Estimation of a Long-Term Variation of a Magnetic-Storm Index Using the Merging Particle Filter

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SUMMARY The Dst index is the most popular measure of a scale of magnetic storms, and it is widely used as a monitor of the conditions of the Earth’s magnetosphere. Since the Dst index contains contributions from multiple magnetospheric phenomena, it is important to distinguish each of the contributions in order to obtain meaningful information about the conditions of the magnetosphere. There have been several efforts which modeled temporal evolution of the Dst index empirically, and these empirical models consider some contributions separately. However, they take only short-term variations into account, and contributions from phenomena which show long-term variations are neglected. In the present study, we have developed a technique for estimating the component of long-term variations of the Dst index using solar wind data and a nonlinear empirical model. The newly-developed technique adopts an algorithm which is similar to the particle filter. This algorithm allows an on-line processing of a long sequence of Dst data, which would enable a real-time estimation of system variables in a nonlinear system model. The estimates of the long-term variations can be used for accurate estimation of other contributions to the Dst index, which would provide credible information about the conditions of the magnetosphere. The framework proposed in the present study could be applied for the purpose of continuous real-time monitoring of the environment of the magnetosphere.

key words: merging particle filter, time-series analysis, geomagnetic data

1. Introduction

Magnetic storm is a global phenomenon which causes great decrease in the magnetic field on the ground at low and mid latitudes. The degree of the decrease is commonly measured by Dst index [1], which is derived from magnetic-field data at four different ground observatories at mid latitudes. The Dst index represents a deviation of the magnetic field at low latitudes from a normal state. The value of Dst is thus near zero in a normal state. During magnetic storm, it decreases typically by 50–100 nT and by hundreds of nT for strong magnetic storm. The Dst index is calculated for every hour, and its hourly data have been accumulated and published for about 50 years since 1957.

It is widely accepted that the decrease of Dst index during magnetic storm is caused by an electric current flowing westward in the magnetosphere, which is called the ring current (Fig. 1; see also, e.g., [2]). It is also well known that an electric current flowing eastward on the outer boundary of the magnetosphere, which is called the magnetopause current, also has a significant effect on the value of Dst as depicted in Fig. 1. The enhancement of the magnetopause current causes the increases of Dst, while the enhancement of the ring current causes the decrease of Dst. Hence, the variations of Dst are mostly a mixture of the ring current effect (RC effect) and the magnetopause current effect (MPC effect).

The variations of the ring current and the magnetopause current are closely associated with the conditions of the solar wind, which streams outward from the sun, although these two currents depend on different parameters of the solar wind. Burton et al. [3] have empirically modeled the temporal evolution of the Dst index using some solar wind parameters. They decomposed the Dst variations into the RC effect and the MPC effect, and then consider the temporal variation of each of two effects separately. They assumed the MPC effect to be given as a function of one of the solar-wind parameters, the solar-wind dynamic pressure Pd. The RC effect was described as a nonlinear evolutive system which requires the solar-wind electric field as an input.

Their model (hereafter we will refer to it as the Burton’s model) was constructed under the assumption that all the variations of Dst are due to the RC effect and/or the MPC effect, and they assumed other contributions to the Dst index to be constant, that is, independent from time. However, it is not trivial whether a variation of another contribution is really negligible or not. In particular, a long-term magnetic field variation which can not be represented by the Burton’s model might make some significant influence over the Dst index. As a matter of fact, we can estimate whether another factor may significantly contribute to Dst variations...
on the basis of the Burton’s model itself using long-term data of the \(D_{st}\) index and the solar wind.

The purpose of the present paper is to show how we can evaluate another contribution to \(D_{st}\) variations and to estimate how large such a contribution is. The estimation of another contribution is conducted using the merging particle filter (MPF) [4]. This algorithm is applicable to nonlinear system models like the Burton’s model and it is effective for applying to a very long-term sequence of data because it is an on-line algorithm which can process long-term data sequentially.

2. Base Model of the Temporal Variation of the \(D_{st}\) Index

The \(D_{st}\) index is influenced by both the RC effect and the MPC effect. Burton et al. took an approach to model the MPC effect as the first step and then eliminate it from \(D_{st}\) to obtain a pure RC effect:

\[
D_{RC} (nT) = D_{st} - b \sqrt{P_d} + c \tag{1}
\]

where \(D_{RC}\) represents the pure RC effect, \(P_d\) denotes the dynamic pressure of the solar wind which is observed by spacecraft out of the magnetosphere, and \(b\) and \(c\) are the parameters which should be given a priori. The second term in the right-hand side of this equation, \(b \sqrt{P_d}\), corresponds to the MPC effect, and the third term \(c\) corresponds to other contributions other than the RC effect and the MPC effect. This means that the Burton’s model assumes that the MPC effect is given by \(b \sqrt{P_d}\) and that other contributions are given by the constant \(c\). The temporal evolution of the pure RC effect \(D_{RC}\) can then be modeled as follows:

\[
\frac{\Delta D_{RC}}{\Delta t} = Q - \frac{D_{RC}}{\tau} \tag{2}
\]

where the parameters \(Q\) and \(\tau\) should be given a priori.

The original paper by Burton et al. provided the optimal value of each constant coefficient in Eqs. (1) and (2). However, the optimal values have been revised by O’Brien and McPherron (2000) [5] using a much longer term of the data. According to them, the parameters \(b\) and \(c\) in Eq. (1) can be given as constants:

\[
b = 7.26 \text{nT/nPa}^{-1/2} \tag{3}
\]

\[
c = 11 \text{nT}. \tag{4}
\]

The parameter \(Q\), which represents the evolution of \(D_{RC}\), can be given as follows:

\[
Q (\text{nT/hour}) = -4.4H(E - 0.49) \tag{5}
\]

where \(H\) denotes the Heaviside function as

\[
H(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
0 & \text{otherwise.} \end{cases} \tag{6}
\]

The variable \(E\) is given as

\[
E = vB_z \tag{7}
\]

where \(v\) is the solar-wind velocity and \(B_z\) is the southward component of the solar-wind magnetic field. Both \(v\) and \(B_z\) can be observed by spacecraft out of the magnetosphere as well as \(P_d\). Finally, \(\tau\) is given as follows:

\[
\tau (\text{hours}) = 2.4 \exp\left(\frac{9.74}{4.69 + H(E)}\right). \tag{8}
\]

Once an initial value of the \(D_{st}\) index is given, we can predict the evolution of \(D_{st}\) sequentially using the model given by Eqs. (1) and (2) as long as the solar-wind data are available. Hence, by comparing the prediction with the actual \(D_{st}\) values, we can validate the Burton’s model and the values of coefficients obtained by O’Brien and McPherron and evaluate contributions from other factors than the RC effect and the MPC effect. In the following sections, we perform the prediction by putting the solar-wind data into the Burton’s model.

3. Modeling of the System

3.1 System Model

In order to perform the prediction of the \(D_{st}\) index, we construct a state space model on the basis of the Burton’s model. We decompose a state of \(D_{st}\) at time \(t\) into three components as

\[
D'_{st} = D'_{RC} + D'_{MPC} + D'_{res} \tag{9}
\]

where \(D'_{st}\) denotes a model \(D_{st}\) value at time \(t\), \(D'_{RC}\) denotes the RC effect on \(D_{st}\), \(D'_{MPC}\) denotes the MPC effect, and \(D'_{res}\) denotes a residual effect other than the RC and MPC effects on \(D_{st}\), which corresponds to \(-c\) in Eq. (1) and is of our main interest in the present paper.

According to Eqs. (2) and (5), we describe a transition of a state of \(D_{RC}\) for an hour as

\[
D'_{RC} = D^{-1}_{RC} - 4.4 H(E^{t-1} - 0.49) - \frac{D^{-1}_{RC}}{\tau^t}. \tag{10}
\]

The MPC effect \(D'_{MPC}\) is assumed to be given as a function of \(P_d\) as

\[
D'_{MPC} = b \sqrt{P_d}. \tag{11}
\]

As for a transition of a state of \(D_{res}\), two models are considered in order to evaluate whether a variation of \(D_{res}\) is negligible or not. One model assumes \(D_{res}\) to be constant as

\[
D'_{res} = -11. \tag{12}
\]

according to Eq. (4). The other model is given as

\[
D'_{res} \sim N(D^{-1}_{res}, 0.01). \tag{13}
\]

We also consider state transitions of the two solar-wind
parameters $P_d$ and $E$. As described above, these two parameters can be observed by spacecraft. However, spacecraft basically observe local structures of the solar wind and the local structures observed by spacecraft do not necessarily agree with large-scale solar-wind structures which controls the conditions of the Earth's magnetosphere. Then, we assume that 'effective' $P_d$ and $E$ are uncertain and we include them in variables to be estimated. Thus, it is necessary to model the state transitions of them. The state of $P_d$ at time $t$ is assumed to obey a log-normal distribution as

$$
\log P_d^t \sim N(\log P_d^{t-1}, 0.02)
$$

(14)

because the solar-wind dynamic pressure $P_d$ can not be less than zero. The variance of $P_d$ was determined by a maximum-likelihood method using a set of OMNI2 hourly solar-wind data $\bar{P}_d$. The OMNI2 solar-wind data were provided on the OMNIWeb database of National Space Science Data Center, NASA (http://omniweb.gsfc.nasa.gov) and they are also referred to Sect. 3.2. A transition of $E$ is represented using a Cauchy distribution in order to allow large jumps which are sometimes observed in $E$. We assume that $E'$ obey the Cauchy distribution with a location parameter of $E'^{-1}$ and a scale parameter of 1. The scale parameter for the transition of $E$ are given subjectively. However, since it is associated with short-term variations, it does not make any significant effects on the estimates of $D_{res}$. In addition, we assume that $\tau$ is time-dependent. We give $\tau'$ as

$$
\tau' = 2.4 \exp \left( \frac{9.74}{4.69 + H(E'^{-1})} \right)
$$

(15)

where

$$
E'^{-1} \sim N(E^{-1}, 0.25).
$$

(16)

3.2 Observation Model

From the system described above, we can obtain observations of three variables. Space craft observations provide the data of $P_d$ and $E$ at each hour, although these data are sometimes lost. We define the observation model of $P_d$ and $E$ at each hour as

$$
\bar{P}_d^t \sim N(P_d^t, 1)
$$

(17)

$$
\bar{E}'^t \sim N(E', 4)
$$

(18)

where $\bar{P}_d^t$ and $\bar{E}'^t$ denote the observations of $P_d$ and $E$ at each hour, respectively. In this study, we refer to the OMNI2 solar-wind hourly data as the data of $\bar{P}_d$ and $\bar{E}'$. Although the original OMNI2 data do not contain $\bar{E}'$ data, we generate $\bar{E}'$ data from data of the solar-wind velocity $\bar{v}'$ and the southward component of the solar-wind magnetic field $\bar{B}_z$ as $\bar{E}' = \bar{v}' \bar{B}_z$. We can also use the values of the $D_{st}$ index at each hour. The data of the $D_{st}$ index are provided by Data Analysis Center for Geomagnetism and Space Magnetism, Kyoto University. The observation model of the $D_{st}$ index is defined as follows:

$$
D_{st}' = D_{st}^t + w_{D_{st}^t}^t
$$

$$
= D_{RC}^t + b \sqrt{P_d^t + D_{res}^t} + w_{D_{st}^t}
$$

(19)

where $D_{st}'$ denotes the observed $D_{st}$ index and $w_{D_{st}^t}$ is the observation error contained in the $D_{st}$ data which obeys the normal distribution $N(0, 25)$.

3.3 State Space Model

For the convenience in the following section, we define a state vector as follows:

$$
x_t = \begin{pmatrix} D_{RC}^t \\ D_{res}^t \\ P_d^t \\ E' \\ \tau' \end{pmatrix}
$$

(20)

Since the MPC effect $D_{MPC}$ is assumed to be given as a function of $P_d^t$, $D_{MPC}$ is not included in a state vector. Using this state vector, we can represent a transition of a state $x_t$ just by a conditional distribution as

$$
x_t \sim p(x_t|x_{t-1}).
$$

(21)

where $p(x_t|x_{t-1})$ can be defined by combining Eqs. (10)–(16). We also define an observation vector $y_t$ as

$$
y_t = \begin{pmatrix} D_{st}' \\ \bar{P}_d^t \\ \bar{E}'^t \end{pmatrix}
$$

(22)

However, the data of $\bar{P}_d$ and $\bar{E}'$ are sometimes missing as mentioned above. In such cases, we redefine an observation vector $y_t$ as

$$
y_t = \begin{pmatrix} \bar{D}_{st}' \end{pmatrix}
$$

(23)

In either case, the observation model described in Sect. 3.2 can be written as

$$
y_t \sim p(y_t|x_t).
$$

(24)

![Fig. 2 Dependencies among the variables considered in the present study. The variables which can be observed are indicated by black boxes with white letters.](image)
In ordinary cases, \( p(y|\mathbf{x}_i) \) can be defined by combining Eqs. (17)–(19). If the data of either \( \tilde{P}_d \) or \( \tilde{E}^i \) are missing, only Eq. (19) is evaluated; that is, \( p(y|\mathbf{x}_i) \) is determined from Eq. (19).

Figure 2 illustrates the dependencies among the variables considered in the present study. The variables included in the state vector \( \mathbf{x}_i \) are enclosed within a shaded box. The observable variables are indicated by black boxes with white letters.

4. Algorithm

We conduct the estimation of the value of the state vector \( \mathbf{x}_i \) at each hour from the observations \( y_i \) on the basis of the system and observation models described in the previous section. As the system model described above is nonlinear, the estimation of a state is done using the merging particle filter (MPF) [4], which is applicable even to nonlinear system models. The MPF is an algorithm based on the particle filter (PF) [6]–[8]. The MPF provides an approximation of the posterior probability density function given a sequence of observations \( \{y_1, \ldots, y_T\} \), like the PF. However, it is much more efficient than the normal PF. In the following, the algorithm of the MPF is briefly reviewed.

In the MPF, a probability density function (PDF) \( p(x) \) is approximated by a set of \( N \) samples \( \{x_1^{(1)}, \ldots, x_N^{(N)}\} \) as also done in the particle filter as

\[
p(x) \approx \frac{1}{N} \sum_{i=1}^{N} \delta(x - x^{(i)}).
\]

Each sample is called a ‘particle’ and an approximation of a PDF using a set of particles is called a ‘particle approximation’. In the normal PF, a set of particles for representing a posterior PDF contains many duplicates of the same particle, which causes highly redundant computational cost. On the other hand, in the MPF, each particle is generated by merging multiple samples in order to maintain the diversity of particles. This allows us to reduce the redundant computational cost.

We sequentially obtain particle approximations of a predictive PDF \( p(x_i|y_{1:t-1}) \) and a filtered PDF \( p(x_i|y_{1:t}) \) at each time step as follows. Suppose that a PDF \( p(x_{t-1}|y_{1:t-1}) \), which is a posterior PDF given the observations until time \( t-1 \), is approximated by a set of particles \( \{x_{t-1|t-1}^{(1)}, \ldots, x_{t-1|t-1}^{(N)}\} \) as

\[
p(x) \approx \frac{1}{N} \sum_{i=1}^{N} \delta(x_{t-1} - x_{t-1|t-1}^{(i)}).
\]

If we obtain a set of particles where each particle \( x_{t-1|t-1}^{(i)} \) is a sample taken from a conditional distribution \( p(x_i|X_{t-1|t-1}^{(i)}) \), this set of samples \( \{x_{t-1|t-1}^{(1)}, \ldots, x_{t-1|t-1}^{(N)}\} \) offers a particle approximation of a predictive PDF at the next time, \( p(x_i|y_{t-1}) \), as

\[
p(x_i|y_{1:t-1}) \approx \frac{1}{N} \sum_{i=1}^{N} \delta(x_i - x_{t-1|t-1}^{(i)}).
\]

An approximation of the filtered PDF \( p(x_i|y_{1:t}) \) can be obtained by incorporating an observation \( y_i \) into the particle approximation of a predictive PDF \( \{x_{t|t-1}^{(i)}\}_{i=1}^{N} \) as follows:

\[
p(x_i|y_{1:t}) = \frac{p(x_i|y_{1:t-1}) p(y_i|x_i)}{\int p(x_i|y_{1:t-1}) p(y_i|x_i) dx_i}
\]

\[
\approx \frac{1}{\sum_{j=1}^{N} p(y_i|x_{t|t-1}^{(j)})} \sum_{i=1}^{N} p(y_i|x_{t|t-1}^{(i)}) \delta(x_i - x_{t|t-1}^{(i)})
\]

\[
= \sum_{i=1}^{N} w_i \delta(x_i - x_{t|t-1}^{(i)})
\]

(28)

where \( p(y_i|x_{t|t-1}^{(i)}) \) is the likelihood of \( x_{t|t-1}^{(i)} \) given the data \( y_i \), which can be calculated according to Eq. (24). The weight \( w_i \) is defined as

\[
w_i = \frac{p(y_i|x_{t|t-1}^{(i)})}{\sum_{j=1}^{N} p(y_i|x_{t|t-1}^{(j)})}.
\]

(29)

Although Eq. (28) offers an approximation of the filtered PDF \( p(x_i|y_{1:t}) \), we generate a new set of particles \( \{x_{t|t}^{(i)}\}_{i=1}^{N} \) which represents the filtered PDF in the following form:

\[
p(x_i|y_{1:t}) \approx \frac{1}{N} \sum_{i=1}^{N} \delta(x_i - x_{t|t}^{(i)}).
\]

(30)

In the MPF, each particle for representing the filtered PDF is obtained by combining multiple particles taken from a set of particles for representing the predictive PDF; that is, a particle \( x_{t|t}^{(i)} \) is obtained from multiple samples from \( \{x_{t|t-1}^{(1)}, \ldots, x_{t|t-1}^{(N)}\} \). Here the number of particles to be combined is taken as 3. Then, in order to obtain a set of \( N \) particles \( \{x_{t|t}^{(1)}, \ldots, x_{t|t}^{(N)}\} \), it is necessary to draw \( 3N \) samples with weights of \( w_i \) \( \{x_{t|t}^{(1)}, \ldots, x_{t|t}^{(3)}, \ldots, x_{t|t}^{(3N)}\} \). Each particle in the new set, \( x_{t|t}^{(i)} \), is generated as a weighted sum of 3 samples contained in this \( 3N \) samples as:

\[
x_{t|t}^{(i)} = \sum_{j=1}^{3} \alpha_j x_{t|t}^{(j)}.
\]

(31)

The set of weights \( \{\alpha_j\}_{j=1}^{3} \) in Eq. (31) is set to satisfy

\[
\sum_{j=1}^{3} \alpha_j = 1,
\]

(32a)

\[
\sum_{j=1}^{3} \alpha_j^2 = 1
\]

(32b)

where \( \alpha_j \in \mathbb{R} \) for all \( j \). Eqs. (32a) and (32b) ensures that the new set of particles \( \{x_{t|t}^{(i)}\}_{i=1}^{N} \) has asymptotically the same average and covariance of the filtered PDF \( p(x_i|y_{1:t}) \) for \( N \rightarrow \infty \).
In particular, the model ever, some long-term variation is also seen in this figure. comes larger than 20 nT, it mostly varies with the amplitude of the September, while the empirical model assumes that Dst is variable according to Eq. (13) with the actual Dst. Although the difference sometimes becomes larger than 20 nT, it mostly varies with the amplitude of about 10 nT during the period from 1995 to 2005. However, some long-term variation is also seen in this figure. In particular, the model Dst is prominently smaller than the actual Dst around the middle of 1999. Figure 4 compares the model Dst for constant Dres with the actual Dst during one year from January 1999 to December 1999. The gray line indicates the actual Dst and the black line indicates the model Dst. From the end of May to July, the model Dst underestimated the actual Dst by about 10 nT.

Figure 5 compares the model Dst for the case that Dres is variable according to Eq. (13) with the actual Dst for the period from January 1999 to December 1999. The gray line indicates the actual Dst and the black line indicates the model Dst. Indeed, while the log-likelihood of the model based on Eq. (12) was about $-1.8 \times 10^{-8}$, the log-likelihood of the model based on Eq. (13) was about $-1.5 \times 10^{-5}$. The dashed line shows the estimated Dres. The residual effect Dres exceeds zero from the middle of the May to the middle of the September, while the empirical model assumes that Dres is negative ($D_{res} = -11$ nT). This result means that the residual effect is more variable than assumed by the previous studies. The Burton’s model sometimes overestimate $D_{res}$ and sometimes underestimate $D_{res}$ by tens of nT, which inevitably causes misestimation of $D_{RC}$. As described in Introduction, $D_{st}$ varies typically by 50–100 nT during magnetic storms. Thus, the improvement of the estimation of $D_{res}$ is not negligible in order to evaluate the RC effect using the $D_{st}$ index.

5. Result

We estimated the temporal evolution of the state vector $x_t$ for the period from 1995 to 2005. The estimation was performed for the period from 1995 because many of the OMNI2 solar-wind data are missed before 1995. An estimate of $x_t$ at each time step was provided from the ensemble mean of particles $[x_{t+1}^{(1)}]^N$, which approximates a filtered PDF $p(x_t|y_{1:t})$. The number of particles N used for the estimation was 1600. In order to evaluate whether the temporal evolution was successfully estimated or not, we compare the $D_{st}$ evolution estimated using the MPF with a sequence of the real $D_{st}$ data.

Figure 3 shows difference between the real $D_{st}$ data and the model $D_{st}$ for the case that $D_{res}$ is assumed to be constant where positive means that the model $D_{st}$ is larger than the actual $D_{st}$. Although the difference sometimes becomes larger than 20 nT, it mostly varies with the amplitude of about 10 nT during the period from 1995 to 2005. However, some long-term variation is also seen in this figure. The estimation was performed for the period from 1995 to 2005. The estimation was performed for the period from 1995 to 2005. The estimation was performed for the period from 1995 to 2005. The estimation was performed for the period from 1995 to 2005. The estimation was performed for the period from 1995 to 2005. The estimation was performed for the period from 1995 to 2005.
operate and with the conditions of the ionosphere. Thus, a real-time monitor of magnetic storm activity and the ring current is important. In the present study, we adopted the merging particle filter (MPF). The MPF is an on-line algorithm applicable for processing even real-time data and it enables an on-line estimation of the long-term residual effect $D_{\text{res}}$. This on-line estimation provides valuable information for real-time estimation of the ring current variation $D_{\text{RC}}$. Thus, the framework adopted in the present paper could be useful for a real-time monitoring of the magnetospheric environment using the real-time edition of the $D_{st}$ index.

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