A Note on Enumeration of 3-Edge-Connected Spanning Subgraphs in Plane Graphs

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SUMMARY  This paper deals with the problem of enumerating 3-edge-connected spanning subgraphs of an input plane graph. In 2018, Yamanaka et al. proposed two enumeration algorithms for such a problem. Their algorithm generates each 2-edge-connected spanning subgraph of a given plane graph with \( n \) vertices in \( O(n^2) \) time, and another one generates each 3-edge-connected spanning subgraph of a general graph with \( m \) edges in \( O(mT) \) time, where \( T \) is the running time to check the \( k \)-edge connectivity of a graph. This paper focuses on the case of the 3-edge-connectivity in a plane graph. We give an algorithm which generates each 3-edge-connected spanning subgraph of the input plane graph in \( O(n^2) \) time. This time complexity is the same as the algorithm by Yamanaka et al., but our algorithm is simpler than theirs.

**key words:** enumeration, algorithm, spanning subgraph, edge-connectivity

1. Introduction

In this paper, we present a simple algorithm that enumerates all the 3-edge-connected spanning subgraphs of a given plane graph with \( n \) vertices in \( O(n^2) \) time, using reverse search method by Avis and Fukuda [1]. For this problem, Yamanaka et al. proposed the algorithm that enumerates all the \( k \)-edge-connected spanning subgraphs of a given general graph with \( n \) vertices and \( m \) edges. Their algorithm generates each \( k \)-edge-connected spanning subgraph of the input graph in \( O(mT) \) time, where \( T \) is the running time to check the \( k \)-edge-connectivity of a graph. From the result by Gabow [3], it can be observed that \( T = O(m + k^2 n \log(n/k)) \) holds. So, their algorithm enumerates all the \( k \)-edge-connected spanning subgraphs in an input plane graph and requires \( O(n^2 \log n) \) time per each. Note that \( m \leq 3n - 6 \) holds for a plane graph. On the other hand, this paper also gives an algorithm that generates each \( k \)-edge-connected spanning subgraph of an input plane graph in \( O(n^2) \) time by using an algorithm which can check \( 3 \)-edge-connectivity in \( O(n) \) time for a given plane graph. In 2017, for a given graph, Mehlhorn et al. already proposed an algorithm for checking \( 3 \)-edge-connectivity in \( O(n) \) time [5]. So, both algorithms require the same time complexity. However, ours is very simple and short. It has an advantage.

2. Preliminary

Let \( G = (V(G), E(G)) \) be an undirected unweighted graph with vertex set \( V(G) \) and edge set \( E(G) \). We always denote \( |V(G)| \) and \( |E(G)| \) by \( n \) and \( m \), respectively. A graph \( G \) is simple if \( G \) has no multi-edges and no self-loop. Throughout this paper, we suppose that graphs are simple unless otherwise noted. An edge set \( S \) of a graph \( G \) is called a bond if \( G - S \) is not connected, while \( G - S' \) is connected for any proper subset \( S' \) of \( S \). A graph \( G \) is \( k \)-edge-connected if \( G \) does not have any bond of size at most \( k - 1 \). Let us remark that, for \( k = 1 \), a 1-edge-connected graph is just a connected graph. A graph \( H = (V(H), E(H)) \) is a subgraph of \( G \) if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \) hold. A subgraph \( H = (V(H), E(H)) \) of \( G \) is spanning if \( V(H) = V(G) \).

Throughout this paper, we assume that the edges in \( E(G) \) are labeled such as \( E(G) = \{e_1, e_2, \ldots, e_m\} \). Let \( e_i \) and \( e_j \), \( i < j \), be two edges in \( G \). We say that \( e_i \) is smaller than \( e_j \), denoted by \( e_i < e_j \). Let \( e \) be an edge of \( G \). For a subgraph \( H \) of \( G \), we denote by \( H - e \) the graph obtained from \( H \) by removing \( e \). We denote by \( H + e \) the graph obtained from \( H \) by inserting \( e \). A graph is planar if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A plane graph is a planar graph with a fixed planar embedding. A plane graph \( G \) divides the plane into connected regions called faces. We put a vertex \( f^* \) in each face \( f \) of \( G \), and for each edge \( e \) in \( G \), we connect by the edge \( e^* \) the vertices \( f^* \) and \( g^* \) in \( G^* \), where \( f \) and \( g \) are the faces of \( G \) whose contours share the edge \( e \). Then we call the obtained graph the dual graph of \( G \), and is denoted by \( G^* \). For an edge set \( S \) in \( G \), let \( S^* = \{e^*|e \in S\} \). A cycle is a circuit in which the only repeated vertices are the first and last vertices. The length of a cycle is the number of edges involved. The following is a well-known lemma, which will be used for our algorithm.

**Lemma 1:** An edge set \( S \) in a connected plane graph \( G \) is bond if and only if \( S^* \) forms a cycle in \( G^* \).

3. Enumerating All \( k \)-Edge-Connected Spanning Subgraphs in Graphs

In this section, we introduce results by Yamanaka et al. [7], [8]. They proposed the algorithm for generating all \( k \)-edge-connected spanning subgraphs. The algorithm based on the reverse search [1] traverses the tree structure in depth-first...
search manner.

Let $G$ be a $k$-edge-connected graph with $k \geq 1$. We denote by $S_k(G)$ the set of $k$-edge-connected spanning subgraphs of $G$. Note that $G$ itself is in $S_k(G)$. To see a tree structure, we define the parent for each $k$-edge-connected spanning subgraph of $G$ except $G$. Let $H$ be a $k$-edge-connected spanning subgraph in $S_k(G) \setminus \{G\}$. Let $sm(H)$ be the smallest edge in $E(G) \setminus E(H)$. Then, we define $par(H) := H + sm(H)$. From the definition, it is easy to observe that $par(H)$ is also $k$-edge-connected spanning subgraph of $G$. By repeatedly finding the parents starting from $H$, we obtain a sequence $H, par(H), par(par(H)), \ldots$ of $k$-edge-connected spanning subgraphs. We call such a sequence the appending sequence of $H$. The sequence starts with $H$ and ends with $G$. In [7] and [8], they proved the following lemma and proposed two algorithms for $k = 2$ and a general case.

**Lemma 2:** Let $H \neq G$ be a $k$-edge-connected spanning subgraph of a graph $G$. Then, the appending sequence of $H$ always ends with $G$.

**Theorem 1:** Let $G$ be a plane graph with $n$ vertices. One can generate every 2-edge-connected spanning subgraph of $G$ in $O(n)$ time for each.

**Theorem 2:** Let $G$ be a graph. One can generate each $k$-edge-connected spanning subgraph of $G$ in $O(mT)$ time for each, where $T$ is the running time to check whether a graph is $k$-edge-connected.

From the result by Gabow [3], it can be observed that $T = O(m + k^2n \log(n/k))$ holds.

Here, we focus on the case of $k = 3$. In 2017, Mehlhorn et al. proposed an algorithm for checking 3-edge-connectivity of an input graph [5]. The time complexity of their algorithm is a linear time. Therefore, by using their checking algorithm, the second algorithm enumerates all the 3-edge-connected spanning subgraphs in an input plane graph and requires $O(n^2)$ time per each.

### 4. Enumerating All 3-Edge-Connected Spanning Subgraphs in Plane Graphs

In this section, we focus on 3-edge-connected spanning subgraphs in plane graphs. For this purpose, as mentioned above, the algorithm obtained by Yamanaka et al.’s one and Mehlhorn et al.’s one generates all 3-edge-connected spanning subgraphs in $O(n^2)$ time per each. Our algorithm also requires the same time complexity per each and bases on Yamanaka et al.’s algorithm whose case is $k = 2$. The key idea is Lemma 1.

To generate all 2-edge-connected subgraphs in a plane graph $G$, Yamanaka et al. used in the proof of Theorem 1 the dual graph $G^*$ of $G$. In their algorithm, they check multiple edges in $G^*$, whose corresponding edges in $G$ cannot be deleted to generate 2-edge-connected subgraphs of $G$ by Lemma 1.

In the case of 3-edge-connected subgraphs, we focus on a cycle of length 3 in the dual graph $G^*$ of $G$ to find those edges that cannot be deleted to keep 3-edge-connectedness.

In the following, we give our main theorem.

**Theorem 3:** Let $G$ be a plane graph with $n$ vertices. One can generate each 3-edge-connected spanning subgraph of $G$ in $O(n^2)$ time per each.

**Proof.** Let us estimate the running time required for a subgraph $H \in S_3(G)$ in the family tree. When $H$ is generated from its parent $par(H)$, the dual graph is updated. This takes $O(n)$ time. To generate a child of $H$, for each edge $e$ in $H$ with $e < sm(H)$, we check 3-edge-connectivity of $H - e$.

Recall that, from Lemma 1, it is sufficient to check cycles whose length is 3 in the dual graph $G^*$ of $G$. To find such cycles efficiently, we check whether or not two end points of each edge $e^*$ have a common neighbor in $G^*$, which requires $O(n)$ time for each edge $e^*$ in $G^*$. If there exists such a vertex, then $e^*$ is contained in a cycle of length 3, and hence we cannot delete the corresponding edge $e$ from $G$. Thus, this check can be done in $O(n)$ time for each and $O(m)$ time in total. Therefore, our algorithm requires $O(n^2)$ time for each 3-edge-connected spanning subgraph in $G$, since $m \leq 3n - 6$ holds in a plane graph. Q.E.D.

### 5. Conclusion

We have proposed the algorithm for enumerating 3-edge-connected spanning subgraphs. Our algorithm enumerates all the 3-edge-connected spanning subgraphs of a given plane graph with $n$ vertices in $O(n^2)$ time for each. On checking 3-edge-connectivity, although it is the same time complexity of the algorithm by Mehlhorn et al. [5], our algorithm is extremely simple, which is also an advantage.

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**References**


