Refinement of Representation Theorems for Context-Free Languages*

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SUMMARY In this paper, we obtain some refinement of representation theorems for context-free languages by using Dyck languages, insertion systems, strictly locally testable languages, and morphisms. For instance, we improved the Chomsky-Schützenberger representation theorem and show that each context-free language L can be represented in the form $L = h(D \cap R)$, where D is a Dyck language, R is a strictly 3-testable language, and h is a morphism. A similar representation for context-free languages can be obtained, using insertion systems of weight $(3, 0)$ and strictly 4-testable languages.

key words: Chomsky-Schützenberger representation theorem, context-free languages

1. Introduction

Representing a class of languages through operations on its subclasses is a traditional topic within formal language theory. We focus on characterizing context-free languages by using insertion systems together with some additional mechanisms.

A traditional topic in insertion systems in which we can use only insertion operations has been the representation of the class of recursively enumerable languages by simple operations on a small subclass of it [1]. From the definition of insertion operations, one would easily imagine that by using only insertion operations, we generate only context-sensitive languages.

It has been shown that using insertion systems together with some morphisms, characterizing recursively enumerable languages is accomplished in [2], [3]. In [4], each recursively enumerable language L can be represented in a similar way to the well-known Chomsky-Schützenberger representation of context-free languages, i.e. $L = h(L(\gamma) \cap D)$, where γ is an insertion system, h is a projection, and D is a Dyck language. They also characterize the class of context-free languages using insertion systems of weight $(3, 0)$ and star languages.

On the other hand, the class of strictly locally testable languages is known as a proper subclass of regular language classes. The equivalence relation between a certain type of splicing languages and strictly locally testable languages is known [5]. In this paper, we make use of strictly locally

testable languages as the additional mechanisms for characterizing context-free languages.

In insertion systems, a pair of the maximum length of inserted strings and the one of context-checking strings, called *weight* is an important parameter for generative powers. As for strictly locally testable languages, the length of local testability-checking is considered.

We prove that each context-free language L can be represented in the form $L = h(D \cap R)$, where D is a Dyck language, h is a morphism, and R is a strictly 3-testable language. A similar representation for context-free languages can be obtained, using insertion systems of weight $(3, 0)$ and strictly 4-testable languages. The optimality of these results, in terms of weight in insertion operations and the parameter of strictly locally testable languages remains to be checked.

2. Preliminaries

In this section, we introduce necessary notation and basic definitions needed in this paper. We assume the reader to be familiar with the rudiments on basic notions in formal language theory (see, e.g., [2], [6]).

2.1 Basic Definitions

For an alphabet V , V^* is the set of all strings of symbols from V which includes the empty string λ . For a string $x \in V^*$, $|x|$ denotes the length of x . Furthermore, $|x|_{V'}$ represents the number of occurrences of the symbols of the set V' in x . For $0 \leq k \leq |x|$, let $Pre_k(x)$ and $Suf_k(x)$ be the prefix and the suffix of x of length k , respectively. For $0 \leq k \leq |x|$, let $Int_k(x)$ be the set of *proper* interior substrings of x of length k , while if $|x| = k$ then $Int_k(x) = \emptyset$.

2.2 Normal Forms of Grammars

A *phrase structure grammar* is a quadruple $G = (N, T, P, S)$, where N is a set of *nonterminal symbols*, T is a set of *terminal symbols*, P is a set of *production rules*, and S in N is the *initial symbol*. A rule in P is of the form $r : \alpha \rightarrow \beta$, where $\alpha \in (N \cup T)^* N (N \cup T)^*$, $\beta \in (N \cup T)^*$, and r is a label such that there are no production rules with the same label, $Lab(P) = \{r \mid r : \alpha \rightarrow \beta \in P\}$. We often omit labels of production rules, if not necessary. For any x and y in $(N \cup T)^*$, if $x = u\alpha v$, $y = u\beta v$, and $r : \alpha \rightarrow \beta \in P$, then we write

$$x \xrightarrow{r}_G y.$$

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We say that x *directly derives* y with respect to G . If there is no confusion, we write $x \Rightarrow y$. The reflexive and transitive closure of $\Rightarrow$ is denoted by $\Rightarrow^*$. A derivation of length $k \geq 0$ from x to y is a sequence $(x_0, x_1, \dots, x_k)$ of strings in $(N \cup T)^*$ such that $x_{i-1} \Rightarrow x_i$ for $i = 1, \dots, k$ and $x = x_0$, $y = x_k$, which is denoted by $x \Rightarrow^k y$.

We define a *language* $L(G)$ generated by a grammar G as follows:

$$L(G) = \{w \in T^* \mid S \Rightarrow_G^* w\}.$$

It is well known that the class of languages generated by the phrase structure grammars is equal to the class of *recursively enumerable languages* RE [6].

A grammar $G = (N, T, P, S)$ is *context-free* if P is a finite set of *context-free rules* of the form $A \rightarrow \alpha$, where $A \in N$ and $\alpha \in (N \cup T)^*$. A language L is a *context-free language* if there is a context-free grammar G such that $L = L(G)$. Let CF be the class of context-free languages.

A context-free grammar $G = (N, T, P, S)$ is in *Chomsky normal form* if each production rule in P is of one of the following forms:

1. $X \rightarrow YZ$, where $X, Y, Z \in N$.
2. $X \rightarrow a$, where $X \in N, a \in T$.
3. $S \rightarrow \lambda$ (only if S does not appear in right-hand sides of rules).

It is well known that, for each context-free language L , there is a context-free grammar in Chomsky normal form generating L [6].

For an alphabet $\Sigma = \{a_1, \dots, a_n\}$ with $n \geq 1$, let $\bar{\Sigma} = \{\bar{a}_1, \dots, \bar{a}_n\}$ be an alphabet such that for each $1 \leq i \leq n$, $\bar{a}_i$ is a barred copy of a_i . Σ and $\bar{\Sigma}$ are considered to be disjoint. Then *Dyck language* D over Σ and $\bar{\Sigma}$ is defined to be the context-free language generated by the grammar $G_D = (\{S\}, \Sigma \cup \bar{\Sigma}, P, S)$, where $P = \{S \rightarrow SS, S \rightarrow aS\bar{a}, S \rightarrow \lambda \mid a \in \Sigma, \bar{a} \in \bar{\Sigma}\}$. Let *Dyck* be a class of Dyck languages.

A grammar $G = (N, T, P, S)$ is *regular* if P is a finite set of rules of the form $X \rightarrow \alpha$, where $X \in N$ and $\alpha \in TN \cup T \cup \{\lambda\}$. A language L is a *regular language* if there is a regular grammar G such that $L = L(G)$. Let REG be the class of regular languages.

We are going to define a strictly locally testable language, which is one of the main objectives of the present work.

Let k be a positive integer. A language L over T is strictly k -testable if there is a triplet $S_k = (A, B, C)$ with $A, B, C \subseteq T^k$ such that for any w with $|w| \geq k$, w is in L iff $Pre_k(w) \in A$, $Suf_k(w) \in B$, $Int_k(w) \subseteq C$.

Note that if L is strictly k -testable, then L is strictly k' -testable for all $k' > k$. Further, the definition of strictly k -testable says nothing about the strings of “length $k - 1$ or less”.

A language L is *strictly locally testable* iff there exists an integer $k \geq 1$ such that L is strictly k -testable. Let $LOC(k)$ be the class of strictly k -testable languages. Then one can prove the following theorem.

Theorem 1: [7] $LOC(1) \subset LOC(2) \subset \dots \subset LOC(k) \subset \dots \subset LOC \subset REG$.

We are now going to define an insertion system. An *insertion system* is a triple $\gamma = (T, P, A_X)$, where T is an alphabet, P is a finite set of *insertion rules* of the form $r : (u, x, v)$ with $u, x, v \in T^*$, A_X is a finite set of strings over T called axioms, and r is a label such that there are no insertion rules with the same label, $Lab(P) = \{r \mid r : (u, x, v) \in P\}$.

We write $\alpha \xRightarrow[r]{\gamma} \beta$ if $\alpha = \alpha_1 u v \alpha_2$ and $\beta = \alpha_1 u x v \alpha_2$ for some insertion rule $r : (u, x, v) \in P$ with $\alpha_1, \alpha_2 \in T^*$. If there is no confusion, we write $\alpha \Rightarrow \beta$. The reflexive and transitive closure of $\Rightarrow$ is defined by $\Rightarrow^*$.

A language generated by γ is defined by

$$L(\gamma) = \{w \in T^* \mid s \xRightarrow[\gamma]^* w, \text{ for some } s \in A_X\}.$$

An insertion system $\gamma = (T, P, A_X)$ is said to be of *weight* (m, n) if

$$\begin{aligned} m &= \max\{|x| \mid (u, x, v) \in P\}, \\ n &= \max\{|u| \mid (u, x, v) \in P \text{ or } (v, x, u) \in P\}. \end{aligned}$$

For $m, n \geq 0$, INS_m^n denotes the class of all languages generated by insertion systems of weight (m', n') with $m' \leq m$ and $n' \leq n$. When the parameter is not bounded, we replace m or n with $*$.

For insertion systems, there exist the following results.

Theorem 2 ([2]):

1. For the class of finite languages FIN and the class of context-sensitive languages CS , $FIN \subset INS_*^0 \subset INS_*^1 \dots \subset INS_*^* \subset CS$.
2. $REG \subset INS_*^*$.
3. $INS_*^1 \subset CF$.
4. CF is incomparable with all INS_*^n ($n \geq 2$), and INS_*^* .
5. INS_*^2 contains non-semilinear languages.

A mapping $h : V^* \rightarrow T^*$ is called *morphism* if $h(\lambda) = \lambda$ and $h(xy) = h(x)h(y)$ for any $x, y \in V^*$. For languages L_1, L_2 , and morphism h , we use the following notation: $h(L_1 \cap L_2) = \{h(w) \mid w \in L_1 \cap L_2\}$. For language classes $\mathcal{L}_1$ and $\mathcal{L}_2$, we introduce the following class of languages:

$$\begin{aligned} H(\mathcal{L}_1 \cap \mathcal{L}_2) \\ = \{h(L_1 \cap L_2) \mid h \text{ is a morphism, } L_i \in \mathcal{L}_i (i = 1, 2)\}. \end{aligned}$$

3. Modifications of Păun's Theorem

We will show how context-free languages can be characterized by insertion systems and strictly locally testable languages. In some respect the proof technique from [4] might be helpful to follow the main proof of this section, in which star languages are used instead of strictly locally testable languages.

Lemma 1: $CF \subseteq H(INS_3^0 \cap LOC(4))$.

Construction: Consider a context-free grammar $G = (N, T, P, S)$ in Chomsky normal form. We construct an insertion system $\gamma = (\Sigma, P_\gamma, \{S\})$, where

$$\begin{aligned} V &= N \cup Lab(P), \\ \Sigma &= V \cup \bar{V} \cup T, \\ P_\gamma &= \{(\lambda, YZr, \lambda), (\lambda, \bar{X}\bar{r}, \lambda) \mid r : X \rightarrow YZ \in P\} \cup \\ &\quad \{(\lambda, ar, \lambda), (\lambda, \bar{X}\bar{r}, \lambda) \mid r : X \rightarrow a \in P\} \cup \\ &\quad \{(\lambda, r, \lambda), (\lambda, \bar{S}\bar{r}, \lambda) \mid r : S \rightarrow \lambda \in P\}. \end{aligned}$$

For the rule $r : X \rightarrow \alpha$ in P , we say that the two insertion rules (λ, ar, λ) and $(\lambda, \bar{X}\bar{r}, \lambda)$ in P_γ are r -pair.

We define the projection $h : \Sigma^* \rightarrow T^*$ by

$$\begin{aligned} h(\lambda) &= \lambda, \\ h(ax) &= ah(x) \quad \text{if } a \in T, \\ h(ax) &= h(x) \quad \text{otherwise.} \end{aligned}$$

Consider $R = A\Sigma^* \cap \Sigma^*B - \Sigma^*C'\Sigma^*$ with $C' = \Sigma^4 - C$, where

$$\begin{aligned} A &= \{arX\bar{X}\bar{r} \mid r : X \rightarrow a \in P\} \cup \\ &\quad \{rS\bar{S}\bar{r} \mid r : S \rightarrow \lambda \in P\}, \\ B &= \{rS\bar{S}\bar{r} \mid r : S \rightarrow \alpha \in P, \alpha \in (N \cup T)^*\}, \\ C &= \{rX\bar{X}\bar{r}, X\bar{X}\bar{r}a, X\bar{X}\bar{r}r_1, \bar{X}\bar{r}ar_1, \\ &\quad \bar{X}\bar{r}r_1Y, \bar{r}_1rX\bar{X}, \bar{r}_1rX\bar{X}, arX\bar{X} \mid \\ &\quad r : X \rightarrow \alpha \in P, r_1 : Y \rightarrow \alpha_1 \in P, a \in T, \\ &\quad \alpha, \alpha_1 \in (N \cup T)^*\}. \end{aligned}$$

Then R is a strictly 4-testable language prescribed by $S_4 = (A, B, C)$. The language R can be characterized by using

$$\Omega = \{rX\bar{X}\bar{r} \mid r : X \rightarrow \alpha \in P, \alpha \in (N \cup T)^*\}$$

such that $R \subset (\Omega \cup T\Omega)^*(B \cup TB)$. A nonterminal symbol X in $rX\bar{X}\bar{r} \in \Omega$ is said to be Ω -blocked. Intuitively, an Ω -blocked nonterminal symbol X in $rX\bar{X}\bar{r}$ means that X has been used for the rule r .

Further, based on γ and R , we define the followings: for each $X \in N$, let

$$\gamma_X = (\Sigma, P_\gamma, \{X\})$$

be an insertion grammar, and let

$$R_X = A\Sigma^* \cap \Sigma^*B_X - \Sigma^*C'\Sigma^*$$

be a strictly 4-testable language, where $B_X = \{rX\bar{X}\bar{r} \mid r : X \rightarrow \alpha \in P, \alpha \in (N \cup T)^*\}$. There is a slight note on the form of $\Sigma^*C'\Sigma^*$ in R_X . Then R_X can be characterized by $R_X \subset (\Omega \cup T\Omega)^*$. For the case $X = S$, $\gamma_S = \gamma$ and $B_S = B$ hold, which implies $R_X = R$ from the definitions of R_X and R .

Idea of the Proof: Since the labels of production rules $Lab(P)$ precisely identify an r -pair insertion rules and $R \subset (\Omega \cup T\Omega)^*(B \cup TB)$ holds, a production rule r in P is simulated by r -pair insertion rules in P_γ . Furthermore, any successful derivation of γ requires the use of r -pair insertion rules in which any consecutive (odd, even) derivations steps

simulate one production r in G . In a successful derivation, a nonterminal symbol X in $rX\bar{X}\bar{r} \in \Omega$ implies that X has been used for the rule r . Finally, the symbol in $\Sigma - T$ which is not used in $L(G)$ is eventually eliminated by the projection h .

To show $L(G) = h(L(\gamma) \cap R)$, we will show the following inclusion.

[$L(G) \subseteq h(L(\gamma) \cap R)$] We will show that for any X in N , if there is a derivation $X \xRightarrow{r_1 \dots r_n}_G a_1 \dots a_l$ with $a_i \in T$ ($1 \leq i \leq l$) then there is a string $w = a_1u_1 \dots a_lu_l$ in $L(\gamma_X) \cap R_X$, where

- for $l \geq 2$, $u_i \in \Omega^+$ ($1 \leq i \leq l-1$), and $u_l \in \Omega^*\{r_1X\bar{X}\bar{r}_1\}$,
- for $l = 1$, $u_1 \in \Omega^*\{r_1X\bar{X}\bar{r}_1\}$,

by induction on the length n of derivations in G .

Base step: Suppose that there is a derivation $X \xRightarrow{r}_G a$. From the definition of P_γ , r -pair insertion rules (λ, ar, λ) and $(\lambda, \bar{X}\bar{r}, \lambda)$ are in P_γ . Therefore, $X \xRightarrow{\gamma_X} X\bar{X}\bar{r} \xRightarrow{\gamma_X} arX\bar{X}\bar{r}$, where $arX\bar{X}\bar{r}$ is in R_X .

Induction step: We suppose that the claim holds for any $k \leq n$.

Consider a derivation $X \xRightarrow{r}_G YZ \xRightarrow{r_1 \dots r_n}_G a_1 \dots a_l a_{h+1} \dots a_l \in T^+$, where $l \geq 2$, $Y \xRightarrow{*}_G a_1 \dots a_h$, $Z \xRightarrow{*}_G a_{h+1} \dots a_l$, and $a_i \in T$ for $1 \leq i \leq l$. For the rule $r : X \rightarrow YZ$ in P , from the definition of P_γ , r -pair insertion rules (λ, YZr, λ) and $(\lambda, \bar{X}\bar{r}, \lambda)$ are in P_γ . From the induction hypothesis, there are strings $a_1u_1 \dots a_hu_h$ and $a_{h+1}u_{h+1} \dots a_lu_l$ such that $a_1u_1 \dots a_hu_h \in L(\gamma_Y) \cap R_Y$, $a_{h+1}u_{h+1} \dots a_lu_l \in L(\gamma_Z) \cap R_Z$, and $u_i \in \Omega^+$ for $1 \leq i \leq l$. Therefore, there is a derivation

$$\begin{aligned} X &\xRightarrow{\gamma_X} X\bar{X}\bar{r} \xRightarrow{\gamma_X} YZrX\bar{X}\bar{r} \\ &\xRightarrow{*}_{\gamma_X} a_1u_1 \dots a_hu_ha_{h+1} \dots a_lu_lrX\bar{X}\bar{r} = w. \end{aligned}$$

The fact w is in R_X can be derived from the definitions of R_X , R_Y , and R_Z . For the projection h , $h(w) = a_1 \dots a_l$ holds. Therefore, for a string w' with $X \xRightarrow{*}_G w'$, there exists a string w such that $w \in L(\gamma_X) \cap R_X$ and $h(w) = w'$.

In the above claim, consider the case $X = S$, then $\gamma_S = \gamma$ and $R_S = R$ hold. Therefore, if a string w' is in $L(G)$, then there is a string w such that $w \in L(\gamma) \cap R$ and $h(w) = w'$.

Conversely, we consider the reverse inclusion.

[$L(G) \supseteq h(L(\gamma) \cap R)$] We will show that for nonempty string w in $L(\gamma) \cap R$, $h(w)$ is in $L(G)$. We start by showing that if a string w is in $L(\gamma_X) \cap R_X$, then there is a derivation $X \xRightarrow{*}_G h(w)$.

Suppose that w is in $L(\gamma_X) \cap R_X$. In order to derive the string w in $R_X \subset (\Omega \cup T\Omega)^*$, without loss of generality, we may assume here that for each r in P the first two steps in γ_X are performed by r -pair insertion rules. For the derivation $X \xRightarrow{2n}_{\gamma_X} w$, we will prove the claim by induction on n .

Base step: Suppose that there is a derivation $X \xRightarrow{\gamma_X} X\bar{X}\bar{r} \xRightarrow{\gamma_X} arX\bar{X}\bar{r} \in L(\gamma_X) \cap R_X$. From the definition of P_γ , for the r -pair insertion rules there is a production rule $r : X \rightarrow a$ in P . Then, there is a derivation $X \xRightarrow{r}_G a$, where $h(arX\bar{X}\bar{r}) = a$.

Induction step: We suppose that the claim holds for

any $k \leq n$. Consider a derivation $X \Rightarrow_{\gamma_X} X\bar{X}\bar{r} \Rightarrow_{\gamma_X} YZrX\bar{X}\bar{r} \Rightarrow_{\gamma_X}^{2n} w$, where $w \in L(\gamma_X) \cap R_X$.

For the r -pair insertion rules (λ, YZr, λ) and $(\lambda, \bar{X}\bar{r}, \lambda)$, by the construction of P_γ , there is the production rule $r : X \rightarrow YZ$ in P .

For w to be in R_X , considering the r -pair insertion rules in γ_X , w is in the form of $u_Y u_Z r X \bar{X} \bar{r}$, where $u_Y u_Z \in \Sigma^*$. Further, the string $u_Y u_Z$ should be in $(\Omega \cup T\Omega)^*$, which implies that the symbols Y and Z must be Ω -blocked. Then r_Y -pair insertion rules $((\lambda, \alpha r_Y, \lambda), (\lambda, \bar{Y}\bar{r}_Y, \lambda))$ and r_Z -pair of insertion rules $((\lambda, \beta r_Z, \lambda), (\lambda, \bar{Z}\bar{r}_Z, \lambda))$ with $\alpha, \beta \in (N \cup T)^*$ are used for Y and Z to be Ω -blocked, respectively. We thus obtain $\alpha r_Y Y \bar{Y} \bar{r}_Y$ and $\beta r_Z Z \bar{Z} \bar{r}_Z$, respectively.

Then, let $u_Y = u_1 r_Y Y \bar{Y} \bar{r}_Y \in (\Omega \cup T\Omega)^*$ and $u_Z = u_2 r_Z Z \bar{Z} \bar{r}_Z \in (\Omega \cup T\Omega)^*$ with $u_1, u_2 \in \Sigma^*$. Since $w = u_Y u_Z r X \bar{X} \bar{r}$ is in R_X , $u_Y \in R_Y$ and $u_Z \in R_Z$ hold from the definition of R_Y and R_Z . Further, from the assumption that w is in $L(\gamma_X)$, u_Y is in $L(\gamma_Y)$ and u_Z is in $L(\gamma_Z)$. As a result, from the induction hypothesis, there are derivations $Y \Rightarrow_G^* h(u_Y)$, $Z \Rightarrow_G^* h(u_Z)$.

Therefore, there is a derivation

$$\begin{aligned} X &\Rightarrow_G YZ \\ &\Rightarrow_G^* h(u_Y)h(u_Z) = h(u_Y u_Z r X \bar{X} \bar{r}) = h(w). \end{aligned}$$

In the above claim, consider the case $X = S$, then $\gamma_S = \gamma$ and $R_X = R$ hold. Therefore, if there is a nonempty string $w \in L(\gamma) \cap R$, then a string $h(w)$ is in $L(G)$.

Proof of Lemma 1: Let us consider the case that λ is in $L(G)$. Since G is in Chomsky normal form, λ is in $L(G)$ if and only if for λ there is a derivation $S \xrightarrow{r} \lambda$. By the construction of P_γ and R , the string λ is in $L(G)$ if and only if $(\lambda, r, \lambda), (\lambda, \bar{S}\bar{r}, \lambda) \in P_\gamma$ and $rS\bar{S}\bar{r} \in A \cap B$. Then there is a derivation $S \Rightarrow_\gamma S\bar{S}\bar{r} \Rightarrow_\gamma rS\bar{S}\bar{r} \in A \cap B$. From the definition of h , $h(rS\bar{S}\bar{r}) = \lambda$. Therefore, λ is in $L(G)$ if and only if λ is in $h(L(\gamma) \cap R)$. Together with the inclusions $L(G) \subseteq h(L(\gamma) \cap R)$ and $L(G) \supseteq h(L(\gamma) \cap R)$, we complete the proof. $\square$

From Theorem 2, the class of context-free languages includes the class of insertion languages INS_3^0 . Together with the fact that the class of context-free languages is closed under intersection with regular languages and morphisms, we have the following lemma.

Lemma 2: $H(INS_3^0 \cap LOC(4)) \subseteq CF$.

From Lemma 1 and Lemma 2, we have the following theorem.

Theorem 3: $CF = H(INS_3^0 \cap LOC(4))$.

Furthermore, from the fact that for arbitrary k with $k \geq 1$, the class of regular languages includes the class of strictly k -testable languages, we have the following corollary.

Corollary 1: $CF = H(INS_3^0 \cap LOC(k))$ ($k \geq 4$).

4. Refinement of the Chomsky-Schützenberger Representation Theorem

In this section, we improve the celebrated Chomsky-Schützenberger representation theorem for the class of context-free languages, in the form $CF = H(Dyck \cap REG)$. To show the equality $CF = H(Dyck \cap LOC(3))$, we first consider the following theorem.

Theorem 4: $H(Dyck \cap REG) \subseteq H(Dyck \cap LOC(3))$.

Construction: Let $h : T^* \rightarrow \Gamma^*$ be a morphism, D be Dyck language over Σ and $\bar{\Sigma}$, and $G = (N, T, P, S)$ with $T = \Sigma \cup \bar{\Sigma}$ be a regular grammar. We construct Dyck language D' , strictly 3-testable languages, and morphism h' as follows.

- Strictly 3-testable languages.

For any $N_1, N_2 \in N$, we construct

$$L(N_1 : N_2) = A(N_1)\Sigma^* \cap \Sigma^* B(N_2) - \Sigma^+ C' \Sigma^+$$

with $C' = \Sigma^3 - C$, where

$$\begin{aligned} A(N_1) &= \{N_1 \bar{N}_1 a \mid N_1 \rightarrow aX \in P, X \in N, a \in T\}, \\ B(N_2) &= \{aN_2 \bar{N}_2 \mid X \rightarrow aN_2 \in P, X \in N, a \in T\}, \\ C &= \{\bar{X}aY, aY\bar{Y}, X\bar{X}a \mid X \rightarrow aY \in P, \\ &\quad X, Y \in N, a \in T\}. \end{aligned}$$

The language $L(N_1 : N_2)$ can be characterized by using $\Gamma = T\{M\bar{M} \mid M \in N\}$ such that $L(N_1 : N_2) \subseteq \{N_1 \bar{N}_1\} \Gamma^* T \{N_2 \bar{N}_2\}$.

By using the new symbols F and $\bar{F}$, for any $N_1 \in N$, we construct

$$\begin{aligned} L(N_1 : F) &= A(N_1 : F)\Sigma^* \cap \Sigma^* B(N_1 : F) - \Sigma^+ C'(F)\Sigma^+ \end{aligned}$$

with $C'(F) = \Sigma^3 - C(F)$, where

$$\begin{aligned} A(N_1 : F) &= \{N_1 \bar{N}_1 a \mid N_1 \rightarrow aX \in P \text{ or } \\ &\quad N_1 \rightarrow a \in P, a \in T, X \in N\} \cup \\ &\quad \{S\bar{S}F \mid N_1 = S, S \rightarrow \lambda \in P\}, \\ B(N_1 : F) &= \{aF\bar{F} \mid X \rightarrow a \in P, a \in T\} \cup \\ &\quad \{SF\bar{F} \mid N_1 = S, S \rightarrow \lambda \in P\}, \\ C(F) &= \{\bar{X}aY, aY\bar{Y}, X\bar{X}a \mid X \rightarrow aY \in P, \\ &\quad X, Y \in N, a \in T\} \cup \\ &\quad \{X\bar{X}a, \bar{X}aF \mid X \rightarrow a \in P, \\ &\quad X \in N, a \in T\}. \end{aligned}$$

From the above definitions, for any $N_1 \in N$, the followings hold:

$$\begin{aligned} A(N_1), B(N_1) &\subset C \subset C(F), \\ A(N_1) &\subseteq A(N_1 : F). \end{aligned}$$

- Dyck language D' .

By using the new symbols F and $\bar{F}$, D' is a Dyck language over $\Sigma \cup N \cup \{F\}$ and $\bar{\Sigma} \cup \bar{N} \cup \{\bar{F}\}$.

- Homomorphism h' .

For $V = T \cup N \cup \bar{N} \cup \{F, \bar{F}\}$, we define $h' : V^* \rightarrow \Gamma^*$ by

$$\begin{aligned} h'(\lambda) &= \lambda \\ h'(ax) &= h'(a)h'(x) \quad \text{if } a \in T, \\ h'(ax) &= h'(x) \quad \text{if } a \notin T. \end{aligned}$$

Idea of the Proof: For any string x in $L(G)$, a derivation $S \Rightarrow_G^* x$ corresponds to a string $y \in D' \cap L(S : F)$. Since Dyck language D' is defined over $\Sigma \cup N \cup F$ and $\bar{\Sigma} \cup \bar{N} \cup \bar{F}$, for Dyck language D over Σ and $\bar{\Sigma}$, any string x in $D \cap L(G)$ is simulated by $D' \cap L(G')$. Finally, the symbol in $N \cup \bar{N} \cup \{F, \bar{F}\}$ is eliminated and the symbol in T is transformed into Γ by h' .

To show $h(D \cap L(G)) = h'(D' \cap L(S : F))$, we consider the following inclusion.

[$h(D \cap L(G)) \subseteq h'(D' \cap L(S : F))$] To show the inclusion, we first prove that for any x which satisfies that $x \in D$, $|x| = 2n$, and $X \Rightarrow_G^* xY$ with $X, Y \in N$, there is a string $y \in D' \cap L(X : Y)$ such that $h'(y) = h(x)$ by induction on n .

Base step: Consider a string $x = a\bar{a} \in D$ such that $X \Rightarrow_G aN_1 \Rightarrow_G a\bar{a}Y$ with $N_1 \in N$. For the production $X \rightarrow aN_1 \in P$, by the construction of $L(X : Y)$, $X\bar{X}a \in A(X)$ and $\bar{X}aN_1, aN_1\bar{N}_1 \in C$ hold. Similarly, for the production $N_1 \rightarrow \bar{a}Y \in P$, $\bar{a}Y\bar{Y} \in B(Y)$ and $\bar{N}_1\bar{a}Y, N_1\bar{N}_1\bar{a} \in C$ hold.

Then a string $y = X\bar{X}aN_1\bar{N}_1\bar{a}Y\bar{Y}$ satisfies that $y \in L(X : Y)$ and $h'(y) = h'(a\bar{a}) = h(a\bar{a}) = h(x)$.

Induction step: We suppose that the claim holds for any $k \leq n$. Consider a string $x \in D$ which satisfies that $X \Rightarrow_G^* xY$ and $|x| = 2(n+1)$. Since x is in D , there are the following two cases for x :

1. $x = \alpha\beta$, where $\alpha, \beta \in D$ and $2 \leq |\alpha|, |\beta| \leq n$.

Let N_1 be a nonterminal symbol in N such that $X \Rightarrow_G^* \alpha N_1 \Rightarrow_G^* \alpha\beta Y$.

For the string $\alpha \in D$ satisfying that $X \Rightarrow_G^* \alpha N_1$ and $2 \leq |\alpha| \leq n$, from the induction hypothesis, there is a string y_1 such that $y_1 = X\bar{X}z_1N_1\bar{N}_1 \in L(X : N_1) \cap D'$ with $z_1 \in D'$ and $h(\alpha) = h'(y_1)$. Similarly for β , there is a string y_2 such that $y_2 = N_1\bar{N}_1z_2Y\bar{Y} \in L(N_1 : Y) \cap D'$ with $z_2 \in D'$ and $h(\beta) = h'(y_2)$.

Consider a string $y = X\bar{X}z_1N_1\bar{N}_1z_2Y\bar{Y}$. From the fact that $A(N_1), B(N_1) \subset C$, y is in $L(X : Y)$. Since z_1 and z_2 are in D' , y is in D' . Further,

$$\begin{aligned} h'(y) &= h'(X\bar{X}z_1N_1\bar{N}_1z_2Y\bar{Y}) \\ &= h'(X\bar{X}z_1N_1\bar{N}_1N_1\bar{N}_1z_2Y\bar{Y}) \\ &= h'(y_1)h'(y_2) = h(\alpha)h(\beta) = h(\alpha\beta). \end{aligned}$$

2. $x = a\bar{a}\bar{a}$, where $\alpha \in D$, $a \in \Sigma$, $\bar{a} \in \bar{\Sigma}$, and $|\alpha| = 2n$.

Let N_1, N_2 be nonterminal symbols in N such that $X \Rightarrow_G^* aN_1 \Rightarrow_G^* a\bar{a}N_2 \Rightarrow_G^* a\bar{a}\bar{a}Y$.

For the string α , from the induction hypothesis, there is a string $y_1 = N_1\bar{N}_1z_1N_2\bar{N}_2$ such that $y_1 \in D'$, $h(\alpha) = h'(y_1)$, and $y_1 \in L(N_1 : N_2)$.

For the production rule $X \rightarrow aN_1$ in P , by the construction of $L(X : Y)$, $X\bar{X}a \in A(X)$ and $\bar{X}aN_1, aN_1\bar{N}_1 \in C$ hold. Similarly, for the production rule $N_2 \rightarrow \bar{a}Y$ in P , $N_2\bar{N}_2\bar{a}Y \in C$ and $aY\bar{Y} \in B(Y)$ hold. From the fact that $A(N_1), B(N_2) \subset C$, a string $y = X\bar{X}ay_1\bar{a}Y\bar{Y} = X\bar{X}aN_1\bar{N}_1z_1N_2\bar{N}_2\bar{a}Y\bar{Y}$ is in $L(X : Y)$. Since z_1 is in D' , y is in D' . Further,

$$\begin{aligned} h'(y) &= h'(X\bar{X}ay_1\bar{a}Y\bar{Y}) \\ &= h'(X\bar{X}a)h'(y_1)h'(\bar{a}Y\bar{Y}) \\ &= h(a)h(\alpha)h(\bar{a}) = h(a\bar{a}\bar{a}). \end{aligned}$$

The inclusion $h(D \cap L(G)) \subseteq h'(D' \cap L(S : F))$ can be proved by considering the case $X = S$ and $Y = F$ in the previous claim.

Note that if λ is in $D \cap L(G)$, then $S\bar{S}F \in A(S : F)$ and $S\bar{S}\bar{F} \in B(S : F)$ hold. The string $S\bar{S}F\bar{F}$ satisfies $S\bar{S}F\bar{F} \in L(S : F) \cap D'$ and $h'(S\bar{S}F\bar{F}) = \lambda = h(\lambda)$.

[$h(D \cap L(G)) \supseteq h'(D' \cap L(S : F))$] We will prove the converse inclusion, starting by showing that for a string $y \in D' \cap L(X : Y)$ with $X, Y \in N$, there is a string x such that $x \in D$, $X \Rightarrow_G^* xY$, and $h(x) = h'(y)$.

For the string $y \in D' \cap L(X : Y)$ such that $|y|_T = 2n$, we will prove the claim by induction on n .

Base step: Consider a string $y = X\bar{X}aN_1\bar{N}_1\bar{a}Y\bar{Y}$ in $D' \cap L(X : Y)$. From the fact $\bar{X}aN_1, \bar{N}_1\bar{a}Y \in C$, production rules $X \rightarrow aN_1$ and $N_1 \rightarrow \bar{a}Y$ are in P . For the string $x = a\bar{a}$, there is a derivation $X \Rightarrow_G aN_1 \Rightarrow_G a\bar{a}Y$ and $h(x) = h'(X\bar{X}aN_1\bar{N}_1\bar{a}Y\bar{Y}) = h'(y)$.

Induction step: We suppose that the claim holds for any $k \leq n$. Consider a string $y \in D' \cap L(X : Y)$ which satisfies that $|y|_T = 2(n+1)$. There are the following cases for y .

- $y = \alpha\beta$, where

$$\begin{aligned} \alpha &= X\bar{X}a_1N_1\bar{N}_1a_2N_2\bar{N}_2 \cdots a_{2i}N_{2i}\bar{N}_{2i} \in D', \\ \beta &= b_1N_{2i+1}\bar{N}_{2i+1}b_2N_{2i+2}\bar{N}_{2i+2} \cdots b_{2j}Y\bar{Y} \in D', \end{aligned}$$

with $i, j \geq 1$ and $i + j = n + 1$.

From the induction hypothesis, for the string α in $D' \cap L(X : N_{2i})$, there is a string $x_1 \in D$ such that $X \Rightarrow_G^* x_1N_{2i}$ and $h(x_1) = h'(\alpha)$.

For the string β in D' , consider a string $\beta' = N_{2i}\bar{N}_{2i}\beta$ in D' . From the induction hypothesis for $\beta' \in L(N_{2i} : Y) \cap D'$, there is a string x_2 such that $x_2 \in D$, $N_{2i} \Rightarrow_G^* x_2Y$ and $h(x_2) = h'(\beta')$.

From the fact that $x_1, x_2 \in D$, the string x_1x_2 is in D , which satisfies that $X \Rightarrow_G^* x_1N_{2i} \Rightarrow_G^* x_1x_2Y$ and $h(x_1x_2) = h(x_1)h(x_2) = h'(\alpha)h'(\beta') = h'(\alpha)h'(\beta) = h'(\alpha\beta)$.

- $y = \alpha\beta$, where

$$\begin{aligned} \alpha &= X\bar{X}a_1N_1\bar{N}_1a_2N_2\bar{N}_2 \cdots a_{2i-1}N_{2i-1}\bar{N}_{2i-1} \in D', \\ \beta &= N_{2i}\bar{N}_{2i}b_1N_{2i+1}\bar{N}_{2i+1}b_2 \cdots b_{2j}Y\bar{Y} \in D', \end{aligned}$$

with $i, j \geq 1$ and $i + j = n + 1$.

In a similar way to the case $y = \alpha\beta$, where $\alpha = X\bar{X}a_1N_1\bar{N}_1a_2N_2\bar{N}_2 \cdots a_{2i}N_{2i}\bar{N}_{2i}$ and $\beta = b_1N_{2i+1}\bar{N}_{2i+1}b_2N_{2i+2}\bar{N}_{2i+2} \cdots b_{2j}Y\bar{Y}$, we can prove the existence of a string x such that $x \in D$, $h(x) = h'(\alpha\beta)$, and $X \Rightarrow_G^* xY$.

- $y = X\bar{X}a_1za_{2n+2}Y\bar{Y}$, where $z = N_1\bar{N}_1a_2 \cdots a_{2n+1}N_{2n+1}\bar{N}_{2n+1} \in D'$, $Y = X$ and $a_{2n+2} = \bar{a}_1$.
For the string $z \in D' \cap L(N_1 : N_{2n+1})$, from the induction hypothesis, there is a string $x_1 \in D$,

$N_1 \Rightarrow_G^* x_1 N_{2n+1}$, and $h(x_1) = h'(z)$.

From the strings $\bar{X}a_1N_1$ and $\bar{N}_{2n+1}\bar{a}_1X$ in C , there are production rules $X \rightarrow a_1N_1$ and $N_{2n+1} \rightarrow \bar{a}_1X$ in P . Therefore, $X \Rightarrow_G a_1N_1 \Rightarrow_G^* a_1x_1N_{2n+1} \Rightarrow_G a_1x_1\bar{a}_1X$ holds. Further, the string $a_1x_1\bar{a}_1$ satisfies that $h(a_1x_1\bar{a}_1) = h(X\bar{X}a_1z\bar{a}_1Y\bar{Y}) = h(y)$ and $a_1x_1\bar{a}_1 \in D$.

The inclusion $h(D \cap L(G)) \supseteq h'(D' \cap L(S : F))$ can be proved by considering the case $X = S$ and $Y = F$ in the previous claim.

Note that if $S\bar{S}F\bar{F} \in D' \cap L(S : F)$, then for the string $S\bar{S}F \in A(S : F)$, by the construction of $L(S : F)$, there is a production $S \rightarrow \lambda$ in P , which implies that $\lambda \in D \cap L(G)$. Further, $h'(S\bar{S}F\bar{F}) = \lambda = h(\lambda)$ holds. $\square$

Since the class of context-free languages is closed under intersection with regular languages and morphisms, $H(\text{Dyck} \cap \text{LOC}(3)) \subseteq CF$ holds from $\text{LOC}(3) \subset \text{REG}$ in Theorem 1. Furthermore, from Theorem 4 and Chomsky-Schützenberger characterization of context-free languages $CF = H(\text{Dyck} \cap \text{REG})$, we have the following theorem.

Theorem 5: $CF = H(\text{Dyck} \cap \text{LOC}(3))$.

From the definition of Dyck language, we can easily show that for any Dyck language D , there is an insertion system γ of weight $(2, 0)$ which satisfies that $D = L(\gamma)$. Therefore, the next result follows from the fact $\text{Dyck} \subset \text{INS}_2^0 \subset CF$ and Theorem 5.

Note 1: $CF = H(\text{INS}_2^0 \cap \text{LOC}(3))$.

Furthermore, from Theorem 2 and the fact that for arbitrary k with $k \geq 1$, the class of regular languages includes the class of strictly k -testable languages, we have the following corollary.

Corollary 2: $CF = H(\text{Dyck} \cap \text{LOC}(k))$ ($k \geq 3$).
 $CF = H(\text{INS}_i^0 \cap \text{LOC}(k))$ ($i \geq 2, k \geq 3$).

5. Conclusion

In this paper, we have contributed to the study of insertion systems with new characterizations of context-free languages. Specifically, we have shown that $CF = H(\text{Dyck} \cap \text{LOC}(k))$ ($k \geq 3$) and $CF = H(\text{INS}_i^0 \cap \text{LOC}(k))$ ($i \geq 2, k \geq 3$).

The following characterizations of regular languages in terms of insertion languages and strictly locally testable languages have shown in [8].

- $H(\text{INS}_1^0 \cap \text{LOC}(1)) \subset \text{REG}$.
- $\text{REG} = H(\text{INS}_1^0 \cap \text{LOC}(k))$ ($k \geq 2$).
- REG and $H(\text{INS}_i^0 \cap \text{LOC}(1))$ are incomparable ($i \geq 2$).
- $\text{REG} \subset H(\text{INS}_i^0 \cap \text{LOC}(k))$ ($i \geq 2, k \geq 2$).

The followings are open problems:

- Can CF be represented as $CF = H(\text{INS}_2^0 \cap \text{LOC}(2))$? This is useful to show that the value of parameter $k = 3$ in the strictly k -testable languages in Note 1 is necessary for expressing context-free languages, or not.

- Can CF be represented as $CF = H(\text{INS}_i^j \cap \text{LOC}(k))$ for $i, j \geq 1$ and $k \geq 1$?

This is useful to see the difference between insertion systems with context-checking and context-free insertion systems.

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