Compressed Sensing in Magnetic Resonance Imaging Using Non-Randomly Under-Sampled Signal in Cartesian Coordinates

Ryo KAZAMA†, Kazuki SEKINE††, Nonmembers, and Satoshi ITO†(a), Member

SUMMARY Image quality depends on the randomness of the k-space signal under-sampling in compressed sensing MRI (CS-MRI), especially for two-dimensional image acquisition. We investigate the feasibility of non-random signal under-sampling CS-MRI to stabilize the quality of reconstructed images and avoid arbitrariness in sampling point selection. Regular signal under-sampling for the phase-encoding direction is adopted, in which sampling points are chosen at equal intervals for the phase-encoding direction while varying the sampling density. Curvelet transform was adopted to remove the aliasing artifacts due to regular signal under-sampling. To increase the incoherence between the measurement matrix and the sparsifying transform function, the scale of the curvelet transform was varied in each iterative image reconstruction step. We evaluated the obtained images by the peak-signal-to-noise ratio and root mean squared error in localized 3×3 pixel regions. Simulation studies and experiments showed that the signal-to-noise ratio and the structural similarity index of reconstructed images were comparable to standard random under-sampling CS. This study demonstrated the feasibility of non-random under-sampling based CS by using the multi-scale curvelet transform as a sparsifying transform function. The technique may help to stabilize the obtained image quality in CS-MRI.

key words: compressed sensing, curvelet transform, L1-norm, sparseness

1. Introduction

Reducing image acquisition time is one of the most important considerations for clinical magnetic resonance imaging (MRI). An effective way to speed up MRI is under-sampling the k-space signal. However, under-sampling violates the Nyquist sampling criterion, resulting in aliasing artifacts in the reconstructed images. Recent advances in signal restoration from measurements sampled below the Nyquist rate, called compressed sensing (CS) [1], [2], have had a major impact on MRI [3]. According to CS theory, acquisition time may be reduced significantly without additional hardware.

CS theory states that successful reconstruction from an under-sampled MR signal requires the restricted isometry property (RIP) condition. The most convenient and practical way to realize the RIP condition is under-sampling the k-space signal in a random manner. In application of CS with Cartesian coordinate sampling, which is the most widely used with MRI in clinical use, random under-sampling is executed only for the phase-encoding direction, since signal acquisition for the read-out direction is very fast compared to the phase-encoding direction. In such cases, most of the energy of images is concentrated close to the k-space origin, therefore variable density random under-sampling in which denser sampling is executed close to k-space origin, is often adopted in CS-MRI. Several methods have been proposed for variable-density random under-sampling. Lustig proposed an under-sampling method in which sampling density scaling was in accordance with a power of the distance from the k-space origin [3]. Vasanawala et al. have shown that signal under-sampling according to a Poisson-disc distribution [4] provides a high degree of incoherence and at the same time a uniform distance between samples [5]. Knoll et al. proposed an adapted random under-sampling method in which sampling patterns were created from power spectra of existing reference data sets. Their results demonstrated an obtained image quality compatible to that of an established model-based strategy with optimization of model parameters [6]. Without this prior knowledge, it is difficult to choose an adequate signal sampling. The appearance of aliasing artifacts in the image will vary depending on the manner of under-sampling, i.e., the distribution of sampling density or randomness of signal acquisition points, even if the amount of acquired signal is the same. It sometimes happens in CS calculation that a certain under-sampling pattern which shows high-quality images for one image sometimes is not good for another MR image due to the size of images in terms of field of view or the distribution of the k-space signal. The selection of the signal sampling pattern is a bothersome problem in CS. One of the drawbacks of CS is the instability of image quality depending on the pattern of k-space random under-sampling.

In this paper, a new image reconstruction method using a non-randomly under-sampled signal is proposed and demonstrated, in which k-space signals are under-sampled regularly while varying the sampling density. The proposed method has an advantage over standard compressed sensing in that the quality of the obtained images is guaranteed to be independent of the randomness of the sampling pattern. To reduce aliasing artifacts caused by under-sampling at equal intervals, we adopt the curvelet transform [7]. The curvelet transform was originally introduced by Candès and Donoho to overcome the drawbacks of wavelet analysis. The transform was designed to represent edges and other singularities along curves much more efficiently than tradi-
tional transforms; therefore, edges and textures of MR images are reconstructed well in the field of image denoising and CS by using the curvelet-based method [8]–[11]. Many sparsifying transform-like wavelets or discrete cosine transform functions divide the input image into orthogonal directions, i.e., the horizontal and vertical directions in the two-dimensional (2D) image case. Strong aliasing artifacts due to under-sampling tend to have strong signals in the sparsiﬁed space and therefore it is very hard to remove those aliasing artifacts by thresholding in the sparsiﬁed space. In contrast to those types of transform, curvelet transform localizes the Fourier transform of an input image near sheared wedges obeying parabolic scaling. Input images are divided for an angular direction and a radial direction in the Fourier domain. Since the direction of image decomposition is not aligned with the under-sampling direction, i.e., the mutual coherencies between the Fourier basis and curvelet basis are small, aliasing artifacts are well decomposed into curvelet space and as a result they can be removed by thresholding in that space. To further improve the performance of aliasing artifact reduction, we introduce the multi-scale curvelet transform in which the scale of image decomposition is varied between the iterative reconstruction steps. A comparison of performances for removing the aliasing artifacts was carried out to examine the effectiveness of a multi-scale curvelet transform.

2. Methods

2.1 Reconstruction Using L1 Minimization

Letting the image of interest be a vector \( \rho \), \( s \) be the observed \( k \)-space data from the scanner, and \( A \) be the measurement matrix, the reconstruction is obtained by solving the following problem:

\[
s = A\rho. \tag{1}\]

Since MR images contain smooth parts and textures, parts of the image may be represented in a highly sparse manner by one transform and others do not. Like the image structures, aliasing artifacts due to under sampling may be strongly detected in a one transform and others do not. So, we propose to solve an optimization problem that involves \( d \) different transforms of the imaging object \( \rho \).

\[
\rho_{cs} = \arg \min_{\rho} \{ \| \Psi_1 \rho \|_1 + \| \Psi_2 \rho \|_1 + \ldots + \| \Psi_d \rho \|_1 \} \\
\text{subject to } s = A\rho, \tag{2}\]

where \( \Psi_1, \Psi_2 \) and \( \Psi_d \) mean the sparsifying transform function having different basis and \( \| \cdot \|_1 \) indicates the L1 norm.

2.2 Curvelet Transform

According to CS theory, two points are the key to successful reconstruction from an under-sampled MR signal [3]. One is the incoherency between the sampling matrix and the basis of the sparsifying transform function, and the second is the sparsity which is introduced by the sparsifying transform function [3]. Therefore, the sparsifying transform function has a very important role in CS.

The curvelet transform [7] was designed to represent edges and other singularities along curves much more efficiently than traditional transforms. Unlike the wavelet transform, it has directional parameters for an angular direction, and the curvelet pyramid contains elements with a very high degree of directional specificity. The curvelet transform has gone through two major revisions. In this paper, the newly constructed version of the curvelet transform, the so-called second-generation curvelet, is used. This new technique is simpler, faster, and less redundant than the original curvelet transform.

Second-generation curvelets are defined in terms of scale \( 2^{-j} \), orientation \( \theta = \pi t2^{-j/2}/2 \) (where \( [a] \) denotes the smallest integer greater than or equal to \( a \)), and position \( x_k^l = R_\theta^{-1}(2^{-j}x_1, 2^{-j/2}x_2) \) by translation and rotation of the mother curvelet \( \varphi_j \) as

\[
\varphi_{j,l,k}(x) = \varphi_j(R_\theta(x - x_k^l)), \tag{3}\]

where \( R_\theta \) is the rotation by \( \theta \) radians, \( \theta \) being an equi-spaced sequence of rotational angles with integer \( l \) such that \( 0 \leq \theta < 2\pi \), and \( x_k = (x_1, x_2) \in \mathbb{Z}^2 \) is the sequence of translation parameters. The waveform \( \hat{\varphi}_j \) is defined by means of its Fourier transform \( \hat{\varphi}_j(\nu) \), written in polar coordinates in the Fourier domain:

\[
\hat{\varphi}_j(r, \theta) = 2^{-j/2} \hat{\varphi}(2^{-j/2}r) e^{j2\pi \theta/2}. \tag{4}\]

The support of \( \hat{\varphi}_j \) is a polar parabolic wedge defined by the support of radial window \( \hat{\varphi} \) and angular window \( \hat{e} \), each of which has a scale-dependent window width in each direction. In continuous frequency \( \nu \), the curvelet coefficients of data \( f(x) \) are defined by inner product

\[
c_{j,l,k} := \langle f, \varphi_{j,l,k} \rangle = \int_{\mathbb{R}^2} f(x) \hat{\varphi}_j(R_\theta \nu) e^{jx_k^l \cdot \nu} \, dx. \tag{5}\]

The new implementation of the curvelet transform, i.e., discrete curvelet transform, takes a 2D image as an input in the form of a Cartesian array. Figure 1 (a) illustrates the continuous curvelet frequency tiling by combining all frequency responses of the curvelet at different scales and orientations. Equation (5) allows the curvelet transform to
be implemented in the frequency domain. In the second-
generation curvelet transform, a 2D DFT is applied to the
image and then “wedge product” is obtained; the result is
wrapped around the origin, and 2D inverse DFT (IDFT) is
applied, resulting in discrete curvelet coefficients.

The discrete curvelet transform that we used can divide
the input $N \times N$ pixel image for higher and lower frequency
levels: $r$ times ($r \leq \log_2 N - 2$) levels for the radial
direction and $4p$ ($p$: integer) levels at the second-coarsest level
for the angular direction. Figure 1(b) shows an example for
$r = 3$ and $p = 4$ where the spatial frequency for the ra-
dial direction is divided into two levels three times, and the
second-coarsest level indicated by the arrow has 16 levels
for the angular direction.

The reconstruction algorithm that we used to solve the
minimization problem shown in Eq. (2) is based on the Split
Bregman method[12] in which multiple sparsifying trans-
form functions are used successively [10],

1) input : $(\rho^0, \rho^0_1, \ldots, \rho^0_d, b^0_1, \ldots b^0_d) = (0, 0, 0, 0, 0)$. (6)

2) $\rho^{m+1} = \frac{1}{2\mu + 1} \Delta^T s^m + \frac{1}{2} \left( I_N \right) - \frac{1}{2\mu + 1} \Delta^T A \sum_{g=1}^{d} \left( \rho^m_g - b^m_g \right)$

3) for $g = 1$ to $d$

   $\rho_g^{m+1} = \Psi^T_g S_{1/\mu} \left( \Psi_g \left( \rho^{m+1}_g - b^m_g \right) \right)$

4) for $g = 1$ to $d$

   $b_g^{m+1} = b_g^m + \rho^{m+1}_g - \rho_g^{m+1}$

endfor

5) $s^{m+1} = s^m + s - A \rho^{m+1}$

where $\Psi_g$ are curvelet transforms and $g$ means the indices of
$d$ kind of curvelet transform; $\mu$ ($0 < \mu$) is coefficients; $b^m_g$
is a variable relating to the subgradient of minimization norm;
$S_{1/\mu}$ is a soft thresholding function with threshold value $1/\mu$,
and $I_N$ is the identity matrix, and $m$ is an index of iteration
step.

3. Reconstruction Experiments

3.1 PSF Evaluation

We used Cartesian grid sampling in 2D image scanning
because it is the most widely used method in commercial
MRI. Under-sampling of the signal is executed only for the
phase-encoding directions. Consideration of a point-spread
function (PSF) analysis is helpful for understanding the be-

Considering the shift of sampling points in the half of $k_y$ direction
($k_y < 0$), then under-sampling function $P_s(k_y)$ can be written
using unit function $u(k_y)$ ($u(k_y) = 1$ for $k_y \geq 0$ and $u(k_y) = 0$
for $k_y < 0$) as:

$$P_s(k_y) = H(k_y) u(k_y) + H(k_y - \Delta k_y) (1 - u(k_y))$$

where $\Delta k_y$ indicate the amount of sampling point shifting
for $k_y$ direction. Let $p_s(y)$ be the inverse transform of $P_s(k_y)$
which corresponds to the PSF of sampling pattern then it can be
described as:

$$p_s(y) = h(y) \ast \left[ \frac{1}{2} \delta(y) + \frac{1}{i2\pi y} + h(y) \exp[i\Delta k_y y] \ast \left( \frac{1}{2} \delta(y) - \frac{1}{i2\pi y} \right) \right],$$

where $\delta(y)$ is the Dirac delta function, ‘$\ast$’ means the convo-
lution integral and $h(y)$ denote the inverse Fourier transform
of $H(k_y)$. Now, we consider the SPR of $p_s(y)$. The issue of
SPR is the maximum of sidelobe signal contained in PSF sig-
nal as shown in Eq. (11). The amplitude of sidelobe signal is
mainly defined by $(1/2)\delta(y)$, since the function $1/(i2\pi y)$ be-
have as edge detection of signal to be convoluted in Eq. (14)
and has small contribution to resultant peak signal. If the
maximum of sidelobe signal is obtained in the point $y_m$, then
the maximum amplitude of $p_s(y_m)$ can be approximated as:

$$|p_s(y_m)| \approx \frac{1}{2} |h(y_m)| \left[ 1 + \exp[i\Delta k_y y_m] \right]$$

$$\leq |h(y_m)|$$

Since PSF of under-sampling function with no sampling point
shifting is $h(y)$, Eq. (15) indicates that maximum of
sidelobe signal, i.e. SPR will be decreased by sampling point
shifting. To reduce the SPR and coherent aliasing artifacts

$$\text{SPR} = \max_{i \neq j} \left| \frac{\text{PSF}(i, j)}{\text{PSF}(i, i)} \right|.$$ (11)

Thresholing in the sparsified space will remove the alias-
ing artifacts caused by under-sampling and thus decrease
SPR. Therefore, the relation between the SPR decrease and
the choice of sparsifying transform function was examined.
In all cases, the threshold value is determined based on the
standard deviation of SPR for random under-sampling SPR,
which is formulated as follows [3]:

$$\sigma_{\text{SPR}} = \sqrt{\frac{(N/n) - 1}{N}},$$ (12)

where $n$ is the number of samples taken and $N$ is the num-
ber of grid points defining the underlying image. Since non-
random under-sampling is adopted in our study, strong co-
herent artifacts appeared and background noise-like artifacts
were smaller than $\sigma_{\text{SPR}}$, which is why we use $\sigma_{\text{SPR}}$ as the
reference value for thresholding.

Let the function that define the manner of under sam-
pling for phase encoding direction $k_y$ be $H(k_y)$. Consider-
ing the shift of sampling points in the half of $k_y$ direction
($k_y < 0$), then under-sampling function $P_s(k_y)$ can be written
using unit function $u(k_y)$ ($u(k_y) = 1$ for $k_y \geq 0$ and $u(k_y) = 0$
for $k_y < 0$) as:

$$P_s(k_y) = H(k_y) u(k_y) + H(k_y - \Delta k_y) (1 - u(k_y))$$

where $\Delta k_y$ indicate the amount of sampling point shifting
for $k_y$ direction. Let $p_s(y)$ be the inverse transform of $P_s(k_y)$
which corresponds to the PSF of sampling pattern then it can be
described as:

$$p_s(y) = h(y) \ast \left[ \frac{1}{2} \delta(y) + \frac{1}{i2\pi y} + h(y) \exp[i\Delta k_y y] \ast \left( \frac{1}{2} \delta(y) - \frac{1}{i2\pi y} \right) \right],$$

where $\delta(y)$ is the Dirac delta function, ‘$\ast$’ means the convo-
lution integral and $h(y)$ denote the inverse Fourier transform
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maximum of sidelobe signal is obtained in the point $y_m$, then
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$$|p_s(y_m)| \approx \frac{1}{2} |h(y_m)| \left[ 1 + \exp[i\Delta k_y y_m] \right]$$

$$\leq |h(y_m)|$$

Since PSF of under-sampling function with no sampling point
shifting is $h(y)$, Eq. (15) indicates that maximum of
sidelobe signal, i.e. SPR will be decreased by sampling point
shifting. To reduce the SPR and coherent aliasing artifacts
on the image, we used sampling point shifting for real-value image evaluation, since all spin phase are aligned before image acquisition and therefore conspicuous artifacts tend to appear due to regular under-sampling. Figure 2(a) shows a non-random under-sampling pattern defined by a 256×256 matrix used in this examination, where white lines indicate the sampling points in k-space. PSF after non-random under-sampling is shown in Fig. 2(b) and the close-up of the central 40×40 point region are shown in Fig. 2(c). Aliasing artifacts are distributed along the y-axis, since under-sampling is executed only for the k_y direction. Figure 2(d) shows a one-dimensional profile of PSF along the y-axis. PSFs after thresholding in the curvelet domain with varying radial scale r and angular scale p are shown in Figs. 2(e)-(g) and PSF after the thresholding in the wavelet domain (Daubechies’ 12-tap wavelet) is shown in Fig. 2(h). Threshold value in the sparsified space is set as \( \sigma_{SPR} \). As shown, SPRs for curvelet domain thresholding are smaller than that for wavelet domain thresholding. Successive curvelet domain thresholding using different scaling parameters (which is what we refer to as “the multi-scale curvelet transform”) is executed in Eq. (8). Table 1 shows the effectiveness of successive curvelet domain thresholding. Initial SPR after the non-random under-sampling is 0.341. SPFs after successive thresholding varying the number or the order of radial scale r fixing the angular scale p = 3 are listed in Table 1 compared with successive wavelet domain thresholding using Daubechies N = 6 wavelet. Since real-value constraint is applied to reconstructed PSFs, SPF values are decreased even though the same curvelet scale or wavelet basis are used. Aliasing artifacts are drastically reduced as the step of curvelet domain thresholding increases with the use of different scaling parameters. After curvelet domain thresholding three times, SPR is reduced from 0.341 to 0.122. It was also shown that the decrease of SPR is not so sensitive to the order of scaling parameter r.

Figure 3 shows the results of another examination of the effectiveness of the multi-scale curvelet transform. Figure 3(a) shows zero-filled reconstructed images (256×256 pixels) of a numerical phantom using the non-random under-sampling shown in Fig. 2(a). Strong aliasing artifacts due to non-random under-sampling are visible in the image. Figure 3(b) shows the image after single-time wavelet domain thresholding (Daubechies’ 12-tap wavelet) and Figs. 3(c), (d), and (e) show the images after single-time curvelet domain thresholding using the same threshold level (255×0.04) with radial scale \( r = 3, 4, 5 \) and 5, respectively, and the fixed angular scale \( p = 3 \). Bottom row shows the images after convergence. Figure 3(f) shows multi-scale curvelet \((r = 3, 4, 5, p = 3)\) and Figs. (g) to (j) correspond to (b) to (e), respectively. Figure 3 indicates that the manner of artifact reduction is different depending on the scaling of curvelet transform. In Fig. 3(c), the original structures of the images remain even though aliasing artifacts strongly remain. In contrast, most of the aliasing artifacts have been removed in Fig. 3(e), even though much of the original structures of the images have been lost. We tried to remove the aliasing artifacts due to non-random under-sampling by introducing the multi-scale curvelet transform in which the scales of image decomposition were varied cyclically in an iterative reconstruction procedure. Figure 3(f) shows the converged images. Almost aliasing artifacts are removed, while some artifacts remained in single-scale curvelet based

![Figure 2](image2.png)

**Fig. 2** PSF of non-random under-sampling and artifacts reduction by wavelet and curvelet domain thresholding: (a) non-random under-sampling pattern, (b) PSF of the sampling pattern (a), (c) close look of (b), (d) one-dimensional profile of PSF (c), (e), (f), (g) after curvelet domain thresholding using different scaling parameters, (h) wavelet domain thresholding.

![Figure 3](image3.png)

**Fig. 3** Obtained images after sparsified domain thresholding. Upper row shows images after single-time thresholding, (a) zero-filled reconstructed images, (b) wavelet domain, (c), (d), (e) curvelet domain using scaling \((r, p) = (3, 3), (4, 3), (5, 3)\), respectively, Bottom row shows converged images, (f) multi-scale curvelet transform \((r = 3, 4, 5, p = 3)\), (g), (h), (i), (j) correspond to (b), (c), (d), (e), respectively. It was shown that the manner of artifacts reduction is different depending on the scaling of curvelet transform.

**Table 1** Decrease of SPR by successive curvelet domain thresholding. Number in the parenthesis (·) means the radial scale \( r \) of curvelet transform. Threshold value is different with that of Fig. 2.

<table>
<thead>
<tr>
<th>initial SPR</th>
<th>Curvelet transform step-1</th>
<th>Curvelet transform step-2</th>
<th>Curvelet transform step-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.341</td>
<td>→ (3) 0.250 → (3) 0.213 → (3) 0.176</td>
<td>→ (4) 0.251 → (4) 0.221 → (4) 0.186</td>
<td>→ (5) 0.251 → (5) 0.192 → (5) 0.161</td>
</tr>
<tr>
<td></td>
<td>→ (4) 0.250 → (4) 0.218 → (5) 0.122</td>
<td>→ (5) 0.213 → (4) 0.180 → (3) 0.124</td>
<td>Wavelet transform (Daubechies N=6)</td>
</tr>
<tr>
<td></td>
<td>→ 0.313 → 0.287 → 0.248</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
was applied to half of the estimation. As for sampling-point shifting, one-point shifting with a Gaussian distribution by varying the standard deviation acquired 35% of patterns (SD-1 to SD-4) were examined. Each sampling pattern used in the proposed method and Fig. 4 (b) shows the distribution of sampling density along the line of the 1st step, real-valued images were used to compare reconstructed images using different types of sparsifying transform functions. MR normal volunteer (5 men) images obtained using a 1.5 T MRI scanner (Vantage, Canon medical, Tokyo, Japan) were used in these reconstruction experiments. In the 2nd step, phase-varying T2 weighted images acquired by fast spin echo method (TE/TR = 352/3500 ms; flip angle 90 degree, 240 × 256 matrix; spatial resolution 1.1 mm) were used to evaluate the effectiveness of the proposed method for complex-value images. Informed consent was obtained from the volunteers. In the numerical experiments, fully sampled echo signal data were acquired first and then the echo signals were under-sampled to simulate reduced sampling in the phase-encoding direction. Since almost all the signal energy is concentrated around the center of k-space, variable density random under-sampling in which denser sampling is executed near the k-space origin is often adopted in applications of CS. Figure 4 (a) shows the sampling pattern used in the proposed method and Fig. 4 (b) shows the distribution of sampling density along the line shown in Fig. 4 (a). The horizontal axis in Fig. 4 (b) shows the indices of the k-space signal for the phase-encoding direction, where an index value of 128 corresponds to the origin of k-space and the right end of the axis is the maximum spatial frequency. The vertical axis shows the sampling density of non-random under-sampling. Since the sampling density is symmetric with respect to the origin of k-space, only half of k-space is shown. Four types of under-sampling patterns (SD-1 to SD-4) were examined. Each sampling pattern acquired 35% of k-space signal and was approximated with a Gaussian distribution by varying the standard deviation. As for sampling-point shifting, one-point shifting was applied to half of the k-space plane, since n_r(n_r > 1) point shifting will make no change for an under-sampling pattern acquiring every n_r-th point signal. In reconstruction experiments, the sparsifying transform function was based on curvelet transform. We used decreasing thresholds [10] in all curvelet domain and initial threshold value 1/μ was set to 4.0 σ_{ε(0)}, where σ_{ε(0)} denotes the standard deviation of reconstruction error due to the k-space under-sampling. The threshold value is decreased by a factor 0.9 with each iteration step, this factor value having been determined from a preliminary examination. Figure 5 shows the results of a peak-signal-to-noise ratio (PSNR) evaluation using 20 kinds of images, where PSNR is defined as follows:

$$\text{PSNR} = 20 \log_{10} \frac{\max[\rho_f]}{\text{RMSE}},$$

in which RMSE is the root mean square of the reconstruction error calculated using the difference of fully scanned image ρ_f and reconstructed image. In each of the result for SD-1 to SD-4, left- and right-hand sides are regular under-sampling without and with sampling-point shifting, respectively. As shown, SPR was reduced in all four sampling patterns by the introduction of sampling-point shifting, thereby improving the resulting PSNR of the reconstructed images. These results indicate that sampling-point shifting is effective for real-value images to reduce aliasing artifacts in equispaced regular sampling CS.

3.3 Comparison of Sparsifying Transform Function

Since PSNR is calculated using the RMSE based on the mean of squared error in the whole image space, we cannot evaluate the magnitude of localized error from PSNR that contaminate the image as noticeable aliasing artifacts. Therefore, we introduce a new measure of image evaluation, localized root mean squared error (LMSE), to evaluate the magnitude of localized error in the image. We define LMSE is follows:

$$\text{LMSE} = \max_{j} \frac{\max_{i} [\sigma_{k}(i, j)]}{\max[\rho_f]},$$

$$\sigma_{k}(i, j) = \left[ \frac{1}{9} \sum_{p, q = -1}^{1} (\rho_f(i+p, j+q) - \rho_c(i+p, j+q))^2 \right]^{1/2}$$

Similar sampling density having the same standard deviation is used for the random under-sampling as that of SD-1. Figures 6 (a) and (b) show PSNR and LMSE obtained...
with the proposed method and random under-sampling CS. The reconstruction methods are shown on the vertical axis as “Non-random” for equi-spaced based non-random under-sampling and “Random” for variable density random under-sampling in which eight different sampling patterns having the same variable sampling density are used. Since PSNR and LMSE of reconstructed images vary depending on the sampling pattern of the signal in random under-sampling, variation of PSNR and LMSE is also shown in those graphs; the two ends of the horizontal thin line on the PSNR and LMSE bars indicate the error bar (± one standard deviation) of the PSNR and LMSE values. “Multi-scale Curvelet” indicates the multi-scale curvelet transform using radial scales of the curvelets cycling successively through the values as $r = 2, 3, 4,$ and 5 fixing the angular scale as $p = 3$. Preliminary experiments varying the order of radial scale values were performed. Results showed was that the variation of the obtained image PSNR is very small, so we used this order throughout this paper. It was shown that the PSNR of the Non-random Multi-scale Curvelet transform is greater than that of Random Wavelet and Random Curvelet. The PSNR of Non-random Multi-scale Curvelet is slightly smaller than that of Random Multi-scale Curvelet; however, the minimum value on the error bar of Multi-scale Curvelet is smaller than the average PSNR of Non-random Multi-scale Curvelet, which means that PSNR for non-random sampling sometimes takes a higher value than that of random under-sampling, depending on the sampling pattern. In Fig. 6(b), LMSE of Non-random Multi-scale Curvelet is much smaller than that of Random Wavelet and is comparable to that of Random Curvelet. The LMSE of Non-random Multi-scale Curvelet is greater than that of Random Multi-

![Fig. 6](image)

**Fig. 6** Comparison of PSNR and LMSE. Both ends of horizontal thin line on the PSNR and LMSE bar in the random under-sampling method indicate the error bar (± one standard deviation) in the variation of sampling pattern.

Figure 7 shows comparisons of PSNR and structural similarity index (SSIM) between Non-random and random under-sampling in the Multi-scale Curvelet method varying the signal reduction factor. SSIM is a quality metric used to measure the similarity between two images [13]. It is considered to be correlated with the quality perception of the human visual system and is defined as follows:

$$\text{SSIM} = \left( \frac{2\mu_r\mu_t + C_1}{\mu_r^2 + \mu_t^2 + C_1} \right) \left( \frac{2\sigma_r\sigma_t + C_2}{\sigma_r^2 + \sigma_t^2 + C_2} \right) \left( \frac{\sigma_{rt} + C_3}{\sigma_r\sigma_t + C_3} \right)$$

(19)

where $r$ and $t$ indicate the reference and test images, respectively. The first term on the right-hand side is the luminance comparison function, which measures the closeness of the mean luminances $\mu_r$ and $\mu_t$ of the two images. The second term is the contrast comparison function, which measures the closeness of the contrast between the two images in terms of the standard deviations $\sigma_r$ and $\sigma_t$. The third term is the structure comparison function, which measures the correlation coefficient between the two images. In this term, $\sigma_{rt}$ is the covariance between the reference and test images. The positive constants $C_1, C_2,$ and $C_3$ are used to avoid a null denominator and are defined as follows:

$$C_1 = (K_1L)^2, \quad C_2 = (K_2L)^2, \quad C_3 = C_2/2$$

(20)

where $L$ is the maximum of pixel value, and $K_1$ and $K_2$ are small constants. We use $K_1 = 0.01$ and $K_1 = 0.03$ following the original paper reporting this index [13]. The vertical bars on the random under-sampling plots show the maximums and minimums of PSNR and SSIM, which depend on the pattern of under-sampling in $k$-space. The proposed method results in PSNR and SSIM values close to the mean for random under-sampling regardless of the signal reduction factor.

Figure 8 shows reconstructed images for each method; Fig. 8(a) shows the fully scanned image and Fig. 8(b) shows...
image of the multi-scale curvelet transform with random under-sampling. Figures 8 (c) to (e) show the reconstructed images using curvelet, Multi-scale curvelet and wavelet transforms, respectively. Error images for each method are shown in Figs. 8 (f) to (i) and enlarged views of red rectangular region shown Fig. (a) are shown in Figs. (j) to (n). Significant artifacts remained in wavelet-based CS as shown in Fig. 8 (i), whereas somewhat smaller artifacts are shown in Figs. 8 (f) to (h). The distribution of the remaining artifacts with the proposed method is different in the point that remained artifacts is not uniform in the image space, as shown in Fig. 8 (h).

3.4 Application to Phase-Varying Images

MR images have spatial phase distribution due to imperfectness of MR equipment or electromagnetic susceptibilities of tissues; therefore, application of the proposed method to phase-varying images was performed. When phase variation on the image is small, spatial phase map can be estimated using low band k-space signal with high accuracy and obtained images can be treated as real-value images after correcting spatial phase variation using the estimated phase map. In those cases, sampling-point shifting is effective to reduce the coherent artifacts. In contrast, CS reconstruction with rapid phase variations faces a different problem, i.e., it is difficult to estimate phase distribution map using low-band k-space signal. Therefore not only the magnitude but also phase regularization is required in the CS framework. The regularizer for phase used in the cost function is non smooth and sometime fails in handle big jumps in the wrapped phase map. To avoid the image degradation due to the estimation error of phase distribution, symmetrical sampling CS was proposed and demonstrated [14]. Since the real and imaginary parts of complex images can be reconstructed independently by synthesizing the signals $\frac{1}{2}[s(k) + s^*(-k)]$ for the real part of the image and $-i\frac{1}{2}[s(k) - s^*(-k)]$ for the imaginary part of the image, the estimation of phase distribution is not necessary and resultant images provides a better quality of images compared to the method [14] that used both magnitude and phase regularizer in the CS framework [15]. We used the symmetric sampling method to proposed non-random sampling CS. A symmetric sampling pattern based on SD-1 was used. Sampling point shifting was not used in this experiments to satisfy the symmetrical sampling. Figure 9 shows a resulting reconstructed image. Figures 9 (a) and (b) show the magnitude and phase map of the fully scanned image, and Fig. 9 (c) shows the image reconstructed by the proposed...
method. Enlarged views of red rectangular region shown is Fig. (a) corresponding to Figs. (a) and (c) are shown shown in Figs. (d) and (e).

4. Discussion

Mutual coherence is the maximum absolute value of the inner product between any two normalized matrixes. Mutual coherence is convenient as a measure of the coherences between two functions. If mutual coherence between the observation function and the sparsifying transform function is small, then the artifacts will be distributed in a random noise-like manner in the sparsified space. Reconstructed images from non-random under-sampling using wavelet transform have strong aliasing artifacts, as shown in Figs. 8 (e) and (i), with a small obtained PSNR relative to that from random under-sampling. Wavelet transform is designed to characterize point-like images and it decomposes input images into the orthogonal directions, namely, the row and column directions. Since one of these directions is parallel with the under-sampling direction in 2D Cartesian coordinates, mutual coherencies between the Fourier basis and wavelet basis become larger, and as a result, coherent aliasing artifacts are likely to remain in the reconstructed images. On the other hand, image decomposition in curvelet transform is executed in the radial and angular directions, and since most of these directions are not parallel with the under-sampling direction, mutual coherencies between the Fourier basis and curvelet basis are small. Since the direction of the curvelet basis is not in line with the Fourier basis, the distribution of PSF after thresholding extends in radial directions with smaller amplitude, as shown in Figs. 2 (e)-(g), even though most artifacts remain along the y-axis in PSF after wavelet domain thresholding. As a result, SPR in curvelet domain thresholding is small and aliasing artifacts are much more likely to be removed compared to wavelet domain thresholding, as shown in Fig. 2. To encourage incoherence between the Fourier basis and curvelet basis, we introduce the multi-scale curvelet transform, in which the scale or the level of image decomposition is varied in an iterative reconstruction procedure. Reconstruction experiments showed that aliasing artifacts were further reduced and the magnitude of artifacts is comparable to those for images obtained by random under-sampling based CS.

The sampling strategy of the proposed method is somewhat similar to that of parallel imaging in the sense that signals are regularly under-sampled to shorten the scan time. Preliminary studies about the application of the proposed CS to parallel imaging SENSE [16] have been performed and achieved promising results for the two-coil parallel imaging case [17]. Further studies are required for the combination of the proposed CS and parallel imaging.

In general, there exists phase variation in MR images due to the inhomogeneity of the static magnetic field or differences of susceptibility in the human tissue. Sampling-point shifting in k-space provides phase shifts of the images according to the shifting theorem of Fourier transform, so the phase of the images given experimentally will decrease the coherency of artifacts due to equi-spaced under-sampling like sampling-point shifting in k-space. Two methods are introduced in this study according to the magnitude of phase variation. One is the case when phase variation on the image is rather small and obtained images can be treated as real-value images after phase correction. Sampling-point shifting is effective to reduce the coherent aliasing artifacts as shown in Fig. 8. In the other case, when phase changes rapidly, symmetrical sampling is effective to obtain images with reduced artifacts. Figure 9 indicated that proposed method has the possibility of being applicable to phase-varying images as well as real-value images.

The proposed algorithm inherits the fast execution speed of projection-based CS reconstruction. Even if the scale of curvelet transform is varied in the iterative reconstruction, the reconstruction time is not lengthened compared to the single-scale curvelet transform reconstruction, since the curvelet scale changes with the iterative reconstruction step and overall the number of curvelet domain thresholding values is the same as that of single-scale curvelet transform.

The selection of a signal sampling pattern in CS is complex work and sometimes fails in image reconstruction due to a mismatch between the sampling pattern and the signal distribution of k-space, especially in 2D imaging. The proposed method has the advantage that the quality of the obtained images is guaranteed to be stable in the sense of PSNR and SSIM as well as visual inspection compared to standard random under-sampling based CS. Future work is a comparison of proposed method with random under-sampling for three- or four-dimensional images. Since randomness of sampling point selection will be much more increased in high-dimensional images, it is expected that the obtained image quality will be guaranteed to be much more stable even in random under-sampling based CS.
5. Conclusion

An alternative approach to CS-MRI that can use a non-randomly under-sampled signal is proposed and demonstrated to guarantee the obtained image quality in terms of, for example, the signal-to-noise ratio and structural similarity index. To increase the incoherencies between the Fourier basis and the sparsifying transform function, we adopted two strategies; one is the use of a curvelet transform as the sparsifying transform function and varying the scaling parameter in the iterative reconstruction procedure, and the other is a sampling-point shifting in k-space for images with small phase variation. Artifacts in the reconstructed images were drastically reduced by the proposed method and comparable PSNR and SSIM values were obtained with random under-sampling based CS using the multi-scale curvelet transform. Since the quality of reconstructed images does not depend on the randomness of the sampling trajectory, use of CS will become more practical. Further study includes examination of the proposed method in practical applications.

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References


Ryo Kazama received his B.S. degrees from the Department of Information Science of Utsunomiya University in 2017. His research interests include system development and noise elimination methods for new MR imaging techniques.

Kazuki Sekine received his B.S. and M.D. degrees from the Utsunomiya University. His research interests include compressed sensing MRI and other new MR imaging techniques.

Stoshi Ito received his B.E. and M.E. degrees from the Department of Electrical Engineering of Utsunomiya University in 1987 and 1989, then joined Toshiba Co. He received Ph.D. degree from the Utsunomiya University in 2001. He has been a professor in the Department of Information Science of Utsunomiya University since 2012. His research interests include MR high-speed imaging techniques using compressed sensing and deep learning, and image reconstruction and image processing methods. He is a member of the International Society of Magnetic Resonance in Medicine (ISMRM), Japanese Society for Magnetic Resonance in Medicine (JSMRMR) and the Japanese Society of Medical Imaging Technology.