Threshold Selection Based on Interval-Valued Fuzzy Sets**

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SUMMARY We propose a thresholding method based on interval-valued fuzzy sets which are used to define the grade of a gray level belonging to one of the two classes, an object and the background of an image. The effectiveness of the proposed method is demonstrated by comparing our classification results on eight test images to results from the conventional methods.

key words: threshold selection, interval-valued fuzzy sets

1. Introduction

Thresholding is a technique to assign each pixel in an image into two classes: object and background. The key issue is to choose an optimal threshold value so that the number of misclassified pixels is kept as low as possible [1]. Several methods for thresholding, including clustering-based methods [2], [3], histogram-based methods [4]–[7], and fuzzy set-based methods [8]–[10] have been proposed and widely used. In particular, fuzzy set-based approaches, including fuzzy entropy-based methods, are useful in finding the ambiguous boundaries between an object and a background. However, these methods based on standard fuzzy sets (i.e., type I fuzzy sets) have a shortcoming: the assignment of a membership degree to pixel in an image is not certain. This drawback was overcome by introducing a threshold selection method based on interval-valued fuzzy sets [11], [12], [13] to define a relationship between object and background. We then apply the proposed method to threshold selection and discuss its effectiveness.

2. Interval-Valued Fuzzy Set-Based Thresholding

2.1 Fuzzy Partitions of an Input Variable

Given an image with \( M \times N \) pixels and \( L \) gray-levels, fuzzy partitions for an input variable are determined by the following procedures:

**Step 1:** Calculate the number of fuzzy partitions \( p \) using statistical information on a gray level histogram.

\[
p = \text{round}\left(\frac{z_{\text{max}} - z_{\text{min}}}{\sigma}\right), \quad 0 \leq z_{\text{min}} < z_{\text{max}} \leq L - 1
\]

(1)

where \( \text{round()} \) denotes a function for rounding-off to the nearest integer; \( z_{\text{min}} \) and \( z_{\text{max}} \) are the minimum and maximum gray level in the histogram, respectively; and \( \sigma \) is the standard deviation in the histogram.

**Step 2:** Generate \( p \) interval-valued fuzzy sets in the gray level distribution \( [z_{\text{min}}, z_{\text{max}}] \).

\[
d = \frac{z_{\text{max}} - z_{\text{min}}}{(p + 1)}
\]

(2)

where \( d \) is the half-width used to define the interval-valued fuzzy membership functions in interval-valued fuzzy sets, as shown in Fig. 1.

In Fig. 1, \( A_j = \{[z, [\mu_j^L(z), \mu_j^U(z)] \mid \mu_j^L, \mu_j^U \rightarrow [0, 1]\} \) where \( \mu_j^L(z) \leq \mu_j^U(z), j = 1, \ldots, p \) are \( j \)-th interval-valued fuzzy sets defined on the histogram, and \( m_1 \) and \( m_2 \) are the margins of the lower and upper fuzzy membership functions based on a half-width \( d \). Thus, the half-width of the \( j \)-th lower and upper fuzzy membership function is \( d - m_1 \) and \( d + m_2 \) respectively, and then the adjustable range of the margin \( m \) \((m = m_1 + m_2, m_1 = m_2)\) cannot exceed the half-width \( d \) because the endpoints of the lower membership functions do not overlap with each other.

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**Fig. 1** Fuzzy partition in the histogram.
2.2 Fuzzy Partitions of an Output Variable

The peak locations in a gray level histogram are useful in selecting a threshold value for segmenting an image into two different regions [14]. The proposed method selects two peaks which are used to generate interval-valued fuzzy sets for an output variable, with the largest variation between the differential values in the histogram occurring when the margin is adjusted. The procedure for this is as follows:

Step 1: Calculate the differential values in the histogram; i.e., the gray level distribution \( z_i \in [z_{\min}, z_{\max}] \).

\[ \Delta h_i = |h(z_i) - h(z_{i+1})|, \quad z_i \in [z_{\min}, z_{\max}] \]  

(3)

Step 2: Find the gray-levels with two peaks among the differential values given by Eq. (3).

\[ x_1 = \text{arg max}_i (\Delta h_i), \quad x_2 = \text{arg max}_i (k_1, k_2), \]

where \( k_1 = \max_{i \in i_{[z_{\min}, z_{\max}]}^{\Delta h_i}} (\Delta h_i), \quad k_2 = \max_{i \in [z_{\min}, z_{\max}]} (\Delta h_i) \)

(4)

Step 3: Generate the interval-valued fuzzy sets \( C_i \) for an output variable \( y \) using Eq. (5).

\[ C_i = \left\{ \left[ \mu_{y}^{L}(y), \mu_{y}^{U}(y) \right] \right\} \]

\[ \mu_{y}^{L}(y) = \begin{cases} y-a_{y}^{L} & \text{if } a_{y}^{L} < y \leq b_{y}^{L} \\ b_{y}^{L} - a_{y}^{L} & \text{if } a_{y}^{L} < y < b_{y}^{L} \\ c_{y}^{L} - y & \text{if } b_{y}^{L} < y < c_{y}^{L} \\ 1 & \text{if } y = b_{y}^{L} \\ 0 & \text{if } y \leq a_{y}^{L} \text{ or } y \geq c_{y}^{L} \end{cases} \]

\[ \mu_{y}^{U}(y) = \begin{cases} y-a_{y}^{U} & \text{if } a_{y}^{U} < y < b_{y}^{U} \\ b_{y}^{U} - a_{y}^{U} & \text{if } a_{y}^{U} < y < b_{y}^{U} \\ c_{y}^{U} - y & \text{if } b_{y}^{U} < y < c_{y}^{U} \\ 1 & \text{if } y = b_{y}^{U} \\ 0 & \text{if } y \leq a_{y}^{U} \text{ or } y \geq c_{y}^{U} \end{cases} \]

where \( a_{y}^{L} = x_1 + m_1 - \sigma, \quad b_{y}^{L} = b_{y}^{U} = x_1, \quad c_{y}^{L} = x_1 - m_1 + \sigma, \quad a_{y}^{U} = x_1 - m_2 - \sigma, \quad c_{y}^{U} = x_1 + m_2 + \sigma \)

(5)

2.3 Rule Generation

We present a rule generation method to define the relationship between the input and output variable when the margin of interval-valued fuzzy sets is adjusted. The fuzzy rules are extracted from the following two conditions based on the mean value \( z_{\text{mean}} \) in the gray level distribution \( [z_{\min}, z_{\max}] \), as considered by [15].

**Condition 1**: If the mean value is not involved at the bounded range of the \( j \)-th interval-valued fuzzy set for the input variable, the output of the \( j \)-th fuzzy rule is given by

\[ y = \begin{cases} C_1 & \text{if } \max \left( A_{j}^{L} \right), \max \left( A_{j}^{U} \right) < z_{\text{mean}} \\ C_2 & \text{if } \min \left( A_{j}^{L} \right), \min \left( A_{j}^{U} \right) > z_{\text{mean}} \end{cases} \]

where \( A_{j}^{L} = \begin{cases} \{ z_{\min}, (j+1) \cdot d - m_1 \} & \text{if } j = 1 \\ \{ (j-1) \cdot d + m_1, z_{\max} \} & \text{if } j = p \\ \{ (j-1) \cdot d + m_1, (j+1) \cdot d - m_1 \} & \text{otherwise} \end{cases} \),

(6)

\[ A_{j}^{L} \text{ and } A_{j}^{U} \text{ are the bounded ranges of the lower and upper fuzzy membership functions in the } j \text{-th interval-valued fuzzy set, respectively.} \]

**Condition 2**: If the mean value is involved at the bounded range of the \( j \)-th interval-valued fuzzy set, the output of the \( j \)-th fuzzy rule is given by

\[ y = \begin{cases} C_1 & \text{if } n_1, n_2 \Rightarrow C_1 \\ C_2 & \text{if } n_1, n_2 \Rightarrow C_2 \\ \text{NA} & \text{otherwise} \end{cases} \]

where \( n_1 = \begin{cases} C_1 & \text{if } \sum_{z_{\leq z_{\text{mean}}}} h(z) > \sum_{z_{> z_{\text{mean}}}} h(z), \\ C_2 & \text{otherwise} \end{cases} \),

(7)

\[ n_2 = \begin{cases} C_1 & \text{if } \sum_{z_{\leq z_{\text{mean}}}} h(z) > \sum_{z_{> z_{\text{mean}}}} h(z), \\ C_2 & \text{otherwise} \end{cases} \]

The symbol \( \Rightarrow \) denotes the fuzzy implication operator, and \( n_1 \) and \( n_2 \) denote the output labels of interval-valued fuzzy membership functions for the if-part in the fuzzy if-then rules. ‘NA’ represents the “not available rule” when the sum of frequencies in the bounded range is different. However, it is useful to observe a suitable margin of interval-valued fuzzy sets for segmenting an image into two different regions.

2.4 Threshold Selection

The following types of fuzzy rules, extracted from the previous section, are used to binarize an image.

\[ R^i : \text{If } z \text{ is } A_{ji}, \quad j = 1, \ldots, p \]

where \( R^i \) is the \( j \)-th fuzzy rule, \( z \) is an input variable (e.g., gray level), \( A_{ji} \) is \( j \)-th antecedent interval-valued fuzzy set, \( y \) is an output variable, and \( C_{ji} \) is the \( l \)-th consequent interval-valued fuzzy set composed of two labels.

The fuzzy outputs of a certain gray level \( z_l \), \((h(z_l) > 0)\) in the histogram are given by

\[ \tilde{C}_{ji} = \left\{ \left[ \mu_{y}^{L}(y), \mu_{y}^{U}(y) \right] \right\} \mu_{y}^{L}(y) \mu_{y}^{U}(y) \rightarrow [0, 1] \]

where \( \mu_{y}^{L}(y) = \max_{j=1,\ldots,p} \min \left[ \mu_{y}^{L}(z), \mu_{y}^{U}(y) \right] \),

\[ \mu_{y}^{U}(y) = \max_{j=1,\ldots,p} \min \left[ \mu_{y}^{L}(z), \mu_{y}^{U}(y) \right] \]

\[ \min \left[ \mu_{y}^{L}(z), \mu_{y}^{U}(y) \right] \text{ and } \mu_{y}^{U}(y) \text{ represent the fuzzy implication operation between the antecedent and consequent parts in the fuzzy if-then rules fired by the gray level } z_l, \mu_{y}^{L}(y) \text{ and } \mu_{y}^{U}(y) \text{ are the fuzzy outputs (i.e., degrees} \]
of fulfillment) of the rules obtained from the max operation after carrying out the fuzzy implication. The inference result \( y^*_i \) is calculated by averaging the outputs obtained from the defuzzification operation.

\[
y^*_i = \left( y^L_i + y^U_i \right) / 2,
\]

where

\[
y^L_i = \sum_y y \cdot \mu_{\tilde{L}}(y) / \sum_y \mu_{\tilde{L}}(y),
\]

\[
y^U_i = \sum_y y \cdot \mu_{\tilde{U}}(y) / \sum_y \mu_{\tilde{U}}(y)
\]

The threshold value \( T \) is then determined by

\[
T = \text{round} \left( \frac{T_{C_1} + T_{C_2}}{2} \right),
\]

where \( T_{C_1} = \max_{y^*_i \in C_1} y^*_i \), \( T_{C_2} = \min_{y^*_i \in C_2} y^*_i \)

\( y^*_i \in C_1 \) and \( y^*_i \in C_2 \) denote the defuzzified output intervals for each label (i.e., its belonging region) in the overall inference result obtained from Eq. (9), as shown in Fig. 2.

3. Experimental Results and Remarks

To show the effectiveness of the proposed method, experiments for the two conventional thresholding methods (Otsu’s method, and Huang and Wang’s method) and the proposed method were performed on eight test images. The binarized images for ‘Color blindness’, ‘Shepp-logan phantom’, and ‘Cell’ are shown in Fig. 3.

Figure 4 shows threshold values determined by the margin of interval-valued fuzzy sets:

\[
m = d \times (K/10), \quad K = 1, 2, \ldots, 10.
\]

In Fig. 4, the symbols ‘○’ and ‘□’ denote the upper \( T_{C_1} \) and lower bounds \( T_{C_2} \) for two classes, and the symbol ‘*’ denotes the threshold values determined within the bounds. For eight images, we observed that the threshold values (‘Butterfly’, ‘Color blindness’, ‘Lena’, ‘Camrerman’, ‘Shepp-logan phantom’, and ‘Cell’) were uniformly saturated in a partial range when the width of the margin in the interval-valued fuzzy sets was smaller than the distance between two peaks. Moreover, the fuzzy rules in six images (‘Butterfly’, ‘Lena’,

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Fig. 2  Threshold value.

Fig. 3  Thresholding results. (a) Original image (‘Color blindness’), (b) original image (‘Shepp-logan phantom’), (c) original image (‘Cell’), (d) histogram of ‘Color blindness’, (e) histogram of ‘Shepp-logan phantom’, (f) histogram of ‘Cell’ histogram, (g) – (i) images thresholded using Otsu’s method, (j) – (l) images thresholded using Huang and Wang’s method, and (m) – (o) images thresholded using the proposed method.

Fig. 4  Variation of threshold values. (a) ‘Color blindness’, (b) ‘Shepp-logan phantom’, and (c) ‘Cell’.
Table 1  Threshold values.

<table>
<thead>
<tr>
<th>Image</th>
<th>Conventional methods</th>
<th>The proposed method</th>
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<tbody>
<tr>
<td></td>
<td>Otsu</td>
<td>Huang and Wang</td>
</tr>
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<td>Butterfly</td>
<td>90</td>
<td>90</td>
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<tr>
<td>Color bleeding</td>
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<tr>
<td>Lena</td>
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<tr>
<td>Girl</td>
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<td>74</td>
</tr>
<tr>
<td>Cameraman</td>
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<td>62</td>
</tr>
<tr>
<td>Fish</td>
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<td>49</td>
</tr>
<tr>
<td>Shepp-logan phantom</td>
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<td>2</td>
</tr>
<tr>
<td>Cell</td>
<td>833</td>
<td>208</td>
</tr>
</tbody>
</table>

‘Girl’, ‘Cameraman’, ‘Fish’, and ‘Cell’) were changed when the margins of interval-valued fuzzy sets were updated as 21.675 (K = 5.1), 11.28 (K = 2.4), 33.52 (K = 9.9), 40.84 (K = 8.3), 30.02 (K = 8.7), and 34.44 (K = 5.6), as described in Sect. 2.3. Accordingly, the threshold values in the proposed method were determined by the previously described properties, as shown in Table 1. From these results, we conclude that the proposed method demonstrated performance superior in visual quality to that of the conventional methods (e.g., ‘Color blindness’ and ‘Shepp-logan phantom’).

4. Conclusions

In this letter, we proposed a novel thresholding method based on interval-valued fuzzy sets. In order to show the influence of margin in interval-valued fuzzy sets, we have presented the threshold values determined by the margin in Eq. (11) on eight images. The effectiveness of the proposed method was demonstrated by comparing the experimental results obtained for the images using the proposed method and two well-known methods. Problems on the conflict rules (‘NA’ of Eq. (7)) are remained to be solved in further studies.

References