A Fast Algorithm for Augmenting Edge-Connectivity by One with Bipartition Constraints

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SUMMARY The k -edge-connectivity augmentation problem with bipartition constraints (k ECABP, for short) is defined by “Given an undirected graph $G = (V, E)$ and a bipartition $\pi = \{V_B, V_W\}$ of V with $V_B \cap V_W = \emptyset$, find an edge set E_f of minimum cardinality, consisting of edges that connect V_B and V_W , such that $G' = (V, E \cup E_f)$ is k -edge-connected.” The problem has applications for security of statistical data stored in a cross tabulated table, and so on. In this paper we propose a fast algorithm for finding an optimal solution to $(\sigma + 1)$ ECABP in $O(|V||E| + |V|^2 \log |V|)$ time when G is σ -edge-connected ($\sigma > 0$), and show that the problem can be solved in linear time if $\sigma \in \{1, 2\}$.

key words: connectivity augmentation problems, edge-connectivity, bipartition constraints, graphs, polynomial time algorithms

1. Introduction

[Background] The k -edge-connectivity augmentation problem (k ECA, for short) is defined by “Given a multigraph $G = (V, E)$, find an edge set E_f of minimum cardinality such that $G' = (V, E \cup E_f)$ is k -edge-connected, where a multigraph means a graph, with unweighted edges, such that multiple edges may exist.” We often denote G' as $G + E_f$, and E_f is called an optimal solution to the problem. There are several applications for construction of a fault-tolerant network, and so on. It is called the k -edge-connectivity augmentation problem with bipartition constraints (k ECABP, for short) when a bipartition $\pi = \{V_B, V_W\}$ of V with $V_B \cap V_W = \emptyset$ is additionally given and we require that E_f consists of edges connecting between V_B and V_W (see Fig. 1).

A bipartite graph is a graph (V, E) such that V is partitioned into two sets V^B and V^W , and any edge $(u, v) \in E$ satisfies a condition ($u \in V^B$ and $v \in V^W$) or ($u \in V^W$ and $v \in V^B$): such a graph is often denoted as $G = (V^B \cup V^W, E)$. If G is bipartite and we set $V_B = V^B$ and $V_W = V^W$ in k ECABP then G' is bipartite.

This problem, denoted as B- k ECABP, is a typical subproblem of k ECABP, where “B-” means that G is a bipartite graph. There are several applications for security of statistical data stored in a cross tabulated table [5], and so on.

[Existing Results] Many algorithms for k ECA have been

proposed. [3] proposed a linear time algorithm for 2ECA, and [4], [9], [15] proposed polynomial time algorithms for k ECA. [5] proposed a linear time algorithm for B-2ECABP, and an $O(\log |V|)$ parallel time algorithm on an EREW PRAM with a linear number of processors. [1] proposed an $O(|V|(|E| + |V| \log |V|) \log |V|)$ time algorithm for k ECAMP that is k ECA with r -partition constraints, where r -partition $\pi_M = \{V_1, \dots, V_r\}$ ($r \geq 2$) of V is given and E_f consists of edges connecting between V_i and V_j ($i \neq j$). Note that, in [1], a given multigraph is handled as an edge-weighted simple one such that, for any pair of vertices u and v , if there are x multiple edges between u and v then it is represented as a simple edge (u, v) with a weight $w((u, v)) = x$. Let M- k ECAMP denote k ECAMP in which G is an r -partite graph, where “M-” means a multipartite graph. Recently, [2] proposed a linear time algorithm for 2ECAMP, and an $O(\log |V|)$ parallel time algorithm on an EREW PRAM with a linear number of processors.

[The Main Results] The main result of the paper is to propose a fast algorithm for obtaining an optimal solution to $(\sigma + 1)$ ECABP in linear time when G is σ -edge-connected and a structural graph $F(G)$ of G is given, where a structural graph $F(G)$ represents all minimum cuts of G . The time complexity of the proposed algorithm is $O(|V||E| + |V|^2 \log |V|)$ because $F(G)$ can be constructed in $O(|V||E| + |V|^2 \log |V|)$ time [10]. Moreover, it follows from our result that, when $\sigma \in \{1, 2\}$, the problem can be solved in linear time because $F(G)$ can be constructed in $O(|V| + |E|)$ time [8], [12]. Note that the proposed algorithm is faster than the algorithm proposed in [1] for $(\sigma + 1)$ ECAMP when

Fig. 1 A multigraph G with $\lambda(G) = 4$, where a closed circle (an open circle, respectively) represents a vertex which belongs to V_B (V_W), where the corresponding edge-weighted simple graph is used in actual handling. The set of dashed lines represents an optimal solution $E_f = \{(1, 8), (3a, 5), (4, 10), (9, 10), (10, 12)\}$ to 5ECABP.

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$r = 2$ and G is σ -edge-connected.

[The Structure of the Paper] The paper is organized as follows. Section 2 provides some definitions and notations. Section 3 shows a lower bound on the cardinality of optimal solutions to this problem. Section 4 presents an algorithm for finding an optimal solution to this problem. We consider its correctness and time complexity in Sects. 4.3 and 4.4, respectively. The concluding remarks are given in Sect. 5.

2. Definitions

[Basic Terminologies of Graphs] An *undirected multigraph* is denoted as $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ are often denoted as V and E , respectively. In this paper, only graphs without loops are considered, and the term “a graph” means an undirected multigraph unless otherwise stated.

Remark 1. It should be noted that conceptually a multigraph is considered, while the corresponding edge-weighted simple graph is used in actual handling in this paper.

An edge that is incident to two vertices u, v in G is denoted by (u, v) . For two disjoint sets $X, X' \subset V$, we denote $(X, X'; G) = \{(u, v) \in E \mid u \in X \text{ and } v \in X'\}$, where it is often written as (X, X') if G is clear from the context. We denote $d_G(X) = |(X, V - X; G)|$ which is called the *degree* of X (in G). For a vertex v , the total number of edges incident to v is called the degree of v and is denoted as $d_G(v)$ (in G). The set $(X, V - X; G)$ is called a k -cut when $|X, V - X| = k$. For an edge set E_f , let $G + E_f$ denote the graph $(V, E \cup E_f)$. For a given bipartition $\pi = (V_B, V_W)$ of V with $V_B \cap V_W = \emptyset$, E_f is said to be *legal* (with respect to π) if E_f consists of edges connecting between V_B and V_W . If π is fixed then “with respect to π ” is omitted.

A *trail* is a sequence of distinct edges $(v_0, v_1), (v_1, v_2), \dots, (v_{r-1}, v_r)$ in which there may appear the same endvertices, and each of v_0 and v_r is called a *terminal vertex* of the trail. It is called a *closed trail* when $v_0 = v_r$. A closed trail is called an *Eulerian closed trail* of G if all edges of G are included. A *path* is a trail such that all vertices $v_0, v_1, \dots, v_r$ are distinct. A *cycle* consists of a path with $r \geq 1$ and an edge (v_r, v_0) .

We say that G is *connected* if there is a path between any pair of vertices. For two vertices $u, v \in V$, let $\lambda(u, v; G)$ denote the maximum number of edge-disjoint paths between u and v in G . The edge-connectivity $\lambda(G)$ of G is defined by $\lambda(G) = \min\{\lambda(u, v; G) \mid u, v \in V\}$, and we say that G is k -edge-connected if $\lambda(G) \geq k$ for a nonnegative integer k . If a set $Z \subseteq V$ is a maximal vertex set such that $\lambda(u, v; G) \geq k$ holds for any pair of vertices $u, v \in Z$ then Z is called a k -edge-connected component (k -component, for short) of G . For a k -component Z , Z is called a *leaf k -component* if $d_G(Z) = \lambda(G)$. Note that distinct k -components are pairwise disjoint.

[Structural Graphs with Bipartition Constraints] A *cactus* is an undirected connected graph in which any pair of cycles share at most one vertex. A structural graph $F(G) = (V(F(G)), E(F(G)))$ [6] of G with $\lambda(G) = \sigma$ (see Fig. 2) is a

Fig. 2 The set of dashed lines represents an optimal solution $E' = \{(1, 8), (3, 5), (4, 10), (9, 10), (10, 12)\}$ for a structural graph $F(G)$ of G in Fig. 1, giving us $\lambda(F(G) + E') = \lambda(F(G)) + 1$.

representation for all minimum cuts of G . $F(G)$ is an edge-weighted cactus of $O(|V|)$ vertices and edges such that each tree edge (a bridge in $F(G)$) has weight $\lambda(G)$ and each cycle edge (an edge included in a cycle) has weight $\lambda(G)/2$. Particularly if $\lambda(G)$ is odd then $F(G)$ is an edge-weighted tree. Each vertex in G maps to exactly one vertex in $F(G)$. Note that any minimum cut of G is represented as either a tree edge or a pair of two cycle edges in the same cycle of $F(G)$, and vice versa. Let $\rho: V(G) \rightarrow V(F(G))$ denote this mapping. We use the following notations: $\rho(X) = \{\rho(v) \mid v \in X\}$ for $X \subseteq V$ and $\rho^{-1}(Y) = \{v \in V \mid \rho(v) \in Y\}$ for $Y \subseteq V(F(G))$. A vertex $y \in V(F(G))$ with $\rho^{-1}(y) = \emptyset$ is called an *empty vertex*. Let $\varepsilon(G) \subseteq V(F(G))$ denote the set of all empty vertices of $F(G)$.

For any cut $(X, V(F(G)) - X; F(G))$, if the summation of weights of all edges in the cut is equal to σ then $(\rho^{-1}(X), V - \rho^{-1}(X); G)$ is a σ -cut (a minimum cut) of G . Conversely, for any σ -cut $(X, V - X; G)$, $F(G)$ has at least one cut $(Y, V(F(G)) - Y; F(G))$ in which the summation of weight of all edges in the cut is equal to σ , where Y is a vertex set of G such that $\rho(X) = Y - \varepsilon(G)$. Each $(\sigma + 1)$ -component S of G is represented as a vertex $\rho(S) \in V(F(G)) - \varepsilon(G)$ of $F(G)$, and, for any vertex $v \in V(F(G)) - \varepsilon(G)$ of $F(G)$, $\rho^{-1}(v)$ is a $(\sigma + 1)$ -component of G . In particular, for $v \in V(F(G)) - \varepsilon(G)$ of $F(G)$, we often denote a vertex that is included in a $(\sigma + 1)$ -component $\rho^{-1}(v)$ as n_v . For a vertex $v \in V(F(G)) - \varepsilon(G)$ of $F(G)$, if the summation of weights of all edges that are incident to v in $F(G)$ equals to σ , then v is called a *leaf* of $F(G)$ and $\rho^{-1}(v)$ is a leaf $(\sigma + 1)$ -component of G . Conversely, for any leaf $(\sigma + 1)$ -component L of G , $\rho(L)$ is a leaf of $F(G)$. Note that any leaf of $F(G)$ is not an empty vertex. Let $LF(G)$ denote the set of all leaves of $F(G)$. It is shown in [10] that $F(G)$ can be constructed in $O(|V||E| + |V|^2 \log |V|)$ time.

If $F(G)$ has any bridge of weight $\lambda(G)$ then we replace such a bridge by a pair of multiple edges, assigning each edge weight $\lambda(G)/2$ even if $\lambda(G)$ is odd. We consider such a pair of multiple edges to be a cycle of length two. We call this graph a *modified cactus*, and we assume $F(G)$ is a modified one in this paper unless otherwise stated. Note that we can handle this modified cactus as a structural graph of G and $\lambda(F(G)) = 2$.

Even if $\lambda(G)$ is odd, finding an edge set E_f such that

$\lambda(G + E_f) = \lambda(G) + 1$ can be solved by fixing an edge set E' such that $\lambda(F(G) + E') = 3$ for a modified cactus $F(G)$ of G . This is because there is a bijection $\xi : E' \rightarrow E_f$ such that $\xi((u, v)) \rightarrow (n_u, n_v)$, where $\rho(n_u) = u$ and $\rho(n_v) = v$.

Given a structural graph $F(G)$ of a graph $G = (V, E)$ with bipartition constraints $\pi = \{V_B, V_W\}$, we classify vertices $v \in V(F(G)) - \varepsilon(G)$ into three members as follows:

- (i) $\rho^{-1}(v) \subseteq V_B$ (v is called a *black vertex of $F(G)$*);
- (ii) $\rho^{-1}(v) \subseteq V_W$ (v is called a *white one of $F(G)$*);
- (iii) $\rho^{-1}(v) \cap V_B \neq \emptyset$ and $\rho^{-1}(v) \cap V_W \neq \emptyset$ (v is called a *hybrid one of $F(G)$*).

Each vertex of V_B or of V_W is called a black vertex or a white one of G , respectively. The set of *black leaves*, *white leaves* or *hybrid leaves* of $F(G)$ is denoted by $BF(G)$, $WF(G)$ or $HF(G)$, respectively. In this paper, without loss of generality, we assume that $V_B \neq \emptyset$, $V_W \neq \emptyset$ and $|BF(G)| \geq |WF(G)|$. If $|BF(G)| > |WF(G)| + |HF(G)|$ holds then $F(G)$ is said to be *B-dominant* [5].

In figures of this paper, a hybrid vertex is represented by a square, and a black one, a white one and an empty one are represented by a closed circle, an open one and a double one, respectively (see Fig. 2).

3. A Lower Bound of a Feasible Solution to $(\sigma + 1)$ ECABP

In the rest of the paper, we set $\sigma = \lambda(G)$.

In this section, a lower bound of a feasible solution to $(\sigma + 1)$ ECABP is given. Since $(\sigma + 1)$ ECABP is a subproblem of k ECAMP, we obtain the following proposition by setting $k = \sigma + 1$ and $r = 2$ for a lower bound shown in [1] on k ECAMP.

Suppose $F(G)$ has a sequence of r pairs ($r \geq 1$) of multiple edges (consisting of a cycle of length 2) from a leaf u_0 to a vertex u_r that is included in a cycle of length at least

Fig. 3 Schematic explanation of leaf-pruning from a leaf u_0 to a vertex u_r ($r \geq 1$) of $F(G)$ in a cycle of length at least three of $F(G)$.

Fig. 4 An example of constructing G_c from a given $F(G)$ (by repeating leaf-pruning).

three. Then let us shrink all vertices $u_0, \dots, u_{r-1}$ to u_r and remove any resulting self-loop, and then rename u_r as u_0 . If u_0 is a black or a white or a hybrid vertex then so is the renamed vertex u_0 . We call this operation *leaf-pruning* (from u_0 to u_r). Figure 3 shows schematic explanation of leaf-pruning.

Proposition 3.1 A lower bound $\mathcal{L}$ on the number of edges required to $(\sigma + 1)$ -edge-connect a given σ -edge-connected graph G with bipartition constraints is stated in (i) or (ii). (i) Let G_c be a graph obtained from $F(G)$ by repeating leaf-pruning (see Fig. 4). If this graph G_c is a simple cycle of length four such that two black leaves and two other ones that are either white or hybrid appear alternately (see Fig. 4) then $\mathcal{L} = 3$. (ii) Otherwise, $\mathcal{L} = \max\{|BF(G)|, |WF(G)|, \lceil |LF(G)|/2 \rceil\}$.

An algorithm to be proposed in the next section finds an edge set whose cardinality is equal to the lower bound of Proposition 3.1, showing that the algorithm finds an optimal solution.

4. A Proposed Algorithm for $(\sigma + 1)$ ECABP

In this section, we propose a fast algorithm for $(\sigma + 1)$ ECABP.

4.1 An Outline of the Proposed Algorithm

From properties of a structural graph, it is enough to consider 3ECABP for $F(G)$ instead of $(\sigma + 1)$ ECABP for G with $\lambda(G) = \sigma$ (see [7], [11]). We call an optimal solution to 3ECABP for $F(G)$ an optimal solution to $F(G)$. Without loss of generality, we can focus on the case where $F(G)$ is a modified cactus with $\lambda(F(G)) = 2$.

In order to efficiently augment the connectivity of G by one, we require that an optimal solution to $F(G)$ should consist of edges (u, v) connecting as many leaves as possible. Furthermore, in order to keep bipartition constraints, the corresponding vertex n_u (n_v , respectively) of G should be a black vertex (a white one) in $\rho^{-1}(u)$ ($\rho^{-1}(v)$).

We outline how to find an optimal solution to $F(G)$. First, in order to narrow the gap between the number of black leaves and that of white ones, each hybrid leaf is regarded as a black leaf or a white one because any hybrid leaf can be treated as a black or a white one. Next, we repeat an operation to decrease the number of leaves by adding an edge connecting a black leaf and a white one. However, if $F(G)$ is *B-dominant* then, after some iterations, there appears the situation where no white leaves are left. Then we find edges each of which connects a black leaf and either a white vertex or a hybrid one (which is not a leaf). Finally, we obtain an optimal solution to $F(G)$ and then we convert it into an optimal solution to G .

Now we explain a *special type of preorder* of a modified cactus $F(G)$. This is introduced in [11] for finding an optimal solution to $F(G)$ efficiently. First, we assign all

simple cycles in the cactus distinct colors. Note that this “color” is different from a color “black” or “white” to represent partition constraints. This coloring can be done in $O(|V| + |E|)$ time utilizing a depth-first search. Next, another depth-first-search starts at an arbitrary vertex in the following manner: if any vertex u is visited the first time via an edge in some simple cycle (for example, its color is red) then, before traversing another edge which is in the red cycle and incident to u , the other edges incident to u are traversed. (Vertices of $F(G)$ in Fig. 2 are numbered according to this preorder.) This search assigns preorder (denoted as $\beta(v)$) to each vertex v from 1 to $|V(F(G))|$. Note that traversing vertices in the order of $\beta(v)$ from 1 to $|V(F(G))|$ makes an Eulerian closed trail $ET(F(G))$ of $F(G)$.

4.2 Description of the Algorithm

The proposed algorithm to find an optimal solution is described as follows.

Algorithm *Sol*_{-($\sigma+1$)ECABP}

Input: A connected graph $G = (V, E)$ with bipartition constraints $\pi = \{V_B, V_W\}$.

Output: An edge set E_f with minimum cardinality such that $(V, E \cup E_f)$ is $(\sigma+1)$ -edge-connected and E_f is legal (with respect to π).

- 1: Construct a structural graph $F(G)=(V(F(G)), E(F(G)))$;
- 2: Compute a special type of preorder $\beta(v)$ for any vertex v of $F(G)$;
- 3: $E' \leftarrow \emptyset$, $E'_1 \leftarrow \emptyset$, $E'_2 \leftarrow \emptyset$, $E'_3 \leftarrow \emptyset$, $B \leftarrow BF(G)$, $W \leftarrow WF(G)$, and $H \leftarrow HF(G)$;
- 4: **if** $H \neq \emptyset$ **then**
- 5: Choose $\min\{[(|LF(G)| - 2|W|)/2], |H|\}$ hybrid leaves arbitrarily, add them to W (regarded as white leaves), and add the other hybrid ones to B (regarded as black leaves) if they exist;
- 6: $H \leftarrow \emptyset$; /* After this step, we have both $|B| \geq |W|$ (because $|W| \leq \lfloor |LF(G)|/2 \rfloor$) and $|LF(G)| = |B| + |W|$. */
- 7: **end if**
- 8: **while** $|B \cup W| \geq 5$ and $W \neq \emptyset$ **do**
- 9: Choose a black leaf b and a white leaf w according to Lemma 4.7 if $|B| = |W| = 3$, or according to Lemma 4.6 otherwise;
 $E'_1 \leftarrow E'_1 \cup \{(b, w)\}$, $B \leftarrow B - \{b\}$ and $W \leftarrow W - \{w\}$;
- 10: **end while**
- 11: **if** $|B \cup W| \geq 5$ and $W = \emptyset$ **then**
- 12: Find any vertex w which is a white one or a hybrid one of $F(G)$;
- 13: $E'_2 \leftarrow E'_2 \cup \{(b, w) \mid b \in B\}$ and $B \leftarrow \emptyset$;
/* See Lemma 4.2. */
- 14: **end if**
- 15: **if** $|B \cup W| = 4$ **then**
- 16: Find an edge set E'_3 according to Lemma 4.5;
- 17: **else if** $|B \cup W| = 3$ **then**
- 18: Find an edge set E'_3 according to Lemma 4.4;
- 19: **else if** $|B \cup W| = 2$ **then**
- 20: Find an edge set E'_3 according to Lemma 4.3;

Fig. 5 Schematic explanation of four vertices v, w, x and y included in $ET(F(G))$ of Lemma 4.1.

- 21: **end if**
- 22: $E' \leftarrow E'_1 \cup E'_2 \cup E'_3$;
- 23: **Output** $E_f = \{(n_b, n_w) \mid (b, w) \in E'\}$;

4.3 Correctness of the Algorithm

We prove correctness of the algorithm by using several lemmas and a theorem.

First, we show the next lemma for a structural graph $F(G)$.

Lemma 4.1 Suppose $|LF(G)| \geq 4$ for a structural graph $F(G)$. If there are distinct four leaves v, w, x, y of $F(G)$ with $\beta(v) < \beta(x) < \beta(w) < \beta(y)$ then we can choose four vertices $n_v, n_w, n_x, n_y \in V(G)$ such that $|LF(G + \{(n_v, n_w)\})| = |LF(G)| - 2$ and $|LF(G + \{(n_x, n_y)\})| = |LF(G)| - 2$.

Proof. An Eulerian closed trail $ET(F(G))$ can be determined by traversing vertices of $F(G)$ in the order of $\beta(v)$ assigned to each vertex $v \in V(F(G))$. Let us separate $ET(F(G))$ into trails P_1, P_2, P_3 and P_4 such that the pairs of terminal vertices are $\{v, x\}$, $\{x, w\}$, $\{w, y\}$ and $\{y, v\}$, respectively (see Fig. 5). Clearly they are pairwise edge-disjoint.

Now we count the number of leaves of $F(G + \{(n_v, n_w)\})$. Let $S \subset V(F(G))$ be any set such that $\{v, w\} \subseteq S$ and $\{x, y\} \subseteq V(F(G)) - S$. Since $F(G)$ has four trails P_i ($1 \leq i \leq 4$) and $d_{F(G)}(S) \geq 4$, $(S, V(F(G)) - S)$ is not a 2-cut of $F(G)$. It follows from properties of a structural graph that $(\rho^{-1}(S), V - \rho^{-1}(S))$ is not a σ -cut of G . Hence $\rho^{-1}(S)$ in $G + \{(n_v, n_w)\}$ cannot be a leaf $(\sigma+1)$ -component, showing that $|LF(G + \{(n_v, n_w)\})| = |LF(G)| - 2$. Similarly, we can prove that $|LF(G + \{(n_x, n_y)\})| = |LF(G)| - 2$. $\square$

A pair $\{v, w\}$ (or a pair $\{x, y\}$) appeared in Lemma 4.1 is called an *augmenting pair* with respect to v, w, x and y .

In the rest of this section, we discuss properties for bipartition constraints. We show the next lemma for a black leaf b and either a white one or a hybrid one w used in Lemmas 4.3 – 4.5 and Step 11 of *Sol*_{-($\sigma+1$)ECABP}.

Lemma 4.2 Suppose $WF(G) \cup HF(G) = \emptyset$ and $|BF(G)| \geq 2$. Then there exists an optimal solution $E_f = \{(n_b, n_w) \mid b \in BF(G)\}$ with $|E_f| = |BF(G)|$ and $\lambda(G + E_f) = \sigma + 1$, where w is any white vertex or hybrid one (that is not a leaf) of $F(G)$.

Proof. Let us consider an edge set $E' = \{(b, w) \mid b \in BF(G)\}$. Then $\lambda(b_1, b_2; F(G) + E') = 3$ for any distinct two black leaves b_1 and b_2 , since $\{(b_1, w), (b_2, w)\} \subseteq E'$ and $F(G) + E'$

has a path connecting b_1 and b_2 and consisting of two edges (b_1, w) and (b_2, w) . Thus, by Proposition 3.1, $E_f = \{(n_b, n_w) \mid b \in BF(G)\}$ is an optimal solution to G and $|E_f| = |BF(G)|$. $\square$

If $|LF(G)| \leq 4$ then we can show the next Lemmas 4.3–4.5, where they are used in Steps 15–19 of the proposed algorithm.

Here we summarize cardinalities of $LF(G)$, $BF(G)$, $WF(G)$ and $HF(G)$, as well as $\Delta = \min\{(|LF(G)| - 2|WF(G)|)/2, |HF(G)|\}$ and $\mathcal{L} = \max\{|BF(G)|, |WF(G)|, \lceil |LF(G)|/2 \rceil\}$ in Table 1. They are used in proofs of Lemmas 4.3, 4.4 and 4.5.

In the rest of this section let us consider situations after Step 6 of $Sol_{\mathcal{L}}(\sigma + 1)ECABP$.

Lemma 4.3 Suppose $|LF(G)| = 2$ for $F(G)$. Then there exists an optimal solution E_f with $|E_f| = 2$ if $|BF(G)| = 2$, or $|E_f| = 1$ if $|BF(G)| \leq 1$.

Proof. If $|BF(G)| = 2$ then the lemma follows from Lemma 4.2. If $|BF(G)| \leq 1$ then $\mathcal{L} = 1$, and Table 1 shows that we can choose a black vertex n_b and a white one n_w , where n_b and n_w are included in $\rho^{-1}(b)$ and $\rho^{-1}(w)$ for the two leaves $b \in B \subseteq BF(G) \cup HF(G)$ and $w \in W \subseteq WF(G) \cup HF(G)$, respectively. Clearly $\lambda(G + \{(n_b, n_w)\}) = \sigma + 1$. The

Table 1 Summary of cardinalities of sets and several values appearing in Lemmas 4.3, 4.4 and 4.5.

ℓf	bf	$wf + hf$	wf	hf	τ	Δ	b'	w'	$\mathcal{L}$	
2	2	0	-	-	-	-	-	-	2	L4.2
2	1	1	1	0	0	0	1	1	1	L4.3
2	1	1	0	1	1	1	1	1	1	L4.3
2	0	2	0	2	1	1	1	1	1	L4.3
3	3	0	-	-	-	-	-	-	3	L4.2
3	2	1	1	0	0	0	2	1	2	L4.4
3	2	1	0	1	1	1	2	1	2	L4.4
3	1	2	1	1	0	0	2	1	2	L4.4
3	1	2	0	2	1	1	2	1	2	L4.4
3	0	3	0	3	1	1	2	1	2	L4.4
4	4	0	-	-	-	-	-	-	4	L4.2
4	3	1	1	0	1	0	3	1	3	L4.5
4	3	1	0	1	2	1	3	1	3	L4.5
4	2	2	2	0	0	0	2	2	*	L4.5
4	2	2	1	1	1	1	2	2	*	L4.5
4	2	2	0	2	2	2	2	2	*	L4.5
4	1	3	1	2	1	1	2	2	*	L4.5
4	1	3	0	3	2	2	2	2	*	L4.5
4	0	4	0	4	2	2	2	2	*	L4.5

Notations:

- $\ell f = |LF(G)|$, $bf = |BF(G)|$, $wf = |WF(G)|$, $hf = |HF(G)|$,
 $\tau = \lceil (|LF(G)| - 2|WF(G)|)/2 \rceil$: the number of hybrid leaves such that we have $b' \geq w'$ even if all of τ hybrid leaves ($|HF(G)|$ hybrid ones if $\tau > |HF(G)|$) are regarded as white ones,
 $\Delta = \min\{\tau, |HF(G)|\}$: the number of hybrid leaves which are regarded as white ones in Step 4 of the proposed algorithm,
 $b' = |BF(G)| + (|HF(G)| - \Delta)$ ($w' = |WF(G)| + \Delta$, respectively): the number of black leaves (white ones) just after Step 4, that is, after regarding Δ hybrid leaves as white ones and $|HF(G)| - \Delta$ hybrid ones as black ones, of the proposed algorithm,
*: representing that $\mathcal{L}$ is given separately in lemmas,
L4.X: representing that $\mathcal{L}$ is given by Lemma 4.X.

lemma follows from Proposition 3.1. $\square$

Lemma 4.4 Suppose $|LF(G)| = 3$ for $F(G)$. Then there exists an optimal solution E_f with $|E_f| = 3$ if $|BF(G)| = 3$, or $|E_f| = 2$ if $|BF(G)| \leq 2$.

Proof. If $|BF(G)| = 3$ then the lemma follows from Lemma 4.2. Next, we consider the case with $|BF(G)| \leq 2$. Then $\mathcal{L} = 2$ and Table 1 shows that we can choose either (i) two black vertices n_{b_1}, n_{b_2} and one white one n_w that are included in $\rho^{-1}(b_1), \rho^{-1}(b_2)$ and $\rho^{-1}(w)$ for the three leaves $b_1, b_2 \in B \subseteq BF(G) \cup HF(G)$ and $w \in W \subseteq WF(G) \cup HF(G)$, respectively; or (ii) one black vertex n_b and two white ones n_{w_1}, n_{w_2} that are included in $\rho^{-1}(b), \rho^{-1}(w_1)$ and $\rho^{-1}(w_2)$ for the three leaves $b \in B \subseteq BF(G) \cup HF(G)$ and $w_1, w_2 \in W \subseteq WF(G) \cup HF(G)$, respectively. Clearly $\lambda(G + \{(n_{b_1}, n_w), (n_{b_2}, n_w)\}) = \sigma + 1$ in the case (i), or $\lambda(G + \{(n_{w_1}, n_b), (n_{w_2}, n_b)\}) = \sigma + 1$ in the case (ii). The lemma follows from Proposition 3.1. $\square$

Lemma 4.5 Suppose $|LF(G)| = 4$ for $F(G)$.

(i) If $|BF(G)| \geq 3$ then there exists an optimal solution E_f with $|E_f| = |BF(G)|$.

(ii) If $|BF(G)| \leq 2$ then let us consider a graph G_c defined in Proposition 3.1.

(ii-1) If G_c is a simple cycle of length four such that two black leaves and two other ones that are either white or hybrid ones appear alternately (see Fig. 6) then there exists an optimal solution E_f with $|E_f| = 3$;

(ii-2) otherwise, there exists an optimal solution E_f with $|E_f| = 2$.

Proof. We consider the two cases (i) and (ii) separately.

[The case (i)] If $|BF(G)| = 4$ then the lemma follows from Lemma 4.2. If $|BF(G)| = 3$ then, by applying Lemma 4.1 to the four leaves, we can obtain an augmenting pair of $\{b, w\}$, where b is a black leaf and w is either a white one or a hybrid one with respect to the four ones. $F(G + \{(n_b, n_w)\})$ has two black leaves. The lemma follows from Lemma 4.2 and Proposition 3.1.

Fig. 6 Schematic explanation of Lemma 4.5, where dash lines represent an optimal solution.

[The case (ii)] There are the two subcases (ii-1) and (ii-2).

(The case (ii-1)) Clearly the lemma follows from Proposition 3.1.

(The case (ii-2)) Table 1 shows that we can choose an augmenting pair $\{b, w\}$ with respect to the four leaves such that there are a black vertex $n_b \in \rho^{-1}(b)$ and a white one $n_w \in \rho^{-1}(w)$ so that $F(G + \{(n_b, n_w)\})$ may have two leaves including at most one black one. The lemma follows from Proposition 3.1 and Lemma 4.3. $\square$

Next, we consider the case with $|LF(G)| \geq 5$. We show the following two lemmas for a black leaf b and a white one w in Step 8 of $Sol_{\leq}(\sigma+1)ECABP$. For simplicity of description, let us call a hybrid leaf $v \in B$ ($v \in W$, respectively) at Step 8 a black leaf (a white one) in the rest of this section, and without loss of generality we denote $B = \{b_1, \dots, b_{|B|}\}$ and $W = \{w_1, \dots, w_{|W|}\}$, where $\beta(b_1) < \dots < \beta(b_{|B|})$ and $\beta(w_1) < \dots < \beta(w_{|W|})$.

Lemma 4.6 Suppose $|B \cup W| \geq 5$ and $W \neq \emptyset$ in Step 8. Then there is a set of four leaves from which we can choose a black leaf b and a white one w of $F(G)$ such that $\{b, w\}$ is an augmenting pair with respect to the four ones.

Proof. We prove the lemma by showing an augmenting pair of a black leaf and a white one with respect to some four ones in $B \cup W$. Let us consider the two cases (i) $|W| = 1$ and (ii) $|W| > 1$, separately.

[The case (i)] We consider the four subcases (i-1)–(i-4) according to the sequences of four leaves w_1, b_1, b_2 and b_3 sorted in increasing order of β . Then we can select an augmenting pair with respect to the four leaves as follows: (i-1) if $\beta(w_1) < \beta(b_1) < \beta(b_2) < \beta(b_3)$ then select w_1 and b_2 ; (i-2) if $\beta(b_1) < \beta(w_1) < \beta(b_2) < \beta(b_3)$ then select w_1 and b_3 ; (i-3) if $\beta(b_1) < \beta(b_2) < \beta(w_1) < \beta(b_3)$ then select b_1 and w_1 ; (i-4) if $\beta(b_1) < \beta(b_2) < \beta(b_3) < \beta(w_1)$ then select b_2 and w_1 .

[The case (ii)] We consider the six subcases (ii-1)–(ii-6) according to the sequence of the four leaves w_1, w_2, b_1 and b_2 sorted in increasing order of β . Then we can select an augmenting pair with respect to the four leaves for the first four cases as follows:

(ii-1) if $\beta(w_1) < \beta(w_2) < \beta(b_1) < \beta(b_2)$ then select w_1 and b_1 ;

(ii-2) if $\beta(b_1) < \beta(b_2) < \beta(w_1) < \beta(w_2)$ then select b_1 and w_1 ;

(ii-3) if $\beta(w_1) < \beta(b_1) < \beta(b_2) < \beta(w_2)$ then select w_1 and b_2 ;

(ii-4) if $\beta(b_1) < \beta(w_1) < \beta(w_2) < \beta(b_2)$ then select b_1 and w_2 .

(ii-5) Now consider the sequence with $\beta(w_1) < \beta(b_1) < \beta(w_2) < \beta(b_2)$. Since $|B \cup W| \geq 5$ and $|B| \geq |W|$, there is $b_{|B|}$ such that $\beta(b_2) < \beta(b_{|B|})$. Hence the pair $\{w_1, b_2\}$ is an augmenting pair with respect to w_1, w_2, b_2 and $b_{|B|}$.

(ii-6) Finally consider the sequence with $\beta(b_1) < \beta(w_1) < \beta(b_2) < \beta(w_2)$. Note that since $|B \cup W| \geq 5$ and $|B| \geq |W|$, b_3 and $b_{|B|}$ ($b_3 = b_{|B|}$ may hold) are different from b_1, w_1, b_2 and w_2 . There are two subcases.

(ii-6-a) $W = \{w_1, w_2\}$ (that is, $w_2 = w_{|W|}$) and $\beta(b_{|B|}) < \beta(w_2)$. The pair $\{w_1, b_3\}$ is an augmenting pair with respect to w_1, b_2, b_3 and w_2 .

(ii-6-b) Any other case except (ii-6-a). There is a leaf v such that v is different from b_1, w_1, b_2 and w_2 , and $\beta(v) = \max\{\beta(b_{|B|}), \beta(w_{|W|})\}$ holds. Then the pair $\{b_1, w_2\}$ is an augmenting pair with respect to b_1, b_2, w_2 and v . $\square$

Even if we select two vertices $n_b \in \rho^{-1}(b)$, $n_w \in \rho^{-1}(w)$ for the pair b, w of Lemma 4.6, we have to avoid the case where G_c constructed from $F(G + \{(n_b, n_w)\})$ by repeating leaf-pruning satisfies Proposition 3.1 (i). This can be done as stated in Lemma 4.7.

Lemma 4.7 Suppose $|B| = |W| = 3$ in Step 8. There are a black leaf b and a white one w satisfying the following two conditions (i) and (ii): (i) the pair $\{b, w\}$ is an augmenting pair; (ii) any G_c constructed from $F(G + \{(n_b, n_w)\})$ by repeating leaf-pruning does not satisfy Proposition 3.1 (i), that is, even if G_c is a cycle of length four, remaining four leaves (two black ones and two white ones) never appear alternately in increasing order of β .

Proof. We can assume that $\beta(b_1) < \beta(w_1)$ without loss of generality. Then there are ten kinds of sequences consisting of the six leaves sorted in increasing order of β . Let us focus on an augmenting pair $\{b_1, w\}$ for some white leaf w with respect to some four leaves including b_1 . In this setting, any choice of $\{b_1, w\}$ in the following two sequences (ii-1) and (ii-2) violates the condition (ii) of the lemma, while we can select $\{b_1, w\}$ satisfying the lemma in the remaining eight sequences.

(ii-1) $\beta(b_1) < \beta(b_2) < \beta(w_1) < \beta(w_2) < \beta(b_3) < \beta(w_3)$;

(ii-2) $\beta(b_1) < \beta(w_1) < \beta(b_2) < \beta(w_2) < \beta(w_3) < \beta(b_3)$.

Choosing another pair, however, gives us a desired one. In the case (ii-1), a pair $\{b_2, w_3\}$ is an augmenting pair with respect to b_1, b_2, w_1 and w_3 , and in the case (ii-2) $\{w_1, b_3\}$ is an augmenting pair with respect to b_1, w_1, w_2 and b_3 . Moreover, for $F(G + \{(n_{b_2}, n_{w_3})\})$ in (ii-1) and $F(G + \{(n_{b_3}, n_{w_1})\})$ in (ii-2), the remaining four leaves satisfy the condition (ii) of the lemma. $\square$

Note that it is not necessary to actually construct a graph G_c even if $|B| = |W| = 3$ at Step 9.

From Lemmas 4.1–4.7, we obtain the next theorem.

Theorem 4.8 For a given graph G with $\lambda(G) = \sigma$ and a bipartition $\{V_B, V_W\}$ of V with $V_B \cap V_W = \emptyset$, $Sol_{\leq}(\sigma+1)ECABP$ finds a legal edge set E_f , with $|E_f| = \mathcal{L} = \max\{|BF(G)|, |WF(G)|, \lceil |LF(G)|/2 \rceil\}$, such that $\lambda(G + E_f) = \sigma + 1$.

Proof. Let b (w , respectively) be a black leaf (a white one) which is chosen in Step 8 or 11. Let B_c (W_c , respectively) denote the current B (W). For the edge set E_f to appear in the proof, its construction shows that it is legal.

For an edge or an edge set found in Steps 8, 11 and 15, we consider the following three cases:

(i) $|LF(G)| \geq 5$ and $|B| = |W|$ just before Step 8;

(ii) $|LF(G)| \geq 5$ and $|B| > |W|$ just before Step 8 (then $F(G)$ is B -dominant);

(iii) $|LF(G)| \leq 4$.

[The case (i)] Note that Step 11 is never executed.

While $|B| = |W| > 3$, we can repeat choosing an augmenting pair $\{b, w\}$ by Lemma 4.6. If the case with $|B| = |W| = 3$ appears then we can select an augmenting pair $\{b, w\}$ by Lemma 4.7.

Now, we consider E'_1 , B_c and W_c just before Step 15. Since Steps 8 and 9 are executed, $|E'_1| = |B| - |B_c| = |W| - |W_c|$, $|B_c| = |W_c| = 2$, and $F(G + E_1)$ has four leaves b_1 , b_2 , w_1 and w_2 with $B_c = \{b_1, b_2\}$ and $W_c = \{w_1, w_2\}$. Let $E_1 = \{(n_b, n_w) \mid (b, w) \in E'_1\}$.

It follows from Lemma 4.7 that two black vertices and two white ones do not appear alternately in any G_c constructed from $F(G + E_1)$ by repeating leaf-pruning even if G_c is a cycle of length four (see Fig. 6).

Thus there exists an optimal solution E_f to $G + E_1$ with $|E_f| = 2$ by Lemma 4.5 (ii-2), and an edge set E'_3 with $|E'_3| = 2$ is found in Step 15. Hence $|E_f| = |LF(G)|/2$ and E_f is a solution to G by Proposition 3.1.

[The case (ii)] Just after Step 8, one of the following subcases (ii-1)–(ii-4) holds:

(ii-1) $|B| > 4$ and $|W| = 0$;

(ii-2) $|B| = 4$ and $|W| = 0$;

(ii-3) $|B| = 3$ and $0 \leq |W| \leq 1$;

(ii-4) $|B| = 2$ and $|W| = 1$.

(The case with $|B| = 2$ and $|W| = 0$ follows from (ii-3) with $|W| = 1$.)

For the case (ii-1), Step 11 is executed and a set E'_2 with $|E'_2| = |B_c|$ is obtained. On the other hand, for each of the cases (ii-2)–(ii-4), Step 15 or 17 is executed and a set E'_3 with $|E'_3| = |B_c|$ is obtained. Thus $|E_f| = |BF(G)|$.

Concerning the optimal solution $E' = E'_1 \cup E'_2 \cup E'_3$ to $F(G)$ shown in Fig. 2, $E'_1 = \{(1, 8), (3, 5)\}$ is obtained in Step 8, and $E'_3 = \{(4, 10), (9, 10), (10, 12)\}$ is obtained in Step 15.

[The case (iii)] In Step 15, we can find an edge set E_f , with $|E_f| = \max\{|BF(G)|, |WF(G)|, \lceil |LF(G)|/2 \rceil\}$, such that $\lambda(G + E_f) = \sigma + 1$ by Lemmas 4.3–4.5. $\square$

4.4 Time Complexity

Let us focus on how to find all edges (b, w) in Step 8 of $Sol_{-(\sigma+1)ECABP}$ in linear time.

Fig. 7 The data structure used in Step 8. Each element with a black circle (a white one, respectively) represents a black leaf (a white one) of $F(G)$.

[Data Structure] Just before Step 8, we make following preparations (see Fig. 7). We provide two linear lists such that each vertex of B or W is kept in one of the lists in increasing order of β (after Step 4, each leaf of $HF(G)$ is included in either B or W). Each element in the lists has three members, and these three members in the element are denoted as $p \rightarrow \beta$, $p \rightarrow \text{color}$ and $p \rightarrow \text{next}$ by using a pointer p to the element of the list. In the element indicated by p and containing a leaf v , we store $\beta(v)$ in $p \rightarrow \beta$, its color (black or white) in $p \rightarrow \text{color}$ and the pointer to the next element in $p \rightarrow \text{next}$. Note that nil is stored in $p \rightarrow \text{next}$ if the element is last one of each list, where it is denoted by a slash in Fig. 7. We also prepare three pointers top , top_b and top_w . For simplicity of description, we often denote the element indicated by a pointer p as a leaf v if v is stored in the element.

The pointer top points to the element containing a leaf with the minimum value of β among $B \cup W$. The pointer top_b (top_w , respectively) points to the element containing a leaf with a minimum value of β among the set B (W) from which top is removed, where if $W = \emptyset$ then $top_w \leftarrow nil$.

[The Details of Step 8] We show the detailed description of Step 8 as follows.

```

1: while  $|B \cup W| \geq 5$  and  $W \neq \emptyset$  do
2:   if  $|B| = |W| = 3$  then
3:     Let  $b$  and  $w$  be leaves specified in Lemma 4.7;
4:   else if  $|W| = 1$  then
5:     Find a black leaf  $b$  and a white leaf  $w$  by (i-1)–(i-4)
      in the proof of Lemma 4.6;
6:   else if  $top \rightarrow \text{color}$  is black /*  $|W| > 1$  */ then
7:     if  $\{W = \{w_1, w_2\} \text{ and } \beta(b_{|B|}) < \beta(w_2) \text{ and } \beta(b_1) < \beta(w_1) < \beta(b_2) < \beta(w_2)\}$  then
8:       Find a black leaf  $b$  and a white leaf  $w$ 
        by (ii-6-a) in the proof of Lemma 4.6;
9:        $top_w \leftarrow top_w \rightarrow \text{next}$ ;
        Delete the element corresponding to  $b_3$  (in
         $top \rightarrow \text{next}$ ) from the list;
10:    else if  $top_b \rightarrow \beta > top_w \rightarrow \beta$  then
11:      /* (ii-4) or (ii-6-b) in the proof of Lemma 4.6 */
       $b \leftarrow top$ ;  $w \leftarrow top_w \rightarrow \text{next}$ ;
       $top \leftarrow top_w$ ;  $top_w \leftarrow (top_w \rightarrow \text{next}) \rightarrow \text{next}$ ;
      /*  $top_w$  may be  $nil$  */
12:    else if  $top_b \rightarrow \beta < top_w \rightarrow \beta$  then
13:      /* (ii-2) in the proof of Lemma 4.6 */
       $b \leftarrow top$ ;  $w \leftarrow top_w$ ;  $top \leftarrow top_b$ ;  $top_b \leftarrow top_b \rightarrow \text{next}$ ;  $top_w \leftarrow top_w \rightarrow \text{next}$ ;
14:    end if
15:  else /*  $top \rightarrow \text{color}$  is white and  $|W| > 1$  */
16:    if  $top_b \rightarrow \beta > top_w \rightarrow \beta$  then
17:      /* (ii-1) in the proof of Lemma 4.6 */
       $b \leftarrow top_b$ ;  $w \leftarrow top$ ;  $top \leftarrow top_w$ ;
       $top_w \leftarrow top_w \rightarrow \text{next}$ ;  $top_b \leftarrow top_b \rightarrow \text{next}$ ;
      /*  $top_w$  may be  $nil$  */
18:    else if  $top_b \rightarrow \beta < top_w \rightarrow \beta$  then
19:      /* (ii-3) or (ii-5) in the proof of Lemma 4.6 */
       $b \leftarrow top_b \rightarrow \text{next}$ ;  $w \leftarrow top$ ;

```

```

top ← top_b; top_b ← (top_b → next) → next;
20:   end if
21:   end if
22:    $E' \leftarrow E' \cup \{(b, w)\};$ 
    $B \leftarrow B - \{b\}$  and  $W \leftarrow W - \{w\};$ 
23: end while

```

It is straightforward to see that this procedure works correctly.

[Total Time Complexity] Clearly the above procedure can find augmenting pairs shown in Lemma 4.6 or Lemma 4.7 in $O(|V|)$ time. A structural graph can be constructed in $O(|V||E| + |V|^2 \log |V|)$ time [10]. Moreover, since all $(\sigma + 1)$ -components are computed in linear time [8], [12]–[14] when $\lambda(G) \in \{1, 2\}$, a structural graph can be constructed in linear time. From above discussion, Proposition 3.1 and Theorem 4.8, we obtain the next theorem.

Theorem 4.9 Algorithm *Sol*_ $(\sigma + 1)$ ECABP computes an optimal solution to $(\sigma + 1)$ ECABP for G with $\sigma = \lambda(G)$ in $O(|V||E| + |V|^2 \log |V|)$ time. Moreover, this is reduced to $O(|V| + |E|)$ time when $\sigma \in \{1, 2\}$.

5. Concluding Remarks

In this paper, we have proposed an $O(|V||E| + |V|^2 \log |V|)$ time algorithm to find an optimal solution to $(\sigma + 1)$ ECABP when $\sigma = \lambda(G)$. Moreover, we have shown that the problem can be solved in linear time when $\sigma \in \{1, 2\}$.

As future research, proposing an efficient algorithm for $(\sigma + 1)$ ECAMP with $\sigma = \lambda(G)$ is left.

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