On Two Problems of Nano-PLA Design

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SUMMARY The logic mapping problem and the problem of finding a largest sub-crossbar with no defects in a nano-crossbar with nonprogrammable-crosspoint defects and disconnected-wire defects are known to be NP-hard. This paper shows that for nano-crossbars with only disconnected-wire defects, the former remains NP-hard, while the latter can be solved in polynomial time.

key words: biclique problem, nano-crossbar, nano-PLA, orthogonal ray graphs, subgraph isomorphism problem

1. Introduction

Implementing a sum-of-product logic function in a conventional programmable logic array (PLA) is a straightforward task of arbitrarily assigning the literals and product terms to the wires of the crossbar and programming the appropriate crosspoints. However, in the case of nano-PLAs, this task is not trivial because of imperfections in the nano-wire crossbar. Defects in nano-wire crossbar have been broadly classified into two types: nonprogrammable-crosspoint defects, in which some crosspoints become unprogrammable, and disconnected-wire defects, in which each horizontal nanowire may not be connected to all vertical nano-wires [5]. The problem of mapping a sum-of-product logic function onto a defective nano-crossbar with nonprogrammable-crosspoint defects and disconnected-wire defects was first considered by Rao, Orailoglu, and Karri [5]. They proposed several heuristics since the problem is NP-hard. The problem of finding a maximum defect-free sub-crossbar in a nano-crossbar with nonprogrammable-crosspoint defects and disconnected-wire defects was first investigated by Tahoori [8]. Since the problem is also NP-hard, several heuristics have been proposed [1], [8].

This paper considers the complexity of the problems for nano-crossbars with only disconnected-wire defects.

1.1 LOGIC MAPPING

Let \( f \) be a logic function in a sum-of-product form. Let \( S \) be a nano-crossbar with disconnected-wire defects. The problem of implementing \( f \) in \( S \) is formulated as LOGIC MAPPING, which is the problem of assigning the literals and product terms of \( f \) to nano-wires of \( S \) so that containment relationships among the literals and product terms can be represented by crosspoint connections in \( S \). A graph model of LOGIC MAPPING can be obtained as follows.

Let \( L_f \) be the set of literals of \( f \), and \( P_f \) be the set of product terms of \( f \). A logic function graph \( G_f \) for \( f \) is a bipartite graph defined as follows: \( V(G_f) = L_f \cup P_f \), and \( (L_f, P_f) \) is a bipartition of \( G_f \); vertices \( l \in L_f \) and \( p \in P_f \) are connected by an edge if and only if literal \( l \) is contained in product term \( p \).

Let \( W_h \) be the set of horizontal nano-wires, and \( W_v \) be the set of vertical nano-wires of \( S \). A crossbar graph \( G_S \) of \( S \) is a bipartite graph defined as follows: \( V(G_S) = W_h \cup W_v \) and \( (W_h, W_v) \) is a bipartition of \( G_S \); vertices \( x \in W_h \) and \( y \in W_v \) are connected by an edge if and only if nano-wires \( x \) and \( y \) have a crosspoint. Then, LOGIC MAPPING can be modeled as the subgraph isomorphism problem, which is to find a subgraph of \( G_S \) isomorphic to \( G_f \). Examples of a logic function \( f \), a defective crossbar \( S \), and their corresponding bipartite graphs \( G_f \) and \( G_S \) are shown in Fig. 1.

1.2 SUB-CROSSBAR

SUB-CROSSBAR is the problem of finding a defect-free sub-crossbar consisting of given numbers of horizontal and vertical wires within the nano-crossbar with disconnected wire defects. SUB-CROSSBAR can be modeled as the \( K_{m,n} \)-biclique problem, which is to find a complete bipartite subgraph \( K_{m,n} \) contained in a crossbar graph \( G_S \).

![Fig. 1 An instance of LOGIC MAPPING and the corresponding graphs.](image-url)
1.3 Our Results

Although it is well known that both the subgraph isomorphism problem and the $K_{m,n}$-biclique problem are NP-hard for bipartite graphs [2, 3], the complexity of LOGIC MAPPING and SUB-CROSSBAR is not immediately clear since the graphs representing surviving sub-crossbars are a special kind of bipartite graph.

A bipartite graph $G$ with a bipartition $(U, V)$ is called an orthogonal ray graph if there exist a set of non-intersecting rays (half-lines) $R_u, u \in U$, parallel to the $x$-axis in the $xy$-plane, and a set of non-intersecting rays $R_v, v \in V$, parallel to the $y$-axis such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if $R_u$ and $R_v$ intersect. An orthogonal ray graph $G$ with a bipartition $(U, V)$ is called a two-directional orthogonal ray graph if $R_u$ is a rightward ray $\{(x, b_u) \mid x \geq a_u\}$ for each $u \in U$, and $R_v$ is an upward ray $\{(a_v, y) \mid y \geq b_v\}$ for each $v \in V$, where $a_u$ and $b_v$ are real numbers for any $w \in U \cup V$.

Nano-wires such as $p$ and $q$ of a defective nano-crossbar shown in Fig. 2 cannot be controlled as they do not reach the boundary of the originally intended nano-crossbar. Since we cannot use such nano-wires, a graph representing a surviving sub-crossbar must be an orthogonal ray graph.

We show in Sect. 3 that LOGIC MAPPING is NP-hard by showing that the subgraph isomorphism problem is NP-hard even for orthogonal ray graphs. We show in Sect. 4 that SUB-CROSSBAR can be solved in polynomial time provided that the vertices of the orthogonal ray graph representing a surviving sub-crossbar are ordered to reflect the top-to-bottom order of horizontal nano-wires and left-to-right order of vertical nano-wires. This is a quite natural condition. We also show in Sect. 4 that in the case of two-directional orthogonal ray graphs, the $K_{m,n}$-biclique problem can be solved in polynomial time without the requirement of such an ordering, thereby providing a purely graph-theoretic solution for an interesting subproblem of SUB-CROSSBAR.

2. Orthogonal Ray Graphs

In this section, we shall discuss some properties of two-directional orthogonal ray graphs that will come in handy in the later sections. Some of the lemmas and theorems in this section also appear in our earlier work [7]. In order to make the paper self-contained, we revisit them and also provide direct explicit proofs for some of them.

The 3-claw is a tree obtained from a complete bipartite graph $K_{1,3}$ by replacing each edge with a path of length 3. (See Fig. 3 (a).)

Lemma 1. The 3-claw is not a 2-directional orthogonal ray graph.

Proof. Assume to the contrary that the 3-claw is a 2-directional orthogonal ray graph. Let the vertices of the 3-claw be named as in Fig. 3 (a). We shall refer to the endpoint of the ray corresponding to a vertex $v$ by $(a_v, b_v)$. Without loss of generality, suppose $R_{u_1}$ is a horizontal ray and that $R_{u_1}, R_{v_2}, R_{v_3}$ intersect with $R_{u_1}$ such that $R_{u_1}$ lies to the right of $R_{v_2}$ and to the left of $R_{v_3}$. (See Fig. 3 (b)). It is easy to observe that $b_{v_3} > b_{v_2} > b_{u_1}$, or else it is not possible to define $R_{u_1}, R_{u_1}$, and $R_{u_3}$. Since $R_{u_1}$ has to be defined such that $a_{u_1} > a_{v_3}$ and $b_{u_1} < b_{v_3}$, it is not possible to define $R_{v_3}$ such that it intersects with $R_{u_1}$ but not with $R_{u_1}$, a contradiction.

A path $P$ in a tree $T$ is called a spine of $T$ if every vertex of $T$ is within distance two from at least one vertex of $P$.

Theorem 1. A tree $T$ has a spine if and only if $T$ contains no 3-claw as a subtree.

Proof. The necessity is obvious. To prove the sufficiency, assume $T$ contains no 3-claw. Let $P$ be a longest path in $T$, and let $V(P) = \{v_1, v_2, \ldots, v_p\}$ and $(v_i, v_{i+1}) \in E(P)$, $1 \leq i \leq p - 1$. We claim that $P$ is a spine. We distinguish two cases: $|V(P)| \leq 6$ and $|V(P)| > 6$.

For the former, it is easy to see that $P$ is a spine because if there is a vertex $v \notin V(P)$ which is at a distance more than two from any vertex in $P$, then the assumption that $P$ is a longest path is contradicted.

We next take the case of $|V(P)| > 6$. Assume $P$ is not a spine. Let $F$ be a forest obtained from $T$ by deleting the edges in $E(P)$. Let $T_i$ be a tree in $F$ containing $v_i$, $1 \leq i \leq p$. Since $P$ is a longest path in $T$, $T_i$ consists of only one vertex, $v_i$, and $T_p$ consists of only one vertex, $v_p$. Also all vertices in $T_2$ and $T_{p-1}$ are within distance one from $v_2$ and $v_{p-1}$, respectively; and all vertices in $T_3$ and $T_{p-2}$ are within distance two from $v_3$ and $v_{p-2}$, respectively. Since we assumed that $P$ is not a spine, there exists an integer $j$ ($4 \leq j \leq p - 3$) such that $T_j$ contains a vertex $w_j$ whose distance from $v_j$ is three. Let $P'$ be the path from $v_j$ to $w_j$. Then the subgraph of $T$ induced by the vertices in $\{v_i \mid j-3 \leq i \leq j+3\} \cup V(P')$ is a 3-claw. This contradicts the assumption that $T$ contains no 3-claw as a subtree, and therefore $P$ is a spine.
**Theorem 2.** A tree $T$ is a 2-directional orthogonal ray tree if and only if $T$ contains no 3-claw as a subtree.

**Proof.** The necessity follows from Lemma 1. We will show the sufficiency. Assume $T$ contains no 3-claw as a subtree. Then from Theorem 1, $T$ contains a spine $P$. Let $V(P) = (v_1, v_2, \ldots, v_p)$, and $(v_i, v_{i+1}) \in E(P), 1 \leq i \leq p-1$. Corresponding to each vertex $v_i$ in $P$, define ray $R_{v_i} = (i, y) \mid y \geq i-1$ if $i$ is odd, and define ray $R_{v_i} = (x, i) \mid x \geq i-1$ if $i$ is even. Let $F$ be a forest obtained from $T$ by deleting the edges in $E(P)$. Let $T_1$ be a tree in $T$ containing $v_1, 1 \leq i \leq p$. Consider $T_1$ to be rooted at $v_1$. Let $w_{ij}, w_{ij_2}, \ldots, w_{iq(i)}$ be the children of $v_i$ in $T_i$, where $q(i)$ is the number of children of $v_i$ in $T_i$. Let $z_{ij_1}, z_{ij_2}, \ldots, z_{ijq(i)}$ be the children of $w_{ij}$ in $T_i$, where $r(j)$ is the number of children of $w_{ij}$ in $T_i$. The rays corresponding to $w_{ij}$ and $z_{ij}$, $(1 \leq i \leq p, 1 \leq j \leq q(i), 1 \leq k \leq r(j))$ can be placed in the region for $T_i$ as shown in Fig. 4. Thus $T$ is a 2-directional orthogonal ray graph. □

**Lemma 2.** A cycle $C_{2m}$ of length $2m$ is a two-directional orthogonal ray graph if and only if $m = 2$.

**Proof.** It is easy to see that $C_4$ is a 2-directional orthogonal ray graph.

We show that $C_{2m}$ is not a 2-directional orthogonal ray graph for any $m \geq 3$. Suppose to the contrary that $C_{2m}$ is a 2-directional orthogonal ray graph for some $m \geq 3$. Let $V(C_{2m}) = \{0, 1, \ldots, 2m - 1\}$ and $E(C_{2m}) = \{(i, i+1 \text{ (mod 2m))}) | 0 \leq i \leq 2m - 1\}$. Suppose without loss of generality that $R_0 = (a_0, y) \mid y \geq b_0$, for some real numbers $a_0$ and $b_0$. Since $(0, 1) \in E(C_{2m})$, $R_1$ intersects with $R_0$ at some point. Similarly, $R_2$ intersects with $R_1$ at some other point. We distinguish two cases.

**Case 1** When $R_2$ intersects with $R_1$ such that $R_2$ is to the left of $R_0$: Then $R_3$ must intersect with $R_2$ such that $R_3$ lies below the endpoint of $R_0$. Similarly, $R_4$ must intersect with $R_3$ such that $R_4$ lies to the left of the endpoint of $R_1$. Continuing in this manner, $R_i (5 \leq i \leq 2m - 1)$ must lie below (to the left of) the endpoint of $R_{i-3}$ for odd (even) $i$. Therefore $R_{2m-1}$ lies in the region below the endpoint of $R_4$. However, $R_0$ is in the region right of $R_2$ and above $R_3$, making it impossible for $R_0$ to intersect with $R_{2m-1}$ without intersecting with $R_3, R_5, \ldots, R_{2m-5}$, a contradiction.

**Case 2** When $R_2$ intersects with $R_4$ such that $R_2$ is to the right of $R_0$: We further distinguish two cases.

**Case 2-1** When $R_1$ intersects with $R_2$ such that $R_1$ is below $R_3$: Then $R_4$ must lie to the left of the endpoint of $R_1$. This confines $R_0$ within the region left of $R_2$ and above $R_3$, making it impossible for ray $R_{2m-1}$ to intersect with $R_0$ without intersecting with $R_2$, a contradiction.

**Case 2-2** When $R_1$ intersects with $R_2$ such that $R_1$ is above $R_3$: This case may be further broken down into two cases depending on whether $R_4$ is to the left of $R_2$ or right of $R_2$. In the former case, $R_4$ gets confined within the region left of $R_2$ and above $R_1$ making it impossible for $R_3$ to intersect with $R_4$ without intersecting with $R_2$, a contradiction. In the latter case, $R_3, R_5, \ldots, R_{2m-1}$ must lie in the region right of $R_2$ and above $R_3$, making it impossible for $R_{2m-1}$ to intersect with $R_0$ without intersecting with $R_2, R_4, \ldots R_{2m-2}$, a contradiction.

Thus we conclude that $C_{2m}$ is not a 2-directional orthogonal ray graph for any $m \geq 3$. □

A bipartite graph is chordal if it contains no induced cycles of length greater than 4. A tree is chordal, by definition. Thus, by Lemma 2 and Theorem 2, we have:

**Theorem 3.** A class of two-directional orthogonal ray graphs is a proper subset of the class of chordal bipartite graphs. □

### 3. Intractability of LOGIC MAPPING

We show in this section the following.

**Theorem 4.** LOGIC MAPPING is NP-hard.

Theorem 4 follows from Theorem 5 below. A decision problem associated with the subgraph isomorphism problem is defined as follows:

**SUBGRAPH ISOMORPHISM**

**INSTANCE:** Graphs $H$ and $G$.

**QUESTION:** Does $G$ contain a subgraph isomorphic to $H$, that is, does there exist a one-to-one mapping $\phi : V(H) \rightarrow V(G)$ such that if $(u, v) \in E(H)$ then $(\phi(u), \phi(v)) \in E(G)$?

**Theorem 5.** SUBGRAPH ISOMORPHISM is NP-complete even if $G$ is a 2-directional orthogonal ray tree and $H$ is a forest.

**Proof.** It is easy to see that the problem is in NP. We show a polynomial-time reduction from 3-PARTITION, which has been shown to be strongly NP-complete in [2]. 3-PARTITION is defined as follows.
3-PARTITION

**INSTANCE:** A finite set $A$ of $3m$ elements, a bound $B \in \mathbb{Z}^+$, and a size $s(a) \in \mathbb{Z}^+$ for each $a \in A$, such that each $s(a)$ satisfies $B/4 < s(a) < B/2$ and such that $\sum_{a \in A} s(a) = mB$.

**QUESTION:** Does $A$ have a 3-partition, that is, can $A$ be partitioned into $m$ disjoint sets $S_1, S_2, \ldots, S_m$ such that, for $1 \leq i \leq m$, $\sum_{a \in S_i} s(a) = B$?

Let $C_1, C_2, \ldots, C_m$ be $B$-vertex paths such that for each $i$ ($1 \leq i \leq m$), $V(C_i) = \{v_{i,j} \mid 1 \leq j \leq B\}$ and $E(C_i) = \{(v_{i,j}, v_{i,(j+1)}) \mid 1 \leq j \leq B - 1\}$. Let $T_1, T_2, \ldots, T_{m-1}$ be complete binary trees of height two rooted at vertices $r_1, r_2, \ldots, r_{m-1}$, respectively. Let $G$ be the graph defined as

$$V(G) = \bigcup_{i=1}^{m} V(C_i) \cup \bigcup_{i=1}^{m-1} V(T_i),$$

$$E(G) = \bigcup_{i=1}^{m} E(C_i) \cup \bigcup_{i=1}^{m-1} E(T_i) \cup \{(r_i, v_{i,B}), (r_i, v_{(i+1),B}) \mid 1 \leq i \leq m-1\}.$$

(See Fig. 5(a).) Since the path in $G$ from $v_{1,1}$ to $v_{m,B}$ is a spine of $G$, it follows from Theorems 1 and 2 that $G$ is a two-directional orthogonal ray tree. Let $H$ be a forest consisting of $m-1$ complete binary trees of height two $T'_1, T'_2, \ldots, T'_{m-1}$, and $3m$ paths $P_1, P_2, \ldots, P_{3m}$, each $P_j$ corresponding to element $a_j$ of $A$ and having $s(a_j)$ vertices. (See Fig. 5(b).) $G$ and $H$ can be constructed in time polynomial in $m$ and $B$.

We next prove that $A$ has a 3-partition if and only if $G$ contains a subgraph isomorphic to $H$.

Suppose first that $A$ can be partitioned into $m$ disjoint subsets $S_1, S_2, \ldots, S_m$ such that for each $i$ ($1 \leq i \leq m$), $\sum_{a \in S_i} s(a) = B$. An isomorphism from $H$ to a subgraph of $G$ can be obtained as follows. Since each path $C_i$ contains $B$ vertices, we can map the paths of $H$ corresponding to the elements of $S_i$ to the path $C_i$ in $G$. Each $T'_j$ in $H$ can be mapped to $T_i$ in $G$. It is easy to see that this is indeed an isomorphism from $H$ to a subgraph of $G$.

Next suppose that $H$ is isomorphic to a subgraph of $G$. Each $T'_j$ $(1 \leq j \leq m-1)$ in $H$ contains two vertices which have degree three and are at a distance two from each other. These vertices must be mapped to the children of vertex $r_i$ of $T_i$ for some $i$ ($1 \leq i \leq m-1$). Therefore, each $T'_j$ in $H$ must be mapped to some $T_i$ in $G$. This means that paths $P'_1, P'_2, \ldots, P'_{3m}$ in $H$ are mapped to paths $C_1, C_2, \ldots, C_m$ in $G$. For $1 \leq i \leq m$, let $S_i$ be the set of elements of $A$ corresponding to the paths of $H$ mapped to $C_i$. Since $C_i$ has $B$ vertices, $\sum_{a \in S_i} s(a) = B$, for all $i$ ($1 \leq i \leq m$). Moreover, since the instance of 3-PARTITION satisfies $\sum_{a \in A} s(a) = mB$, we can conclude that $\sum_{a \in S_i} s(a) = B$ for all $i$ ($1 \leq i \leq m$). Therefore $A$ has a 3-partition. \(\square\)

4. Tractability of SQUARE SUB-CROSSBAR

Let $\mathcal{H}$ be a set of non-intersecting horizontal rays, and let $\mathcal{V}$ be a set of non-intersecting vertical rays. Let $\mathcal{K}_h \subseteq \mathcal{H}$ and $\mathcal{K}_v \subseteq \mathcal{V}$. $\mathcal{K}_h \cup \mathcal{K}_v$ is called a $|\mathcal{K}_h| \times |\mathcal{K}_v|$ sub-crossbar of $\mathcal{H} \cup \mathcal{V}$ if each $X \in \mathcal{K}_h$ intersects every $Y \in \mathcal{K}_v$. For a ray $R$, we shall denote the $x$ and $y$-coordinates of its endpoints by $x(R)$ and $y(R)$, respectively. We associate with $\mathcal{H} \cup \mathcal{V}$, a sequence $X_{H\cup V}$ of the rays of $\mathcal{H} \cup \mathcal{V}$ sorted in the increasing order of $x$-coordinate values of the end points – ties are broken such that if a horizontal ray and a vertical ray have the same $x$-coordinate value, then the horizontal ray appears before the vertical ray in the sequence. We also associate with $\mathcal{H} \cup \mathcal{V}$, a sequence $Y_{H\cup V}$ of the rays of $\mathcal{H} \cup \mathcal{V}$ sorted in the increasing order of $y$-coordinate values of the end points – ties are broken such that if a vertical ray and a horizontal ray have the same $y$-coordinate value, then the vertical ray appears before the horizontal ray in the sequence.

Our earlier observation that a nano-wire crossbar can be represented by a set of orthogonal rays allows us to use the terms “nano-wires” and “rays” interchangeably. Then an alternate, equivalent definition of SUB-CROSSBAR is as follows:

**SUB-CROSSBAR**

**INSTANCE:** A set $\mathcal{H}$ of horizontal rays, a set $\mathcal{V}$ of vertical rays, and positive integers $k_h$ and $k_v$. 
Algorithm 1.

**Input:** A set of rightward rays $\mathcal{R}$ and a set of upward rays $\mathcal{U}$, sequences $X_{\mathcal{RU}}$ and $Y_{\mathcal{RU}}$, and positive integers $k_r$ and $k_u$.

**Output:** A $k_r \times k_u$ sub-crossbar of $\mathcal{R} \cup \mathcal{U}$, if one exists. NO, otherwise.

**Step 1:** Compute set $R = \{R_1, R_2, \ldots, R_{|\mathcal{R}|}\}$ of the rays of $\mathcal{R}$ such that $y(R_i) < y(R_{i+1})$ ($1 \leq i \leq |\mathcal{R}| - 1$); compute sequence $U = (U_1, U_2, \ldots, U_{|\mathcal{U}|})$ of the rays of $\mathcal{U}$ such that $x(U_j) < x(U_{j+1})$ ($1 \leq j \leq |\mathcal{U}| - 1$).

**Step 2:** Set $uEnd(i) = \{U_i \mid y(U_i) \leq y(R_1)\}$ and for each $i \in \{2, 3, \ldots, |\mathcal{R}|\}$, set $uEnd(i) = \{U_i \mid y(R_{i-1}) < y(U_i) \leq y(R_i)\}$; set $rEnd(1) = \{R_1 \mid x(R_1) < x(U_1)\}$ and for each $j \in \{2, 3, \ldots, |\mathcal{U}|\}$, set $rEnd(j) = \{R_i \mid x(U_{j-1}) < x(R_i) \leq x(U_j)\}$.

**Step 3:** Initialize $h = 0, v = 0, rCross = 0, uCross = 0$.

**Step 4:** if $rCross < k_r$

\[
v = v + 1;
\]

if $v > |\mathcal{U}| - k_u$ output NO and halt.

Set $rCross = rCross + |rEnd(v)|$.

**Step 5:** if $uCross < k_u$

\[
h = h + 1;
\]

if $h > |\mathcal{R}| - k_r$ output NO and halt.

Set $uCross = uCross + |uEnd(h)|$.

**Step 6:** if $rCross \geq k_r$ and $uCross \geq k_u$, then output $\bigcup_{i=1}^{h} rEnd(i)$ and $\bigcup_{i=1}^{v} uEnd(i)$ and halt.

**Step 7:** if $rCross < k_r$

\[
\text{remove } U_v \text{ from one of the sets } uEnd(i) (1 \leq i \leq |\mathcal{R}|) \text{ which contains it;}
\]

if $y(U_v) < y(R_h)$, then $uCross = uCross - 1$.

**Step 8:** if $uCross < k_u$

\[
\text{remove } R_h \text{ from one of the sets } rEnd(j) (1 \leq i \leq |\mathcal{U}|) \text{ which contains it;}
\]

if $x(R_h) < x(U_v)$, then $rCross = rCross - 1$.

**Step 9:** Return to Step 4.

**Question:** Show a $k_h \times k_v$ sub-crossbar of $\mathcal{H} \cup \mathcal{V}$, if any.

An interesting subproblem of SUB-CROSSBAR in which the instance is restricted to rightward and upward rays can be defined as follows:

**2-SUB-CROSSBAR**

**INSTANCE:** A set $\mathcal{R}$ of rightward rays, a set $\mathcal{U}$ of upward rays, and positive integers $k_r$ and $k_u$.

**Question:** Show a $k_r \times k_u$ sub-crossbar of $\mathcal{R} \cup \mathcal{U}$, if any.

In the following subsections, we will discuss algorithms to solve these problems.

**4.1 Algorithms for 2-SUB-CROSSBAR**

Kloks and Kratsch [4] showed the following.

**Lemma 3.** [4] A chordal bipartite graph with $n$ vertices and $m$ edges contains at most $m$ maximal complete bipartite subgraphs which can be enumerated in $O(\min(m \log n, n^2))$ time.

From Lemma 3 and Theorem 3, we have:

**Lemma 4.** The $K_{m,n}$ biclique problem can be solved in $O(\min(m \log n, n^2))$ time for $n$-vertex, $m$-edge 2-directional orthogonal ray graphs.

Since the graph representing $\mathcal{R} \cup \mathcal{U}$ is a 2-directional orthogonal ray graph, we have the following theorem from the above lemma.

**Theorem 6.** 2-SUB-CROSSBAR can be solved in $O(\min(m \log n, n^2))$ time for a crossbar, where $n = |\mathcal{R}| + |\mathcal{U}|$ and $m$ is the number of crosspoints.

This is a purely graph theoretic approach, which assumes no information about the endpoints of rays. Taka-hashi [9] showed that a computational geometry approach utilizing the coordinates of the endpoints yields a faster algorithm of time complexity $O(n \log n)$. We present Algorithm 1 (See Fig. 6), which is a linear-time algorithm to solve 2-SUB-CROSSBAR given that sequences $X_{\mathcal{RU}}$ and $Y_{\mathcal{RU}}$ are provided. Since $X_{\mathcal{RU}}$ and $Y_{\mathcal{RU}}$ can be computed in $O(n \log n)$ time, Algorithm 1 can be easily extended to solve 2-SUB-CROSSBAR in $O(n \log n)$ time. However, the main purpose of introducing Algorithm 1 is to use it as a subroutine to solve SUB-CROSSBAR, as shown in the next subsection.

A brief description of Algorithm 1 follows. Algorithm 1 begins with some preprocessing operations, in which the sequences $R$, $U$ and the sets $uEnd(i)(1 \leq i \leq |\mathcal{R}|)$, $rEnd(j)(1 \leq j \leq |\mathcal{U}|)$ are computed (see Steps 1 and 2). To search for a $k_r \times k_u$ sub-crossbar, Algorithm 1 uses two sweep lines to perform a left-to-right, bottom-to-top scan of the rays. The horizontal sweep line stops at $R_1, R_2, \ldots$, and it is represented by variable $h$, which indicates that it is at the position of ray $R_h$. The vertical sweep line stops at $U_1, U_2, \ldots$, and it is represented by variable $v$, which indicates that it is at the position of ray $U_v$. At each stop, the following processes are carried out. The number $rCross$ of horizontal rays that cross the vertical sweep line and lie in the area above, and including, the horizontal sweep line is...
4.2 Algorithm for SUB-CROSSBAR

Let $\mathcal{H}$ be a set of non-intersecting horizontal rays and $\mathcal{V}$ be a set of non-intersecting vertical rays. For two rays $H \in \mathcal{H}$ and $V \in \mathcal{V}$ which intersect, say at point $(p_x, p_y)$, define

$$\mathcal{H}_{HV} = \{ R \mid R \in \mathcal{H}, R \text{ intersects } V, \text{ and } y(R) \leq p_y \}.$$

Similarly, define

$$\mathcal{V}_{HV} = \{ R \mid R \in \mathcal{V}, R \text{ intersects } H, \text{ and } x(R) \leq p_x \}.$$

Let $B$ be the bottommost ray in $\mathcal{H}_{HV}$, and let $L$ be the leftmost ray in $\mathcal{V}_{HV}$. For each ray $R \in \mathcal{H}_{HV}$, define ray $R'$ such that if $R$ is a rightward ray, $R' = R$; if $R$ is a leftward ray, $R'$ is a rightward ray with $x(R') = x(L)$ and $y(R') = y(R)$. And for each ray $R \in \mathcal{V}_{HV}$, define ray $R'$ such that if $R$ is an upward ray, $R' = R$; if $R$ is a downward ray, $R'$ is an upward ray with $x(R') = x(R)$ and $y(R') = y(B)$. Finally, define

$$\mathcal{H'}_{HV} = \{ R' \mid R \in \mathcal{H}_{HV} \}$$

and

$$\mathcal{V'}_{HV} = \{ R' \mid R \in \mathcal{V}_{HV} \}.$$

Figure 7 shows an example of $\mathcal{H}_{HV}$, $\mathcal{V}_{HV}$, $\mathcal{H'}_{HV}$, and $\mathcal{V'}_{HV}$. The following observation is obvious from the definitions above.

**Observation 1.** Two rays in $\mathcal{H}_{HV} \cup \mathcal{V'}_{HV}$ intersect if and only if their corresponding rays in $\mathcal{H}_{HV} \cup \mathcal{V}_{HV}$ intersect. □

**Observation 2.** $\mathcal{H} \cup \mathcal{V}$ contains a $k_h \times k_v$ sub-crossbar if and only if there exists a pair of intersecting rays $H \in \mathcal{H}$ and $V \in \mathcal{V}$ such that $\mathcal{H'}_{HV} \cup \mathcal{V'}_{HV}$ contains a $k_h \times k_v$ sub-crossbar.
Proof. The sufficiency is immediate from Observation 1. To see the necessity, set \( H \) and \( V \) to be the topmost and rightmost rays, respectively of a \( k_h \times k_v \) sub-crossbar of \( H \cup V \).

Since \( H'_{HV} \) contains only rightward rays and \( V'_{HV} \) contains only upward rays, we can use Algorithm 1 to find a \( k_h \times k_v \) sub-crossbar in \( H'_{HV} \cup V'_{HV} \). Algorithm 2 which solves SUB-CROSSBAR is shown in Fig. 8. It exhaustively checks all pairs of intersecting rays to determine if there exists a pair \( H \in H' \) and \( V \in V' \) such that \( H'_{HV} \cup V'_{HV} \) contains a \( k_h \times k_v \) sub-crossbar.

Let \( n = |H| + |V| \). Step 1 can be performed in \( O(n \log n) \) time. Step 2 takes \( O(n^2) \) time. The items in Step 4 can be computed in \( O(n) \) time from the sequences obtained in Step 1. Step 5 takes \( O(n) \) time. Steps 3 through 6 are repeated \( O(n^2) \) time. Then it follows from Observation 2 and Theorem 7 that:

**Theorem 8.** Algorithm 2 solves SUB-CROSSBAR in \( O((|H| + |V|)^3) \) time.

5. Concluding Remarks

The complexity of SUBGRAPH ISOMORPHISM in which \( G \) is a 2-directional orthogonal ray graph and \( H \) is a connected graph is open. Note that if both \( G \) and \( H \) are trees, then SUBGRAPH ISOMORPHISM is polynomial-time solvable [2]. Reducing the time complexity of SUB-CROSSBAR is another interesting open question.

A preliminary version of this paper has appeared in [6].

References


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