy bt-CGMY	7.276172	0.249951	4.463557	0.225336
Clear bt-VG	10.974434	0.202534	8.509217	0.361526
Clear bt-NIG	7.715236	0.404925	8.244162	0.327514
Clear bt-CGMY	9.041845	0.293794	4.018898	0.258982

The results in Table 6 indicate that the least squares Monte Carlo approach has a better pricing accuracy than the binomial tree method regardless of whether pricing occurs in a fuzzy or clear environment. The simulation results of the least squares Monte Carlo approach under a fuzzy environment are better than the results under a clear environment, and among the different models of the least squares Monte Carlo approach, the NIG model achieves the most prominent enhancing effect. Comparing the pricing effects of the VG, NIG and CGMY models, the NIG model in the least squares Monte Carlo approach has the best effect, followed by the CGMY model, with the VG model exhibiting the poorest effect; in the binomial tree method, the CGMY model has the best effect, followed by the VG and NIG models. The above outcomes show that fuzzy environment has great significance in the study of option pricing theory.

6.4 Improvement of the Least Squares Monte Carlo Approach

In the evaluation of the improvement effect of quasi-random number and Brownian Bridge approach on the least squares Monte Carlo approach, we are mainly concerned with whether the improved method has increased the convergence speed. Therefore, this section uses the NIG-GJR-GARCH model as an example in the dynamic analysis of

Fig. 12 Convergence process of the least squares Monte Carlo approach and the improved calculation method.

the convergence process of the pricing result. The results of a comparison of the convergence speeds of different calculation methods are shown in Fig. 12. It appears that under the least squares Monte Carlo simulation, it takes at least 5,000 simulations before the option pricing result can converge accurately to the average value. On the other hand, using the improved method, even within only 2,000 simulations, the pricing result can be kept within the reliable range. This result indicates that for option pricing, the improved method can effectively increase the convergence speed, reduce the number of simulations required, and increase the pricing efficiency.

7. Conclusions and Future Directions

Due to the existence of the problem of the optimal stopping time, studies regarding American option pricing problems are far more complicated than those corresponding to European option pricing problems, and the traditional B-S model is not suitable for American option pricing. Taking into account the stochastic volatility of the underlying asset price, leverage effect, stochastic jump and other characteristics, this study constructed a Levy-GJR-GARCH American option pricing model based on an infinite pure jump process. We also incorporated fuzzy theory and set the underlying asset price volatility as a parabolic fuzzy number, established the fuzzy least squares Monte Carlo approach for American put options and the fuzzy binomial tree model, and deduced the model's pricing calculation method. Lastly, using the S&P 100 Index and data for the corresponding American put options, we empirically tested the theoretical model; comparatively analysed the pricing effect of the VG, NIG, and CGMY models under a fuzzy environment; and compared the results with the option pricing under a clear environment. The main conclusions are as follows: 1. there is significant volatility clustering and strong leverage effects and random jump characteristics in the S&P 100 Index prices. 2. The option price increases as the length of the time to expiration of options increases and as the ex-

ercise price increases; in addition, the simulation accuracy for short-term option prices is greater than for medium- and long-term options. 3. Under a fuzzy environment, the market price of short-term options can be better covered by the fuzzy interval of the VG, NIG and CGMY models, and the membership function curve of the option price is asymmetric with an inclined left tendency, whereas the fuzzy interval narrows as the level set α and the exponent of membership function n increase. 4. The model's simulation accuracy is significantly greater under a fuzzy environment than under a clear environment, and the pricing accuracy of the least squares Monte Carlo approach is greater than that of the binomial tree method, whereas among different Levy processes, the NIG and CGMY models yield better simulation results than the VG model. 5. The convergence efficiency is significantly improved by using quasi-random numbers and the Brownian Bridge approach to improve the least squares Monte Carlo approach. The subject of this research was relatively straightforward American put options, without taking into account the pricing of more complex American options, and the membership functions of fuzzy variables were assumed to have fixed forms, without considering the situation in which the membership degree is also a fuzzy number. Therefore, future research may focus on incorporating fuzzy theory in the analysis of the pricing of new American option types and other derivative products, finding a more reasonable method to study the fuzzy variable's membership function and other related problems.

References

- [1] H.E. Johnson, "An analytic approximation for the American put price," *Journal of Financial and Quantitative Analysis*, vol.18, no.1, pp.141–148, 1983.
- [2] R. Geske and H.E. Johnson, "The American put option valued analytically," *Journal of Finance*, vol.39, no.5, pp.1511–1524, 1984.
- [3] J.C. Cox, S.A. Ross, and M. Rubinstein, "Option pricing: A simplified approach," *Journal of Financial Economics*, vol.7, no.3, pp.229–263, 1979.
- [4] M.J. Brennan and E.S. Schwartz, "Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis," *Journal of Financial and Quantitative Analysis*, vol.13, no.3, pp.461–474, 1978.
- [5] F.A. Longstaff and E.S. Schwartz, "Valuing American options by simulation: A simple least-squares approach," *Review of Financial Studies*, vol.14, no.1, pp.113–147, 2001.
- [6] J.-C. Duan, "The GARCH option pricing model," *Mathematical Finance*, vol.5, no.1, pp.13–32, 1995.
- [7] M. Saez, "Option pricing under stochastic volatility and stochastic interest rate in the Spanish case," *Applied Financial Economics*, vol.7, no.4, pp.379–394, 1997.
- [8] L. Stentoft, "American option pricing using simulation: An introduction with an application to the GARCH option pricing mode," *Handbook of Research Methods and Applications in Empirical Finance*, A. Bell, C. Brooks, and M. Prokopczuk, eds., Edward Elgar, 2012. Available at SSRN: <https://ssrn.com/abstract=2021193>
- [9] S.Z. Levendorskiĭ, "Early exercise boundary and option prices in Lévy driven models," *Quantitative Finance*, vol.4, no.5, pp.525–547, 2004.
- [10] R.A. Maller, D.H. Solomon, and A. Szimayer, "A multinomial approximation for American option prices in Lévy process models," *Mathematical Finance*, vol.16, no.4, pp.613–633, 2006.
- [11] D.B. Madan and E. Seneta, "The variance gamma (V.G.) model for share market returns," *Journal of Business*, vol.63, no.4, pp.511–524, 1990.
- [12] O.E. Barndorff-Nielsen, "Processes of normal inverse Gaussian type," *Finance and Stochastics*, vol.2, no.1, pp.41–68, 1997.
- [13] P. Carr, H. Geman, D.B. Madan, and M. Yor, "The fine structure of asset returns: An empirical investigation," *Journal of Business*, vol.75, no.2, pp.305–332, 2002.
- [14] E.A. Daal and D.B. Madan, "An empirical examination of the variance-gamma model for foreign currency options," *Journal of Business*, vol.78, no.6, pp.2121–2152, 2005.
- [15] A.N. Avramidis and P. L'Ecuyer, "Efficient Monte Carlo and quasi-Monte Carlo option pricing under the variance gamma model," *Management Science*, vol.52, no.12, pp.1930–1944, 2006.
- [16] S. Song, J. Jeong, and J. Song, "Asymptotic option pricing under pure-jump Lévy processes via nonlinear regression," *Journal of the Korean Statistical Society*, vol.40, no.2, pp.227–238, 2011.
- [17] P. Christoffersen, R. Elkamhi, B. Feunou, and K. Jacobs, "Option valuation with conditional heteroskedasticity and nonnormality," *Review of Financial Studies*, vol.23, no.5, pp.2139–2183, 2010.
- [18] S.J. Byun and B. Min, "Conditional volatility and the GARCH option pricing model with non-normal innovations," *Journal of Futures Markets*, vol.33, no.1, pp.1–28, 2013.
- [19] C.-F. Lee, G.-H. Tzeng, and S.-Y. Wang, "A new application of fuzzy set theory to the Black-Scholes option pricing model," *Expert Systems with Applications*, vol.29, no.2, pp.330–342, 2005.
- [20] L. Maciel, F. Gomide, and R. Ballini, "Evolving fuzzy-GARCH approach for financial volatility modeling and forecasting," *Computational Economics*, vol.48, no.3, pp.379–398, 2016.
- [21] W.-Q. Liu and S.-H. Li, "European option pricing model in a stochastic and fuzzy environment," *Applied Mathematics—A Journal of Chinese Universities*, vol.28, no.3, pp.321–334, 2013.
- [22] Z.-Y. Feng, J.T.-S. Cheng, Y.-H. Liu, and I.-M. Jiang, "Options pricing with time changed Lévy processes under imprecise information," *Fuzzy Optimization and Decision Making*, vol.14, no.1, pp.97–119, 2015.
- [23] R. Breen, "The accelerated binomial option pricing model," *Journal of Financial and Quantitative Analysis*, vol.26, no.2, pp.153–164, 1991.
- [24] M. Broadie and J. Detemple, "American option valuation: New bounds, approximations, and a comparison of existing methods," *Review of Financial Studies*, vol.9, no.4, pp.1211–1250, 1996.
- [25] E. Clément, D. Lamberton, and P. Protter, "An analysis of a least squares regression method for American option pricing," *Finance and Stochastics*, vol.6, no.4, pp.449–471, 2002.
- [26] L. Stentoft, "Assessing the least squares Monte-Carlo approach to American option valuation," *Review of Derivatives Research*, vol.7, no.2, pp.129–168, 2004.
- [27] L. Stentoft, "Value function approximation or stopping time approximation: A comparison of two recent numerical methods for American option pricing using simulation and regression," *Journal of Computational Finance*, vol.18, no.1, pp.65–120, 2014.
- [28] J. Kienitz and D. Wetterau, *Financial modelling: Theory, implementation and practice with MATLAB source*, John Wiley & Sons, 2012.
- [29] Y. Yoshida, M. Yasuda, J.-I. Nakagami, and M. Kurano, "A new evaluation of mean value for fuzzy numbers and its application to American put option under uncertainty," *Fuzzy Sets and Systems*, vol.157, no.19, pp.2614–2626, 2006.
- [30] S. Muzzioli and H. Reynaerts, "American option pricing with imprecise risk-neutral probabilities," *International Journal of Approximate Reasoning*, vol.49, no.1, pp.140–147, 2008.
- [31] L.A. Zadeh, "Fuzzy sets as a basis for a theory of possibility," *Fuzzy Sets and Systems*, vol.100, Supplement 1, pp.9–34, 1999.
- [32] B. Liu and Y.-K. Liu, "Expected value of fuzzy variable and fuzzy expected value models," *IEEE Trans. Fuzzy Syst.*, vol.10, no.4, pp.445–450, 2002.
- [33] B. Liu, "Uncertainty theory," *Uncertainty Theory*, pp.205–234,

Springer, 2007.

- [34] L.R. Glosten, R. Jagannathan, and D.E. Runkle, "On the relation between the expected value and the volatility of the nominal excess return on stocks," *Journal of Finance*, vol.48, no.5, pp.1779–1801, 1993.
- [35] P. Christoffersen, R. Elkamhi, B. Feunou, and K. Jacobs, "Option valuation with conditional heteroskedasticity and nonnormality," *Review of Financial Studies*, vol.23, no.5, pp.2139–2183, 2010.
- [36] I.M. Sobol, "Uniformly distributed sequences with an additional uniform property," *USSR Computational Mathematics and Mathematical Physics*, vol.16, no.5, pp.236–242, 1976.

Huiming Zhang received an M.Sc. degree from Xidian University, Xi'an, China. He is currently working towards a Ph.D. at the Graduate School of Information, Production and Systems of Waseda University, Japan. His research interests include, financial engineering, economics analysis, fuzzy set theory, management engineering and meta-heuristic algorithms.

Junzo Watada received his B.Sc. and M.Sc. degrees in electrical engineering from Osaka City University, Japan, and his Ph.D. degree from Osaka Prefecture University, Japan. Currently, he is a professor in the Computer and Information Sciences Department, PETRONAS University of Technology, and a Professor Emeritus at Waseda University. He received the Henri Coanda Medal Award from Inventico in Romania in 2002. He is a Life Fellow of the Japan Society for Fuzzy Theory and intelligent

informatics (SOFT). He is a Co-principal Editor, a Co-Editor and an Associate Editor of various international journals, including *IDT journal*, *ICIC Information Sciences*, *International Journal of Systems and Control Engineering*, and *Fuzzy Optimization & Decision Making*. His professional interests include soft computing, tracking systems, knowledge engineering, financial engineering and management engineering.