Program Transformation Templates for Tupling Based on Term Rewriting

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SUMMARY Chiba et al. (2006) proposed a framework of program transformation of term rewriting systems by developed templates. Contrast to the previous framework of program transformation by templates based on lambda calculus, this framework provides a method to verify the correctness of transformation automatically. Tupling (Bird, 1980) is a well-known technique to eliminate redundant recursive calls for improving efficiency of programs. In Chiba et al.’s framework, however, one can not use tuple symbols to construct developed templates. Thus their framework is not capable of tupling transformations. In this paper, we propose a more flexible notion of templates so that a wider variety of transformations, including tupling transformations, can be handled.

key words: program transformation, tupling, term rewriting

1. Introduction

Several techniques for optimizing functional programs by transformation have been developed [4], [9], [14], [16], [23]. One of the general such frameworks is a program transformation by templates proposed by Huet and Lang [16]. In this framework, a program is transformed according to a given program transformation template—a template consists of program schemas for input and output programs and a set of equations that expresses the precondition of operators contained in the programs. In contrast to specific program transformations such as fusion [14], [23] or tupling [3], [9], [15], the program transformation by template provides a general framework for treating various types of program transformations by preparing suitable templates in a uniform way.

In Huet and Lang’s framework, the programs and program schemas are given by second-order simply-typed lambda terms. A program transformation is carried out (automatically) by the second-order matching and the correctness is established (manually) based on the denotational semantics. For the former part, matching algorithms for the program transformation by template have been improved to achieve more flexible and efficient transformations [11], [13], [20], [24]. In contrast, the latter part, namely the verification of the correctness of program transformation, has been paid less attention. But in general, the second-order matching often provides many matching solutions and not all of them are appropriate for program transformations. Therefore, the verification of the correctness is expected to be important for this framework.

Chiba et al. proposed a framework of program transformation by templates based on term rewriting [5]–[8]. In this framework, programs and program schemas are given by term rewriting systems (TRSs for short) and TRS patterns—a TRS in which pattern variables (which will be instantiated for program transformations) are used in the place of function symbols. They gave a criterion for the correctness of transformations by developed templates. The criterion consists of some properties of input and output TRSs such as confluence, sufficient completeness and of inductive validity of the instantiated precondition. The former two properties can be checked automatically in some classes and automated inductive theorem proving methods can be applied for the last property. Thus, once a suitable developed template is constructed, this result provides a method to verify the correctness of transformations automatically.

Tupling [3], [9], [15], which improves efficiency of programs by eliminating redundant recursive calls, is one of the well-known program transformation techniques in functional programming. Chiba et al.’s framework, however, is not capable of tupling, since the notion of developed templates is not expressive enough to handle tupling transformations. More precisely, one can not deal with tuple symbols, which play an essential role in tupling transformations, in the construction of developed templates. In this paper, we give a more flexible notion of templates called correct templates so that a wider variety of transformations, including tupling transformations, can be handled. We introduce a notion of carrying of signatures for term homomorphisms to give a criterion to guarantee the correctness of program transformation by those templates.

The rest of the paper is organized as follows. In Sect. 2, we first explain the framework of transformation by templates based on term rewriting and then give some basic notions and notations used in this paper. In Sect. 3, we explain tupling transformation in term rewriting. In Sect. 4, we explain a method to show the correctness of program transformation based on equivalent transformations. In Sect. 5, we introduce notions of correct templates and carrying of signatures. We then prove a new criterion to guarantee correctness of transformations. Several templates for tupling transformations are also presented. In Sect. 6, we compare our new framework with the old one. We conclude in Sect. 7.
2. Transformation by Templates

In this section, we first explain the framework of program transformation by templates based on term rewriting [5]–[7].

Let us describe a program transformation by the following template \( \langle \mathcal{P}, \mathcal{P}', \mathcal{H} \rangle \).

\[
\mathcal{P} = \begin{cases} 
    f(a) & \rightarrow b \\
    f(c(u_1, v_1)) & \rightarrow g(u_1, f(v_1)) \\
    g(b, u_2) & \rightarrow u_2 \\
    g(d(u_3, v_3), w_3) & \rightarrow d(u_3, g(v_3, w_3)) 
\end{cases}
\]

\[
\mathcal{P}' = \begin{cases} 
    f(f_1(a, u_5)) & \rightarrow u_5 \\
    f_1(c(u_6, v_6), w_6) & \rightarrow f_1(v_1, g(w_6, u_6)) \\
    g(b, u_7) & \rightarrow u_7 \\
    g(d(u_8, v_8), w_8) & \rightarrow d(u_8, g(v_8, w_8)) 
\end{cases}
\]

\[
\mathcal{H} = \begin{cases} 
    g(b, u_1) & \approx g(u_1, b) \\
    g(g(h_2, v_2), w_2) & \approx g(u_2, g(v_2, w_2)) 
\end{cases}
\]

The TRS pattern \( \mathcal{P} \) is the source and \( \mathcal{P}' \) is the target, i.e. \( \mathcal{P} \) is a schema of input programs to be transformed and \( \mathcal{P}' \) is a schema whose instantiations become the output programs. The set \( \mathcal{H} \) of equations is a schema of hypothesis whose instantiations are used to verify the correctness of transformations. This template is for a program transformation from recursive programs to iterative programs.

Consider, for example, the following TRS \( \mathcal{R}_{\text{sum}} \) which specifies a program that computes the summation of a list of natural numbers. Here the natural numbers 0, 1, 2, ..., are expressed as 0, s(0), s(s(0)), ... .

\[
\mathcal{R}_{\text{sum}} = \begin{cases} 
    \text{sum}([]) & \rightarrow 0 \\
    \text{sum}(x_1, y_1) & \rightarrow +(x_1, \text{sum}(y_1)) \\
    +(0, x_2) & \rightarrow x_2 \\
    +(s(x_3), y_3) & \rightarrow s(+((x_3), y_3)) 
\end{cases}
\]

The TRS \( \mathcal{R}_{\text{sum}} \) computes the summation of a list using a recursive call. For instance, \( \text{sum}(1:(2:(3:(4:(5:[]))))) \rightarrow_{\mathcal{R}_{\text{sum}}} +(1, +(2, +(3, +(4, +(5, \text{sum}([]))))) \rightarrow_{\mathcal{R}_{\text{sum}}} 15. \)

To transform this program to the iterative form, we perform a pattern matching with the source \( \mathcal{P} \) against the input TRS \( \mathcal{R}_{\text{sum}} \). Using the matching algorithm presented in [6], [7], one finds the following term homomorphism \( \varphi \) satisfying \( \mathcal{R}_{\text{sum}} = \varphi(\mathcal{P}) \).

\[
\varphi = \begin{cases} 
    f \mapsto \text{sum}(\text{\texttt{d}}) \quad a \mapsto [1] \quad b \mapsto 0 \\
    g \mapsto +(\text{\texttt{C}}(\text{\texttt{d}}, \text{\texttt{d}})) \quad c \mapsto \text{\texttt{C}}(\text{\texttt{d}}, \text{\texttt{d}}) \\
    f_1 \mapsto \text{\texttt{sum}}(\text{\texttt{d}}(\text{\texttt{d}})) \quad d \mapsto \text{\texttt{d}}(\text{\texttt{d}}) \\
    u_1 \mapsto x_1 \quad u_2 \mapsto x_2 \quad u_4 \mapsto x_4 \\
    u_5 \mapsto x_5 \quad u_6 \mapsto x_6 \quad u_7 \mapsto x_7 \\
    v_1 \mapsto y_1 \quad v_3 \mapsto x_3 \quad v_6 \mapsto y_6 \\
    v_8 \mapsto y_8 \quad w_3 \mapsto y_3 \quad w_6 \mapsto y_6 \\
    w_9 \mapsto z_8 
\end{cases}
\]

The output TRS \( \mathcal{R}'_{\text{sum}} \), the iterative form \( \dagger \) of \( \mathcal{R}_{\text{sum}} \), is obtained by applying \( \varphi \) to the target \( \mathcal{P}' \) i.e. \( \mathcal{R}'_{\text{sum}} = \varphi(\mathcal{P}') \).

\[
\begin{align*}
\mathcal{R}'_{\text{sum}} = & \begin{cases} 
\text{sum}(x_4) & \rightarrow \text{sum}(x_4, 0) \\
\text{sum}(y_5, z_6) & \rightarrow \text{sum}(y_5, +(z_6, x_6)) \\
+(0, x_7) & \rightarrow x_7 \\
+(s(y_8), z_8) & \rightarrow s(+((y_8), z_8)) 
\end{cases}
\end{align*}
\]

The term homomorphism \( \varphi \) is also used to generate the following set \( \mathcal{E}_{\text{sum}} = \varphi(\mathcal{H}) \) of equations.

\[
\mathcal{E}_{\text{sum}} = \begin{cases} 
+(0, x_1) & \approx +(x_1, 0) \\
+(+(x_2, v_2), v_2) & \approx +(x_2, +(v_2, w_2)) 
\end{cases}
\]

It is required to ensure the correctness of the transformation that these equations are inductive consequences of \( \mathcal{R}_{\text{sum}} \). In the course of the transformation the inductive validity of these equations is verified, for example, by automated inductive theorem proving methods. In this case, using the rewriting induction [19], it is successfully proved automatically that the equations in \( \mathcal{E}_{\text{sum}} \) are inductive consequences of \( \mathcal{R}_{\text{sum}} \).

The correctness of the transformation, i.e. the equivalence of the input TRS \( \mathcal{R}_{\text{sum}} \) and the output \( \mathcal{R}'_{\text{sum}} \), is guaranteed if \( \mathcal{R}_{\text{sum}} \) is confluent and \( \mathcal{R}_{\text{sum}} \) and \( \mathcal{R}'_{\text{sum}} \) are sufficiently complete [7]—these conditions can be checked, for example, by checking the joinability of the critical pairs [2] and the complement substitution algorithm [18], [21], respectively, provided that \( \mathcal{R}_{\text{sum}} \) and \( \mathcal{R}'_{\text{sum}} \) are terminating. Termination of a TRS is an undecidable property but various termination proving methods are known and are still actively investigated. In this case, the termination of \( \mathcal{R}_{\text{sum}} \) and \( \mathcal{R}'_{\text{sum}} \) is successfully proved automatically using the lexicographic path order [2].

We now give some definitions to formalize program transformation by template based on term rewriting that will be used in this paper. Familiarity with term rewriting will be helpful in what follows (see e.g. [2]).

Let \( \mathcal{F}, \mathcal{X} \) and \( \mathcal{Y} \) be the sets of function symbols, pattern variables and local variables, respectively. Any \( p \in \mathcal{F} \cup \mathcal{X} \) has its arity, denoted by \( \text{arity}(p) \). The set \( T(\mathcal{F} \cup \mathcal{X}, \mathcal{Y}) \) of term patterns is defined by (1) \( \mathcal{Y} \subseteq T(\mathcal{F} \cup \mathcal{X}, \mathcal{Y}) \); and (2) \( \forall (t_1, \ldots, t_n) \subseteq T(\mathcal{F} \cup \mathcal{X}, \mathcal{Y}) \) for any \( p \in \mathcal{F} \cup \mathcal{X} \) such that \( \text{arity}(p) = n \) and \( t_1, \ldots, t_n \in T(\mathcal{F} \cup \mathcal{X}, \mathcal{Y}) \). A term pattern without pattern variables is called a term. The set of terms is denoted by \( T(\mathcal{F}, \mathcal{Y}) \). The sets of function symbols, pattern variables and local variables in a term pattern \( s \) are denoted by \( \mathcal{F}(s) \), \( \mathcal{X}(s) \) and \( \mathcal{Y}(s) \), respectively. A ground term (pattern) is the one without local variables. We abbreviate \( T(\mathcal{F} \cup \mathcal{X}, \mathcal{Y}), T(\mathcal{F}, \mathcal{X}), \text{etc.} \) as \( T(\mathcal{F}, \mathcal{Y}) \), \( T(\mathcal{F}) \), etc.

A substitution \( \theta \) is a mapping from \( \mathcal{Y} \) to \( T(\mathcal{F} \cup \mathcal{X}, \mathcal{Y}) \). A substitution \( \theta \) is extended to a mapping \( \bar{\theta} \) over term pattern \( T(\mathcal{F} \cup \mathcal{X}, \mathcal{Y}) \) like this: (1) \( \bar{\theta}(x) = \theta(x) \) if \( x \in \mathcal{Y} \), (2) \( \bar{\theta}(p(s_1, \ldots, s_n)) = p(\bar{\theta}(s_1), \ldots, \bar{\theta}(s_n)) \). We usually identify \( \bar{\theta} \) and \( \theta \). We denote \( st \) instead of \( \theta(st) \). The domain of a substitution \( \theta \) (denoted by \( \text{dom}(\theta) \)) is defined by \( \text{dom}(\theta) = \{x \in \mathcal{Y} \mid x \neq \theta(x) \} \).

\( \dagger \)Precisely speaking, only the rules for \( \text{sum} \) are changed to iterative and those for \( + \) remain recursive.
Consider special (indexed) constants \( \alpha_i \) (\( i \geq 1 \)) called holes such that \( \alpha_i \notin \mathcal{F} \). An (indexed) context \( C \) is an element of \( T(\mathcal{F} \cup \mathcal{I} \cup \{ \alpha_i \mid i \geq 1 \}, \mathcal{V}) \). \( C[s_1, \ldots, s_n] \) is the result of \( C \) replacing \( \alpha_i \) by \( s_i, \ldots, s_n \) from left to right. \( C[s_1, \ldots, s_n] \) is the result of \( C \) replacing \( \alpha_i \) by \( s_i \) for \( i = 1, \ldots, n \) (indexed replacement). (For example, \( \text{f}(\alpha_1, \text{g}(\alpha_2), \text{h}(\alpha_1, y)) = \text{f}(\text{g}(x), \text{h}(\alpha_1, y)) \) because \( \alpha \) context with precisely one hole is denoted by \( C[\square] \). The set of contexts is denoted by \( T^C(\mathcal{F} \cup \mathcal{I} \cup \{ \alpha \mid i \leq n \}, \mathcal{V}) \). \( T^C(\mathcal{F}) \) and \( T^C(\mathcal{F}) \) are defined in the same way as \( T(\mathcal{F}) \).

A pair \( (l, r) \) of term patterns is a rewrite rule if \( l \notin \mathcal{V} \) and \( \mathcal{V}(l) \supseteq \mathcal{V}(r) \). We usually write the rewrite rule \( (l, r) \) as \( l \rightarrow r \); \( l \) is the left-hand-side (lhs for short) of the rule and \( r \) is the right-hand-side (rhs for short). A term rewriting system \((\mathcal{F}, \mathcal{R})\) for some \( \mathcal{V} \) is an \( \mathcal{R} \) such that \( \mathcal{F} \) is a TRS pattern over \( \mathcal{F} \) and \( \mathcal{R} \) is a TRS pattern over \( \mathcal{F} \). We assume that the set \( \mathcal{R} \) is the set of rewrite rules of \( \mathcal{F} \).

We note that confluence and termination of TRSs are strongly normalizing (SN) (\( \mathcal{R} \)) if there exists no infinite reduction \( s_1 \rightarrow_R s_2 \rightarrow_R s_3 \rightarrow_R \cdots \). We note that confluence and termination of TRSs are undecidable problems in general. However, confluence of terminating TRSs can be detected by checking the joinability of critical pairs of \( \mathcal{R} \).

A transformation template (or just template) is a triple \( (\mathcal{P}, \mathcal{P}', \mathcal{H}) \) of TRS patterns \( \mathcal{P}, \mathcal{P}' \) and a set \( \mathcal{H} \) of equations. The TRS patterns \( \mathcal{P} \) and \( \mathcal{P}' \) are the source and the target of the template, respectively, and the set \( \mathcal{H} \) of equations is the hypothesis of the template. A mapping \( \varphi \) from \( \mathcal{P} \) to \( \mathcal{P}' \) is a term homomorphism if \( \varphi(p) \neq \varphi(p') \) for any \( p \) in \( \mathcal{P} \). If \( \varphi(x) \neq \varphi(y) \) for any \( x \) and \( y \) in \( \mathcal{P} \), then \( \varphi(x) \neq \varphi(y) \). Here, the domains of \( \varphi \) over pattern variables and local variables are defined by \( \text{dom}_\mathcal{P}(\varphi) = \{ p \in \mathcal{P} \mid \varphi(p) \neq \varphi(p', \ldots, p_{\text{arity}(p)}) \} \) and \( \text{dom}_\mathcal{P}(\varphi) = \{ x \in \mathcal{P} \mid \varphi(x) \neq x \} \), respectively. A term homomorphism \( \varphi \) is extended to a mapping \( \varphi \) over \( T(\mathcal{F} \cup \mathcal{I}, \mathcal{V}) \) as follows:

\[
\varphi(s) = \begin{cases} 
\varphi(s) & \text{if } s = x \in \mathcal{P} \\
\varphi(f(s_1, \ldots, s_n)) & \text{if } s = f(s_1, \ldots, s_n), \ f \in \mathcal{F} \\
\varphi(p)(\varphi(s_1), \ldots, \varphi(s_n)) & \text{if } s = p(s_1, \ldots, s_n), \ p \in \mathcal{P}' 
\end{cases}
\]

A term homomorphism is extended to a mapping on rewrite rules and equations in the obvious way. For term homomorphisms \( \varphi \) and \( \varphi' \) such that \( \text{dom}(\varphi) \cap \text{dom}(\varphi') = \emptyset \), \( \varphi \cup \varphi' \) is a term homomorphism defined by \( (\varphi \cup \varphi')(p) = \varphi(p) \) if \( p \in \text{dom}(\varphi) \) and \( \varphi(p) \) otherwise. The following basic property of the term homomorphisms will be used later.

**Proposition 1** (Proposition 1 of [6]): Let \( \varphi \) be a term homomorphism, \( \mathcal{P} \) a TRS pattern and \( \mathcal{H} \) a set of equations. If \( s \rightarrow_R t \) (\( s \leftrightarrow_R t \) then we have \( \varphi(s) \rightarrow_{\mathcal{P}} \varphi(t) \) (resp. \( \varphi(s) \leftrightarrow_{\mathcal{P}} \varphi(t) \)).

\[\text{This is very usual (see e.g. Sect. 3 of [2]).}\]
3. Tupling Transformations for Term Rewriting Systems

Tupling is one of the well-known program transformation methods in the field of functional programming [3], [9], [10], [15]. In this section, we give some examples of tupling transformations in term rewriting.

Let us consider one source of inefficiency in programs—namely, inefficient programs may perform the same computations while efficient programs share this same computations and use the result many times. Such a typical program arises from a usual definition of Fibonacci number.

\[
R_{fib}\begin{cases}
  fib(0) & \rightarrow s(0) \\
  fib(s(0)) & \rightarrow s(0) \\
  fib(s(s(x))) & \rightarrow +(fib(s(x)), fib(x)) \\
  +(s(x), y) & \rightarrow s(+(x, y)) \\
  +(0, x) & \rightarrow x \\
\end{cases}
\]

In this program, the function calls from \(fib\) are described in a directed graph as follows.

\[
\text{fib}(5) \rightarrow \text{fib}(4) \rightarrow \text{fib}(3) \rightarrow \text{fib}(2) \rightarrow \ldots \]

\[
\text{fib}(3) \rightarrow \text{fib}(2) \rightarrow \ldots \]

We see that \(fib(3)\) is computed twice and \(fib(2)\) is computed three times and these computations are performed independently. Such multiple computations of the same expression are the reason of exponential time computation of \(fib(n)\).

Tupling eliminates such multiple computations of the same expression called in different contexts. One of the basic methods of tupling in functional programs is operated as follows [9]. First one needs to find some suitable expressions that should be computed in simultaneously. Then one prepares a new function definition which computes these expressions together. Finally, one makes original functions derived from this new function. The name of “tupling” comes from the tuple symbols “(•)” used for coupling the expressions that are simultaneously computed.

In our example, by tupling, one gets the following \(R'_{fib}\) from \(R_{fib}\).

\[
R'_{fib}\begin{cases}
  fib(0) & \rightarrow s(0) \\
  fib(s(0)) & \rightarrow s(0) \\
  fib(s(s(x))) & \rightarrow +(fib(s(x)), fib(x)) \\
  +(s(x), y) & \rightarrow s(+(x, y)) \\
  +(0, x) & \rightarrow x \\
\end{cases}
\]

In \(R'_{fib}\), \(fibpair(n)\) computes \(fib(n)\) and \(fib(n-1)\) simultaneously. \(step((x, y))\) computes \((x + y, x)\). If \(x, y\) are values then this can be computed efficiently. In term rewriting, this is done by sharing evaluation of \(x\) in rhs of the rule \(step(x) \rightarrow (+(\pi_1(x), \pi_2(x)), \pi_1(x))\) (based on e.g. term graph rewriting framework). Thus, \(fib(n)\) can be computed much efficiently using \(R'_{fib}\) instead of \(R_{fib}\).

Let us see another example of tupling transformation. In the following TRS \(R_{factlist}\), \(factlist(n)\) computes a list \([(n - 1)!], \ldots, 0]\).

\[
R_{factlist}\begin{cases}
  factlist(0) & \rightarrow [] \\
  factlist(s(x)) & \rightarrow fact(x):factlist(x) \\
  fact(0) & \rightarrow s(0) \\
  fact(s(x)) & \rightarrow \times(s(x), \text{fact}(x)) \\
  \times(0, y) & \rightarrow 0 \\
  \times(s(x), y) & \rightarrow +(y, \times(x, y)) \\
  +(0, x) & \rightarrow x \\
  +(s(x), y) & \rightarrow s(+(x, y)) \\
\end{cases}
\]

The dependency of function calls from \(factlist(4)\) can be described by the following graph:

\[
\text{factlist}(4) \rightarrow \text{factlist}(3) \rightarrow \text{factlist}(2) \rightarrow \text{factlist}(1) \rightarrow \text{factlist}(0)
\]

\[
\text{fact}(0) \rightarrow \text{fact}(0)
\]

\[
\text{step}(x, (y, s(x))) \rightarrow \text{step}(y, \text{fact}(x))
\]

\[
\text{step}(x, (y, s(x))) \rightarrow (\pi_1(y), \pi_2(y), \pi_3(x, \pi_2(y)))
\]

\[
\times(0, y) \rightarrow 0 \\
\times(s(x), y) \rightarrow +(y, \times(x, y)) \\
+(0, x) \rightarrow x \\
+(s(x), y) \rightarrow s(+(x, y)) \\
\pi_1((x, y)) \rightarrow x \\
\pi_2((x, y)) \rightarrow y \\
\]

In \(R'_{factlist}\), \(factpair(n)\) computes \(factlist(n)\) and \(fact(n)\) simultaneously. \(step(x, (y, s(x)))\) computes \((y, y, s\times x)\). If \(x, y\) are values then this can be computed efficiently.

4. Correctness of Program Transformation

A first question that arises is how correctness of such program transformation can be verified. In this section, we explain a method based on the equivalent transformation [6], [7], [22] on which our framework is based.

To discuss correctness of program transformations, we need to argue about the notion of equivalence of TRSs suitable for program transformations. Let \(G\) be a set of function symbols such that \(\mathcal{F} \subseteq G \subseteq \mathcal{F}\). Two TRSs \(R\) and \(R'\) are said to be equivalent for \(G\) (\(R \equiv_G R'\)), if for any
s ∈ T(𝒟) and t ∈ T(𝒟), s →R t iff s →R∗ t holds [6, 7].
We note that program transformation may introduce auxiliary function symbols such as \(\text{sum1}, \text{factpair}\), etc. This is why one has to restrict his/her attention to some set \(𝒟\) of function symbols. Equivalent transformation [6], [7], [22] can be used to establish the equivalence of two TRSs.

**Definition 1**: Let \(R_0\) be a left-linear CS over \(𝒟_0\) and \(E\) a set of equations over \(𝒟_0\). An equivalent transformation sequence \(\omega\) under \(E\) is a sequence \(R_0, \ldots, R_n\) of TRSs (over \(𝒟_0, \ldots, 𝒟_n\), respectively) such that \(R_{k+1}\) is obtained from \(R_k\) by applying one of the following inference rules:

(I) **Introduction**

\[R_{k+1} = R_k \cup \{f(x_1, \ldots, x_n) \rightarrow r\}\]

provided that \(f \notin 𝒟_k\), and \(r \in T(𝒟_0, \{x_1, \ldots, x_n\})\), where \(x_1, \ldots, x_n\) are mutually distinct variables. We put \(𝒟_{k+1} = 𝒟_k \cup \{f\}\).

(A) **Addition**

\[R_{k+1} = R_k \cup \{l \rightarrow r\}\]

provided \(l \in R_{k+1} \land \exists \cdot E \ r\) holds.

(E) **Elimination**

\[R_{k+1} = R_k \setminus \{l \rightarrow r\}\]

If this is the case, we write \(R_k \Rightarrow R_{k+1}\). (In the Addition and Elimination rules, \(𝒟_{k+1}\) can be any set of function symbols such that \(𝒟_{k+1} \subseteq 𝒟_k\) provided that \(R_{k+1}\) is a TRS over \(𝒟_{k+1}\).) The reflexive transitive closure of \(\Rightarrow\) is denoted by \(\Rightarrow\). We indicate the rule of \(\Rightarrow\) by \(\Rightarrow\), \(\Rightarrow\), or \(\Rightarrow\).

Finally, we say there exists an equivalent transformation from \(R\) to \(R'\) under \(E\) if there exists an equivalent transformation sequence \(\Rightarrow\) from \(R\) to \(R'\) under \(E\).

The next proposition is the basis of correctness of program transformation by template based on term rewriting.

**Proposition 2** (Theorem 4.4 of [7]): Let \(𝒟\) and \(𝒟'\) be sets of function symbols such that \(𝒟 \subseteq 𝒟', 𝒟' \subseteq 𝒟\). Let \(R\) be a left-linear CS over \(𝒟\), \(E\) a set of equations over \(𝒟\), and \(R'\) a TRS over \(𝒟'\). Suppose that \(R, 𝒟, E \vdash_{\text{ind}} E\) and there exists an equivalent transformation from \(R\) to \(R'\) under \(E\). Then, \(\text{CR}(R) \sqcap \text{SC}(𝒟, 𝒟') \sqcap \text{SC}(R', 𝒟')\) imply \(R \equiv_{\text{str}(𝒟', 𝒟')} R'\).

The transformation from \(R_{\text{factlist}}\) to \(R'_{\text{factlist}}\) in the previous section is based on the tupling transformation of [9]. We are now going to demonstrate how the same transformation is obtained by the method of equivalent transformations (Definition 1) and how the correctness of the transformation is obtained by Proposition 2.

First, we show an equivalent transformation from \(R_{\text{factlist}}\) to \(R'_{\text{factlist}}\) under the empty set \(\emptyset\), where \(R_x = \{\pi_1((x,y), x, \pi_2((x,y))) \rightarrow y\}\).

1. Let \(R_0 = R_{\text{factlist}}\).

2. Let \(R_1 = R_0 \cup \{\text{factpair}(x) \rightarrow \langle \text{factlist}(x), \text{fact}(x)\rangle\}\). Here, \(\text{factpair}\) is a fresh function symbol. Thus, \(R_0 \Rightarrow R_1\) by the Introduction rule.

3. Let \(R_2 = R_1 \cup [\text{step}(x, y) \rightarrow \langle \pi_1(y), \pi_2(y), x \times \pi_2(y)\rangle]\). Here, \(\text{step}\) is a fresh function symbol. Thus, \(R_1 \Rightarrow R_2\) by the Introduction rule.

4. Let \(R_3 = R_2 \cup \langle \text{factpair}(0), \text{fact}(0) \rangle \rightarrow \langle \emptyset, \text{fact}(0)\rangle\). Here, \(\text{factpair}(0) \rightarrow \langle \emptyset, \text{fact}(0)\rangle\). Thus, \(R_2 \Rightarrow R_3\) by the Addition rule.

5. Let \(R_4 = R_3 \cup \{\text{factpair}(s(x)) \rightarrow \text{step}(s(x), \text{factpair}(x))\}\). Here, we have \(\text{factpair}(s(x)) \rightarrow \langle \emptyset, \text{fact}(s(x))\rangle\). Thus, \(R_3 \Rightarrow R_4\) by the Addition rule.

6. Let \(R_5 = R_4 \cup \langle \text{factpair}(0), \text{fact}(0) \rangle \rightarrow \langle \emptyset, \text{fact}(0)\rangle\). Here, \(\text{factpair}(0) \rightarrow \langle \emptyset, \text{fact}(0)\rangle\). Thus, \(R_4 \Rightarrow R_5\) by the Addition rule.

8. Finally, applying the Elimination rule five times to \(R_6\), we obtain \(R'_{\text{factlist}}\).

Thus \(R_{\text{factlist}} \Rightarrow R'_{\text{factlist}}\) under \(\emptyset\), i.e. there is an equivalent transformation from \(R_{\text{factlist}}\) to \(R'_{\text{factlist}}\) under \(\emptyset\).

Next, we show the correctness of the transformation based on Proposition 2. For this, we need to consider the many-sorted signature additionally to guarantee the sufficient completeness. We consider the following natural many-sorted signature factlist: \(\text{Nat} \rightarrow \text{List}, \text{fact} : \text{Nat} \rightarrow \text{Nat}, \times : \text{Nat} \times \text{Nat} \rightarrow \text{Nat}, + : \text{Nat} \times \text{Nat} \rightarrow \text{Nat}, \times : \text{Nat} \times \text{List} \rightarrow \text{List}, \_ : \text{List}, s : \text{Nat} \rightarrow \text{Nat}, 0 : \text{Nat}, \cdot : \text{Nat} \times \text{Nat} \rightarrow \text{Pair}, \pi_1 : \text{Pair} \rightarrow \text{Nat}, \pi_2 : \text{Pair} \rightarrow \text{Nat}, \text{factpair} : \text{Nat} \rightarrow \text{Pair}, \text{step} : \text{Nat} \times \text{Pair} \rightarrow \text{Pair}.

We now show \(R_{\text{factlist}} \cup R_x \equiv_{\text{str}(𝒟', 𝒟')} R'_{\text{factlist}}\) where \(𝒟' = \{\text{factpair}, \text{fact}, +, \times, \dashv, \emptyset, \_ \}, s, 0, \pi_1, \pi_2, \cdot\} \) and \(𝒟' \subseteq 𝒟\). Since there is an equivalent transformation from \(R_{\text{factlist}}\) to \(R'_{\text{factlist}}\) under \(\emptyset\), it suffices to check the rest of the conditions of Proposition 2.

It is easy to see that \(R_{\text{factlist}} \cup R_x\) is a left-linear CS. Since \(E = \emptyset\), \(𝒟', 𝒟, 𝒟 \vdash_{\text{ind}} E\) holds trivially. To show \(\text{CR}(R_{\text{factlist}} \cup R_x), \text{SC}(R_{\text{factlist}} \cup R_x, 𝒟)\) and \(\text{SC}(R'_{\text{factlist}}', 𝒟')\), we use the fact \(\text{SN}(R_{\text{factlist}} \cup R_x)\) and \(\text{SN}(R'_{\text{factlist}}')\)—they can be shown automatically using the lexicographic path order. Then \(\text{CR}(R_{\text{factlist}} \cup R_x)\) follows since there is no crit-

\(^{1}\text{Recall a remark after the definition of sufficient completeness in Sect. 2.}\)
5. Extended Templates and Its Correctness

Since it is difficult to find a suitable equivalent transformation sequence automatically for each program transformation, the framework in the previous section is not directly applicable to automated verification of the correctness of program transformations.

In the framework of program transformation by templates, this can be partially achieved by abstracting the similar equivalent transformations of TRSs into a transformation of templates. The idea is to prepare a template constructed by an abstract equivalent transformation and to lift up from an abstract equivalent transformation $P \Rightarrow P'$ to a concrete equivalent transformation $\varphi(P) \Rightarrow \varphi(P')$.

To make such a lift up possible, the notions of developed templates and CS homomorphisms have been introduced in [5]–[8]. As explained later, however, these notions turn out to be too restrictive to deal with tupling transformations, notwithstanding the successful tupling transformation based on equivalent transformation in the previous section. This motivates us to extend the notion of developed templates. We first give an equivalent transformation of templates which extends the one in [5]–[8].

Definition 2: Let $P_0$ be a TRS pattern over a set $\Sigma_0 \subseteq \mathcal{F} \cup \mathcal{X}$ and $\mathcal{H}$ a set of equations over $\Sigma_0$. An abstract equivalent transformation sequence under $\mathcal{H}$ is a sequence $P_0, \ldots, P_n$ of TRS patterns (over $\Sigma_0, \ldots, \Sigma_n$, respectively) such that $P_{k+1}$ is obtained from $P_k$ by applying one of the following inference rules:

(I) Introduction

$$P_{k+1} = P_k \cup \{p(x_1, \ldots, x_n) \rightarrow r\}$$

provided that $p \in \mathcal{X} \setminus \Sigma_k$ and $r \in T(\Sigma_k, \{x_1, \ldots, x_n\})$, where $x_1, \ldots, x_n$ are mutually distinct variables. We put $\Sigma_{k+1} = \Sigma_k \cup \{p\}$.

(A) Addition

$$P_{k+1} = P_k \cup \{l \rightarrow r\}$$

provided $l \not\vDash_{P \cup \mathcal{H}} r$ holds.

(E) Elimination

$$P_{k+1} = P_k \setminus \{l \rightarrow r\}$$

If this is the case, we write $P_k \Rightarrow P_{k+1}$.

In the Addition and Elimination rules, $\Sigma_{k+1}$ can be any set such that $P_{k+1} \subseteq \Sigma_k$ provided that $P_{k+1}$ is a TRS pattern over $\Sigma_{k+1}$.) The reflexive transitive closure of $\Rightarrow$ is denoted by $\Rightarrow$. We indicate the rule of $\Rightarrow$ by $\Rightarrow_A$, $\Rightarrow_B$, or $\Rightarrow_C$. Finally, we say that $(P, P', \mathcal{H})$ is a correct template if there exists an abstract equivalent transformation sequence $P \Rightarrow \cdots \Rightarrow P' \subseteq \mathcal{H}$ under $\mathcal{H}$.

Unfortunately, the existence of abstract equivalent transformation $P \Rightarrow P'$ under $\mathcal{H}$ does not imply that of equivalent transformation $\varphi(P) \Rightarrow \varphi(P')$ under $\varphi(\mathcal{H})$ for any term homomorphism $\varphi$. More specifically, the application of Introduction rule is not preserved. For example, $\varphi(p(x_1, \ldots, x_n)) \rightarrow \varphi(r)$ may be (1) $f(x_1) \rightarrow g(x_1, x_2)$, (2) $g(x_1, x_2) \rightarrow r$, (3) $f(l(x_1)) \rightarrow r$, or (4) $x_1 \rightarrow r$, all of which do not satisfy the conditions of Introduction rule for an equivalent transformation. Namely, for (1), the variable condition $(\forall(r) \subseteq V(l))$ is violated; for (2), $l$ is not linear; for (3), the proper subterm of $l$ is not a variable; and for (4), the root symbol of $l$ is not a function symbol. Similarly, one needs to guarantee $f \not\in \mathcal{F}_k$ and $\mathcal{F}(r) \subseteq \mathcal{F}_k$. Therefore, in order to guarantee the successful lift up of abstract equivalent transformation sequences, one has to impose some conditions on the term homomorphism $\varphi$ to use. The following condition of term homomorphisms is helpful to specify the needed conditions.

Definition 3: Let $\Sigma \subseteq \mathcal{F} \cup \mathcal{X}$ and $\mathcal{I} \subseteq \mathcal{F}$. A term homomorphism $\varphi$ carries $\Sigma$ to $\mathcal{I}$ if $\Sigma \cap \mathcal{I} \subseteq \mathcal{I}$ and $\varphi(p) \in T(\Sigma \cup \mathcal{H}_{\text{arity}(p)})$ for all $p \in \Sigma \cap \mathcal{I}$.

Note that immediately follows from this definition that $s \in T(\Sigma, \mathcal{V})$ implies $\varphi(s) \in T(\mathcal{I}, \mathcal{V})$ for any term homomorphism $\varphi$ that carries $\Sigma$ to $\mathcal{I}$. We also note that, for term homomorphism $\varphi$ with a finite domain, it is decidable whether $\varphi$ carries $\Sigma$ to $\mathcal{I}$.

Lemma 1: Let $\Sigma \subseteq \mathcal{F} \cup \mathcal{X}$ and $P$ and $\mathcal{H}$ be a TRS pattern and a set of equations over $\Sigma$, respectively. Suppose $\Sigma' \subseteq \mathcal{F} \cup \mathcal{X}$, $P'$ is a TRS pattern over $\Sigma'$, and that there is an abstract equivalent transformation sequence under $\mathcal{H}$ from $P$ to $P'$. Let $\mathcal{I} \subseteq \mathcal{F}$ and $\varphi$ a term homomorphism such that (1) $\varphi$ carries $\Sigma$ to $\mathcal{I}$, (2) $\varphi(P)$ is a left-linear CS, and (3) for each $p \in \mathcal{X} \setminus \Sigma$, $\varphi(p) = f_p(\Sigma_1, \ldots, \Sigma_{\text{arity}(p)})$ with mutually distinct $f_p \in \mathcal{F}_\Delta \setminus \Sigma$. Then there exists an equivalent transformation sequence $\varphi(P) \Rightarrow \varphi(P')$ under $\varphi(\mathcal{H})$.

Proof. We lift up the abstract equivalent transformation sequence $P = P_0 \Rightarrow \cdots \Rightarrow P_n = P'$ (over $\Sigma = \Sigma_0, \ldots, \Sigma_n = \Sigma'$, respectively) to the equivalent transformation sequence $\varphi(P) = \varphi(P_0) \Rightarrow \cdots \Rightarrow \varphi(P_n) = \varphi(P')$ (over $\mathcal{I} \subseteq \mathcal{F}_0, \ldots, \mathcal{F}_n$, respectively) using the corresponding transformation rules. For the transformation $P_0 \Rightarrow P_k$,
we have \( \Sigma_{i+1} = \Sigma_i \cup \{p_i+1\} \) (0 ≤ i < k), for some \( p_i+1 \notin \Sigma_i \). For each i, we put \( \mathcal{F}_{i+1} = \mathcal{F}_i \cup \{p_i+1\} \) where \( p_i+1 = \text{root}(\varphi(p_i+1)) \). We simultaneously show by induction on \( k \) that (a) \( \varphi(P_0) \Rightarrow \varphi(P_k) \) and (b) if \( P_0 \Rightarrow_i P_k \) then \( \varphi \) carries \( \Sigma \) to \( \mathcal{F}_k \).

The base step for (b) follows from the condition (1). For the induction step, we divide the cases by the last inference rule applied to \( P_0 \Rightarrow_i P_k \).

1. The case of \( P_k \Rightarrow_i P_{k+1} \). Let \( P_{k+1} = P_k \cup \{p_{k+1}(x_1, \ldots, x_n) \rightarrow r\} \). By the condition of the Introduction rule \( x_1, \ldots, x_n \) are mutually distinct variables and \( p_{k+1} \in \mathcal{F}_i \setminus \Sigma_k \) and \( r \in T(\Sigma_k, \{x_1, \ldots, x_n\}) \).

2. The case of \( \mathcal{F}_k \Rightarrow \mathcal{F}_{k+1} \). Then \( \mathcal{F}_{k+1} = \mathcal{F}_k \cup \{f_{p_{k+1}}\} \) \( \varphi \) carries \( \Sigma \) to \( \mathcal{F}_k \) by the induction hypothesis and \( r \in T(\Sigma_k, \{x_1, \ldots, x_n\}) \), we have \( \varphi(r) \in T(\mathcal{F}_k, \{y_1, \ldots, y_n\}) \) and\( \varphi(P_k) \Rightarrow_i \varphi(P_{k+1}) \).

The conditions of suitable transformations are summarized in the following definition.

**Definition 4:** Let \( \langle P, P', H \rangle \) be a correct template where \( P \) and \( P' \) are TRS patterns over \( \Sigma \) and \( \Sigma' \) respectively. A TRS \( R \) over \( \mathcal{G} \) is transformed to a TRS \( R' \) over \( \mathcal{G}' \) by the correct template \( \langle P, P', H \rangle \) if there exist a term homomorphism \( \varphi \) and a TRS \( R_{\text{com}} \) over \( \mathcal{G} \) such that:

1. \( R = \varphi(P) \cup R_{\text{com}} \).
2. \( \varphi \) carries \( \Sigma \) to \( \mathcal{G} \).
3. \( \mathcal{G}' = \varphi_{\text{out}}(P') \cup R_{\text{com}} \), where \( \varphi_{\text{out}} = \varphi \cup \{p \rightarrow f_p(\mathcal{D}_1, \ldots, \mathcal{D}_{\text{arity}(p)}) \mid p \in \mathcal{P} \setminus \Sigma \} \) where each \( f_p \) is a fresh function symbol, and
4. \( \mathcal{G} \vdash \text{true} \varphi(H) \).

Note that although \( \mathcal{G}' \) can be obtained from the templates and term homomorphism \( \varphi \), our definition of transformation by correct templates also imposes the additional conditions 2 and 4 which are used to guarantee the correctness of the transformation.

The next theorem gives a sufficient condition to guarantee the correctness of TRS transformations by correct templates.

**Theorem 1:** Let \( \mathcal{F}_c \subseteq \mathcal{G}, \mathcal{G}' \subseteq \mathcal{F} \). If a left-linear CS \( R \) over \( \mathcal{G} \) is transformed to a TRS \( R' \) over \( \mathcal{G}' \) by a correct template \( \langle P, P', H \rangle \), then \( \text{CR}(R) \land \text{SC}(R, \mathcal{G}) \land \text{SC}(R', \mathcal{G}') \) implies \( R \equiv_{\mathcal{G}, \mathcal{G}'} R' \).

**Proof:** Suppose that a left-linear CS \( R \) over \( \mathcal{G} \) is transformed to a TRS \( R' \) over \( \mathcal{G}' \) by a correct template \( \langle P, P', H \rangle \). Then there exist a term homomorphism \( \varphi \) and a TRS \( R_{\text{com}} \) over \( \mathcal{G} \) that satisfies conditions of Definition 4.

By the condition 3, we have \( \varphi(P) = \varphi_{\text{out}}(P) \).
Furthermore, since \( R_{\text{com}} \) is over \( \mathcal{G} \), we have \( R = \varphi(P) \cup R_{\text{com}} \) and \( R' = \varphi(P') \cup R_{\text{com}} \).
Then, by conditions 2, 3, 4, it follows from Lemma 1 that there is an equivalent transformation sequence from \( R \) to \( R' \) under \( \varphi_{\text{out}}(H) (= \varphi(H)) \). Then the claim follows from Proposition 2.

Below we present several correct templates for tupling transformations.

**Example 1:** Let \( \langle P, P', H \rangle \) be a template where

\[
\begin{align*}
P(a) & \rightarrow b \\
P(c(x, y)) & \rightarrow h(e(x, c(x, y)), p(y), q(y)) \\
q(a) & \rightarrow d \\
q(c(x, y)) & \rightarrow k(e(x, c(x, y)), p(y), q(y)) \\
p(x) & \rightarrow \pi_1(rx(x)) \\
q(x) & \rightarrow \pi_2(rx(x)) \\
r(a) & \rightarrow \langle b, d \rangle \\
r(c(x, y)) & \rightarrow \langle e(x, c(x, y)), r(y) \rangle \\
r1(x, y) & \rightarrow \langle h(x, \pi_1(x, y), \pi_2(y)), k(x, \pi_1(x, y), \pi_2(y)) \rangle \\
\pi_1(x, y) & \rightarrow x \\
\pi_2(x, y) & \rightarrow y \\
H & = \emptyset
\end{align*}
\]

Here, \( \pi_1, \pi_2, \) and \( \langle \rangle \) are function symbols. We now show that the template \( \langle P, P', H \rangle \) is a correct template.

1. Let \( P_0 = P \).
2. Let \( P_1 = P_0 \cup \{r(x) \rightarrow \langle p(x), q(x) \rangle \} \). Here, \( r \) is a fresh pattern variable. Thus, \( P_0 \Rightarrow P_1 \) by the Introduction rule.
3. Let \( P_2 = P_1 \cup \{r1(x, y) \rightarrow \langle h(x, \pi_1(x, y), \pi_2(y)), k(x, \pi_1(x, y), \pi_2(y)) \rangle \} \). Here, \( r1 \) is a fresh pattern variable. Thus, \( P_1 \Rightarrow P_2 \) by the Introduction rule.
4. Let \( P_3 = P_2 \cup \{r(a) \rightarrow \langle b, d \rangle \} \). Here, we have \( r(a) \rightarrow p_2 \langle p(a), q(a) \rangle \rightarrow r_2 \langle b, d \rangle \). Thus, \( P_2 \Rightarrow P_3 \) by the Addition rule.
5. Let \( P_4 = P_3 \cup \{r(c(x, y)) \rightarrow r1(e(x, c(x, y)), r(y)) \} \). Here, we have \( r(c(x, y)) \rightarrow p_1 \langle p(c(x, y)), q(c(x, y)) \rangle \).

\[\text{We regard the verification of these conditions as a part of verification of correctness of the program transformation by templates.}\]
Example 2: A list of numbers is said to be steep if each element is greater than the sum of the elements that follow it[12]. The following TRS \( R_{\text{steep}} \) specifies a program which checks whether the input list is steep.

\[
\begin{align*}
\text{steep}([ ]) & \rightarrow \text{tt} \\
\text{steep}(x,{x:x}) & \rightarrow \text{and} (\text{gt}(x,\text{sum}(x)), \text{steep}(x)) \\
\text{sum}([ ]) & \rightarrow \text{0} \\
\text{sum}(x,{x:y}) & \rightarrow + (x, \text{sum}(y)) \\
\pi_1((x),y) & \rightarrow x \\
\pi_2((x),y) & \rightarrow y
\end{align*}
\]

where \( R_{\text{com}} \) is a correct template.

\[
\begin{align*}
\text{gt}(0,0) & \rightarrow \text{ff} \\
\text{gt}(0,\text{s}(x)) & \rightarrow \text{ff} \\
\text{gt}(\text{tt},\text{tt}) & \rightarrow \text{tt} \\
\text{gt}(\text{tt},\text{ff}) & \rightarrow \text{ff} \\
\text{gt}(\text{ff},\text{tt}) & \rightarrow \text{ff} \\
\text{gt}(\text{ff},\text{ff}) & \rightarrow \text{ff} \\
\text{gt}(\text{s}(x),y) & \rightarrow \text{gt}(\text{s}(x),y)
\end{align*}
\]

Let \((\mathcal{P}, \mathcal{P}', \mathcal{H})\) be the template in Example 1. The following term homomorphism \( \varphi \) is a matcher between the source \( \mathcal{P} \) and the TRS \( R_{\text{steep}} \setminus R_{\text{com}} \), that is, \( \varphi(\mathcal{P}) = R_{\text{steep}} \setminus R_{\text{com}} \).

It is easy to check that the above term homomorphism \( \varphi \) carries \([p, a, b, c, e, h, q, d, k, \pi_1, \pi_2, \cdot] \) to \([\text{steep}, \text{sum}, \text{gt}, \text{and}, +, [], \text{tt}, \text{ff}, \text{s}, 0, \pi_1, \pi_2, \cdot] \). Since \( \mathcal{H} = 0, R_{\text{steep}}, \mathcal{F} \vdash_{\text{ind}} \varphi(\mathcal{H}) \) holds trivially. Let \( R'_{\text{steep}} = R_{\text{com}} \cup \varphi(\mathcal{P}') \)
Thus, \( \tilde{P} \) is a correct template.

Finally, applying the Elimination rule twice to \( P \) have a fresh pattern variable. Thus, \( \tilde{P} \) is a correct template.

Let \( \bar{r} = (d, c) \). Here, we have \( \bar{r}(a) \rightarrow_{\bar{P}_2} \langle \bar{q}(\bar{a}(x)), \bar{p}(\bar{a}(x)) \rangle \rightarrow_{\bar{P}_2} (d, c) \). Thus, \( \bar{P}_2 \Rightarrow \bar{P}_3 \) by the Addition rule.

Let \( \bar{P}_4 = \bar{P}_3 \cup \{ \bar{r}(b(x)) \rightarrow \bar{r}(\bar{r}(\bar{b}(x))) \} \). Here, we have \( \bar{r}(b(x)) \rightarrow_{\bar{P}_4} \langle \bar{p}(\bar{b}(x)), \bar{b}(x) \rangle \rightarrow_{\bar{P}_4} \langle \bar{q}(\bar{p}(\bar{b}(x))), \bar{b}(x) \rangle \rightarrow_{\bar{P}_4} \langle \bar{q}(\bar{p}(\bar{b}(x))), \bar{p}(\bar{b}(x)) \rangle \rightarrow_{\bar{P}_4} \langle \bar{p}(\bar{b}(x)), \bar{b}(x) \rangle \rightarrow_{\bar{P}_4} \langle \bar{p}(\bar{b}(x)), \bar{p}(\bar{b}(x)) \rangle \rightarrow_{\bar{P}_4} \langle \bar{q}(\bar{p}(\bar{b}(x))), \bar{p}(\bar{b}(x)) \rangle \rightarrow_{\bar{P}_4} \langle \bar{r}(\bar{r}(\bar{b}(x))) \rangle \rightarrow_{\bar{P}_4} \bar{r}(\bar{r}(\bar{b}(x))) \).

Thus, \( \bar{P}_4 \Rightarrow \bar{P}_3 \) by the Addition rule.

Let \( \bar{P}_5 = \bar{P}_4 \cup \{ \bar{p}(\bar{b}(x)) \rightarrow \bar{p}(\bar{r}(\bar{r}(\bar{b}(x)))) \} \). Here, we have \( \bar{p}(\bar{b}(x)) \rightarrow_{\bar{P}_5} \bar{p}(\bar{r}(\bar{r}(\bar{b}(x)))) \).

Thus, \( \bar{P}_4 \Rightarrow \bar{P}_5 \) by the Addition rule.

Finally, applying the Elimination rule twice to \( \bar{P}_5 \), we obtain \( \bar{P}' \).

The transformation from TRS \( \mathcal{R}_{ib} \cup \mathcal{R}_c \) to \( \mathcal{R}_{ib} \) in Sect. 3 is obtained by this template as follows. By matching the source \( P \) and the TRS \( (\mathcal{R}_{ib} \cup \mathcal{R}_c) \backslash \mathcal{R}_{com} \), we obtain the term homomorphism \( \varphi = \) \[
\begin{align*}
p & \mapsto \text{fib}(\square_1) \\
a & \mapsto 0 \\
c & \mapsto s(0) \\
b & \mapsto s(\square_1) \\
d & \mapsto s(0) \\
q & \mapsto +(\square_1, \square_2) 
\end{align*}
\]
where \( \mathcal{R}_{com} = \{ +0(x), x \rightarrow x, +(s(x), y) \rightarrow s((x, y)) \} \).

It is easily checked that \( \varphi \) carries \( \{ p, a, c, b, q, \pi_1, \pi_2, \langle \cdot \rangle \} \) to \( \{ \text{fib}, s, 0, \pi_1, \pi_2, \langle \cdot \rangle \} \) and \( \tilde{H} = \{ \text{fib}, s, 0, \pi_1, \pi_2, \langle \cdot \rangle \} \) holds trivially. Since \( \mathcal{R}_{ib} = \mathcal{R}_{com} \cup \tilde{\varphi}(\mathcal{P}') \) where \( \tilde{\varphi} = \varphi \circ (r \mapsto \text{fibpair}(\square_1), r1 \mapsto \text{step}(\square_1)) \), the TRS \( \mathcal{R}_{ib} \cup \mathcal{R}_c \) is transformed to the TRS \( \mathcal{R}_{ib} \).

The termination of the TRSs \( \mathcal{R}_{ib} \cup \mathcal{R}_c \) can be shown automatically by lexicographic path order; from this, \( CR(\mathcal{R}_{ib} \cup \mathcal{R}_c) \) and \( SC(\mathcal{R}_{ib} \cup \mathcal{R}_c, \mathcal{P}) \) are checked by computing the critical pairs and the complement substitution algorithm (under the assumption of many-sorted signature). Therefore, the correctness of the transformation is guaranteed by Theorem 1, that is, \( \mathcal{R}_{ib} \cup \mathcal{R}_c \Rightarrow [\text{fib}, s, 0, \pi_1, \pi_2, \langle \cdot \rangle] \mathcal{R}_{ib} \).

6. Comparison with the Previous Framework

As mentioned earlier, in our previous framework, we guarantee the successful lift up of abstract equivalent transformation sequence by the notions of CS homomorphism and developed templates. In this section, we compare our new framework with the previous one given in [6], [7].

Definition 5 ([6], [7]): Suppose \( \mathcal{P} \) is partitioned into sets \( \mathcal{P}_d, \mathcal{P}_c \) of defined pattern symbols and constructor pattern symbols. A term homomorphism \( \varphi \) is said to be a CS homomorphism if (1) for any \( p \in \mathcal{P}_d \) of arity \( n \), \( \varphi(p) = f(\bar{p}_1, \ldots, \bar{p}_n) \) for some \( f \in \mathcal{F}_d \) and mutually distinct indices \( i_1, \ldots, i_n \); and (2) for any \( p, q \in \mathcal{P}_d \), \( p \neq q \) implies \( \text{root}(\varphi(p)) \neq \text{root}(\varphi(q)) \). A template is developed if there is an abstract equivalent transformation sequence such that \( p(x_1, \ldots, x_n) \rightarrow r \) in the Introduction rule is restricted as \( p \in \mathcal{P}_d \setminus \Sigma_k \) and \( r \in T(\Sigma_k \cap \mathcal{P}_d, \mathcal{F}_d) \).

It is easy to see that \( \mathcal{P} \Rightarrow \mathcal{P}' \) under the restricted Introduction rule implies \( \varphi(\mathcal{P}) \overset{I}{\Rightarrow} \varphi(\mathcal{P}') \) for any CS homomorphism \( \varphi \) [6], [7].

As demonstrated in Examples 1 and 3, templates for tupling transformations need to introduce rewrite rules \( r(\cdot) \rightarrow (s_1, \ldots, s_n) \) and \( r(\cdot) \rightarrow (s'_1, \ldots, s'_m) \) using a tuple symbol \( \langle \cdot \rangle \) by the Introduction rule. However, since \( \langle \cdot \rangle \in \mathcal{F}_c \), one can not introduce such a rewrite rule by the condition of abstract equivalent transformation sequence for developed templates that rhs of the introduced rule should be contained in \( T(\Sigma_k \cap \mathcal{P}_d, \mathcal{F}_d) \). Hence templates for tupling transformations can not be expressed by developed templates. This is why the previous framework of [5]–[8] fails to handle tupling transformations.

In the previous framework, templates are constructed on \( T(\mathcal{P}, \mathcal{F}) \) and thus function symbols are excluded. Then one may wonder why a pattern variable \( q \in \mathcal{P}_c \) had not been allowed in the Introduction rule for the developed templates. In general, if one allows a pattern variable \( q \in \mathcal{P}_c \) to occur in the rhs \( r \) of the rewrite rule introduced by the Introduction
rule, then, for a CS homomorphism \( \varphi, P \Rightarrow P' \) may not imply \( \varphi(P) \Rightarrow \varphi(P') \). For example, let \( P(x,y) \Rightarrow Q(x,r(x,y)) \) be a rewrite rule in the input template and \( \varphi = \{ P \mapsto f(□, □), Q \mapsto g(□), r \mapsto h(□) \} \) be a matcher. Then one can not guarantee \( h \in F_k \) because \( h \notin \mathcal{F}(\varphi(Q(x,r(x,y)))) \).

One can not exclude the case of \( r \in \Sigma_k \) but \( \mathcal{F}(\varphi(r)) \notin F_k \).

In our new criterion, such a case is forbidden by introducing the notion of carrying of signature.

7. Conclusion

We extended a framework of program transformation by templates based on term rewriting. We introduced a notion of correct templates by which a wider variety of transformations by template, including tupling transformations, can be handled. We proved a criterion to guarantee the correctness of program transformations by these templates. We gave several correct templates for tupling transformations and demonstrated some examples of tupling transformations by templates.

RAPT \(^5\) is an implementation of program transformation by templates based on term rewriting. For a given TRS \( R \) and a correct template, it automatically verifies the correctness of transformation of \( R \) by that template, and outputs the result of transformation. We have implemented the new version of RAPT based on the framework presented in this paper. For all examples in this paper, program transformations and their verification have been operated in RAPT successfully. To complete the each program transformation and verification of Examples 2, 3, 4, it took 820, 620 and 40 msec, respectively, on a workstation with UltraSPARC IIIi 1.34 GHz CPU. Furthermore, it is confirmed experimentally that the new version is also capable of all examples prepared for the old version.

Another implementation of program transformation using templates is the MAG system \(^{13},^{20} \). MAG supports transformations that include modifications of expressions and matching with the help of hypothesis; its target also includes higher-order programs. In contrast, the target of RAPT is limited to first-order term rewriting systems but it cooperates with automated theorem-proving techniques of term rewriting to verify the correctness of transformations.

In general, constructing a suitable template for a particular program transformation is operated manually and it is often not easy to construct a correct template that is widely applicable to many programs. To improve this situation, we are working toward the development of a method for constructing possible candidates of template automatically \(^8\).

Another possible future extension is to extend the framework to deal with higher-order functions. We expect that the equivalent transformation technique can be extended to higher-order TRSs without much difficulties. However, an integration of automated verification methods for properties such as sufficient completeness, and an incorporation of higher-order inductive theorem proving \(^1,^{17} \) seem yet not clear. The extension toward this direction remains as a future work.

References


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