PAPER

An Efficient Algorithm for Node-Weighted Tree Partitioning with Subtrees' Weights in a Given Range

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SUMMARY Tree partitioning arises in many parallel and distributed computing applications and storage systems. Some operator scheduling problems need to partition a tree into a number of vertex-disjoint subtrees such that some constraints are satisfied and some criteria are optimized. Given a tree T with each vertex or node assigned a nonnegative integer weight, two nonnegative integers l and u ($l < u$), and a positive integer p , we consider the following tree partitioning problems: partitioning T into minimum number of subtrees or p subtrees, with the condition that the sum of node weights in each subtree is at most u and at least l . To solve the two problems, we provide a fast polynomial-time algorithm, including a pre-processing method and another bottom-up scheme with dynamic programming. With experimental studies, we show that our algorithm outperforms another prior algorithm presented by Ito et al. greatly.

key words: tree partition, operator scheduling, dynamic programming, distributed computing

1. Introduction

Pipelined operator scheduling is an important problem in the area of parallel and distributed computing [1], [2]. For a large class of applications, the computing tasks can be represented as pipelined operator graphs, and an efficient operator schedule is critical to reach the best possible performance for distributed computing. Here we investigate a special class of operator scheduling problems for trees. Consider an arbitrary tree T with nonnegative integer weights on the nodes. Each node v in T is called an operator and assigned a weight $w(v)$. We aim to partition T into disjoint p clusters by deleting edges from T such that each cluster is a subtree and the total weight of each subtree is in a given range $[l, u]$, where l and u are two nonnegative integers and $l < u$. Such a partition is called a p - $[l, u]$ partition of T . In this paper, we mainly deal with the following problem: partitioning T into minimum number of node-disjoint subtrees satisfying that the total weight of each subtree is in the given range $[l, u]$, which is called *minimum $[l, u]$ partition problem*. We present a polynomial-time algorithm to solve the problem. The concept of the algorithm is as follows. Firstly, we adopt a preprocessing approach to transform the original tree T into a number of new trees for which the weight of each leaf node is in $[l, u]$. Then we use a dynamic pro-

gramming formulation to find a *minimum $[l, u]$ partition* for each of these new trees, where a few pruning methods are presented to speed up the computing process. At last, we combine *minimum $[l, u]$ partitions* for these new trees and get the *minimum $[l, u]$ partition* for T as the final result. The proposed algorithm only needs to be modified slightly to solve the p - $[l, u]$ partition problem. A series of experiments are conducted to evaluate our algorithm and compare it with the state-of-the-art polynomial-time algorithm presented by Ito et al. [3]. Experimental results reveal that our proposed algorithm runs much faster than the algorithm presented by Ito et al. and reduce the total computing time greatly in different cases.

The paper is structured as follows. Section 2 contains a review of related works. In Sect. 3 we formalize the problem and present a pre-processing algorithm to simplify the original problem. Based on the pre-processing step, we then give a polynomial-time algorithm for the *minimum $[l, u]$ partition* problem in Sect. 4. Section 5 presents a slightly modified algorithm to solve the p - $[l, u]$ partition problem. Section 6 evaluates our novel algorithm. Finally, Sect. 7 concludes this paper.

2. Related Work

Our work is mainly related to the research of pipelined operator scheduling. Pipelined operator tree (POT) scheduling is an important problem in the area of parallel query optimization. A schedule assigns n operators of a tree to p processors. Weights associated with nodes and edges of a tree represent respectively the cost of operators and communication between them. There are a few kinds of tree scheduling problem depending on different application fields and different optimization objectives. Some scheduling don't consider the communication costs of processors, while others do. To evaluate the performance of a schedule, various optimization criteria are used.

The optimization target of some scheduling problem [1], [4], [5] is to find a schedule with minimum response time (or makespan). The response time of a schedule is the maximum processor load. This kind of tree scheduling is NP-hard since even the special case with all communication costs being zero is a classical multiprocessor scheduling problem. S.E. Hambrusch [6] considers partitioning the vertices of an n -vertex tree into p disjoint clusters so that the number of vertices in a cluster and the number of subtrees in a cluster are minimized. For this NP-hard problem,

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S.E. Hambruch presents greedy heuristics for k -clustering the vertices of a tree when cluster sizes need to lie in an upper limit defined by k . Kanne and Moerkotte [7] present a partitioning model called sibling partitioning, which considers the subtrees that are siblings as one part. They propose a linear time algorithm that optimally partitions an ordered, labeled, weighted tree such that each partition does not exceed a fixed weight limit. Jan-Jan Wu [8] proposes an optimal partitioning algorithm for metadata trees in the area of distributed storage systems, focusing on how to partition a hierarchical metadata structure into multiple metadata servers evenly. In [8], a dynamic programming approach is proposed in minimizing the workload imbalance among metadata servers. Ito et al. [3] present the $[l, u]$ partition problem for trees and obtained a polynomial-time algorithm to solve p - $[l, u]$ partition and *minimum*- $[l, u]$ partition problems.

3. Problem Statement and Simplification

3.1 Terms and Definitions

Consider a tree $T = (V, E)$ rooted with node r , where V is a set of nodes and E is a set of edges. For any node v of T , we denote v 's parent node by $v.parent$ and denote the subtree of T which is rooted at v by T_v . Each node of T is assigned a nonnegative integer weight. The weight of a subtree is defined as the sum of node weights in the subtree. The weight of node v and subtree T_v are denoted by $w(v)$ and $w(T_v)$ respectively. The number of nodes for T is denoted by $|T|$ or n ($n = |T|$). Assume we select $k - 1$ nodes from a subtree T_v and break T_v into k non-empty subtrees by cutting down the $k - 1$ edges which connect the $k - 1$ nodes with their parents. The set of the $k - 1$ nodes together with v is called a k -partition of T_v , denoted by $P_{T_v}^k$. Given a k -partition $P_{T_v}^k$, we denote the set of partitioned subtrees by $F(P_{T_v}^k)$ and call it *partition forest*. The weight of the subtree rooted with a node x in $F(P_{T_v}^k)$ is denoted by $W_x(P_{T_v}^k)$. The unique subtree which contains v for a $P_{T_v}^k$ is called *pivot subtree* (*pivot tree* for short), and its weight is denoted by $W_v(P_{T_v}^k)$.

Definition 1 (k - $[l, u]$ partition). A k -partition for a tree T_v is called a k - $[l, u]$ partition, if the weight of each partitioned subtree is in $[l, u]$. Then, a k - $[l, u]$ partition for T_v is called *feasible*.

There may exist many (J for example) k - $[l, u]$ partitions for a tree T_v , we can denote each of them by $P_{T_v}^k(j)$, $1 \leq j \leq J$. A $P_{T_v}^k(j)$ is called *leanest* if the weight of *pivot tree* is minimum among all k - $[l, u]$ partitions.

Definition 2 (*minimum* $[l, u]$ partition). P_T^k is a *minimum* $[l, u]$ partition if k is the minimum integer existed for k - $[l, u]$ partition of a tree T .

The problem to be solved in this paper is formally stated as follows: (1) Given a tree T with a weight range $[l, u]$, find a *minimum* $[l, u]$ partition for T . (2) Given a tree T with a weight range $[l, u]$ and a number p , find a p - $[l, u]$ partition for T .

Before solving our problems, we present some properties which will be useful for optimizing our algorithm later.

Property 1. Assume p - $[l, u]$ partition for a tree T is *feasible*. Consider a leaf node v of T with $w(v) > l$. If we replace v 's weight with a new integer w' , $l \leq w' < w(v)$, T 's p - $[l, u]$ partition is still *feasible*.

From *property 1*, we can easily deduce the following property.

Property 2. Assume p - $[l, u]$ partition for tree T is *feasible*. Let T_v be a subtree of T and $P_{T_v}^k(j)$ be a k - $[l, u]$ partition of T_v , $k < p$. We suppose $P_{T_v}^k(j) = \{x \in P_T^p(j') \mid x \in T_v\}$ holds for some $P_T^p(j')$. If $P_{T_v}^k(j)$ is not the *leanest* among all k - $[l, u]$ partition of T_v , then there must be another $P_T^p(j'')$ for which $P_{T_v}^{k(l)} = \{x \in P_T^p(j'') \mid x \in T_v\}$ holds, where $P_{T_v}^{k(l)}$ denotes the *leanest* k - $[l, u]$ partition of T_v .

3.2 Problem Simplification

Definition 3. A tree or subtree is called *fitting* if its total weight is in $[l, u]$. A node is called *fitting* if its weight is in $[l, u]$. A tree is called *leaf-fitting tree* (*lf-tree* for short) if all leaf nodes are *fitting*.

To reduce the scale of tree partitioning problems, we first translate an initial tree T into much smaller *lf-trees*. Now we present two properties that one can verify easily.

Property 3. Consider a leaf node v in tree T and suppose $w(v) < l$. If $w(v) + w(v.parent) > u$, then p - $[l, u]$ partition of T doesn't exist for any p . If $w(v) + w(v.parent) \leq u$, then v and its parent must belong to the same subtree for T 's any p - $[l, u]$ partition.

Property 4. Let $\hat{T}$ be a *lf-tree*. Suppose there is a leaf node or an internal node v' of $\hat{T}$ and $w(v') + w(v'.parent) > u$. If a k - $[l, u]$ partition $P_{\hat{T}}^k$ of $\hat{T}$ exists for any integer k , then we have: $v' \in P_{\hat{T}}^k$.

Motivated by the properties above, we use a pre-processing algorithm (Algorithm 1) to glue together certain leaf nodes of T iteratively into their parent nodes with their weight adding together, and then partition some subtrees from the global tree, provided that doing so would not affect the final solution of our problems.

The pre-processing algorithm iteratively fuses pairs of nodes $(v, v.parent)$ for which the condition in *property 3* holds (line 3, 14, 22, 23 of Algorithm 1). Once we fuse a node v into its parent, we update the parent's weight as $w(v.parent) = w(v.parent) + w(v)$. After the fusing process, we turn the original T into a new tree $\hat{T}$ in which all leaf nodes are *fitting*. Then, we check each internal node's children, and partition those nodes for which the condition in *property 4* holds (line 10 to 11). By the iterative fusing and partitioning, we obtain at last a few new *lf-trees*, in which all leaf nodes are *fitting* and $w(v) + w(v.parent) \leq u$ for any node v . Figure 1 shows an example tree T_v and the *lf-trees* after pre-processing. Here we assume $l = 5$ and $u = 14$.

In Fig. 1, each node is represented as an oval with its identifier and weight. All *fitting* nodes are represented as a black oval. Clearly, nodes $v_1, v_3, v_{22}, v_{23}, v_{41}, v_{43}$ are fused

Algorithm 1 Pre-processing.**Require:** T tree;**Ensure:** $\tilde{T}S$ (a set of *lf-trees*);

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1: for all nodes  $v$  of  $T$  in post order do
2:   if  $v$  is leaf then
3:     LeafProcess( $v$ );
4:   else
5:     if  $w(v) > u$  then
6:        $\tilde{T}S = \phi$ ;
7:       return;
8:     end if
9:     for any  $v$ 's child  $v_i$  with  $w(v) + w(v_i) > u$  do
10:      cut off  $v_i$  from  $T$ ;
11:      add  $T_{v_i}$  into  $\tilde{T}S$ ;
12:    end for
13:    if  $v$  is leaf then
14:      LeafProcess( $v$ );
15:    end if
16:  end if
17: end for
18: return  $\tilde{T}S$ ;
19:
20: function : LeafProcess( $v$ )
21: if  $w(v) < l$  then
22:    $w(v.parent) = w(v.parent) + w(v)$ ;
23:   cut off  $v$  from  $T$ ;
24: end if
25: if  $w(v) > u$  then
26:    $\tilde{T}S = \phi$ ;
27:   return;
28: end if

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Fig. 1 (a) Tree T_v , and (b) *lf-subtrees* after pre-processing to T_v , where $l = 5$ and $u = 14$.

into their parents, since they are not *fitting* leaf nodes. The edge (v, v_4) is deleted, since $w(v) + w(v_4) > u$. Our experiments in Sect. 6 show that this pre-processing step decreases the scale of problem greatly, and help to speed up the process of arriving at the final solution. The pre-processing algorithm has a runtime complexity of $O(n)$.

4. Minimum $[l, u]$ Partition Problem for Arbitrary Tree

4.1 Problem Analysis of p - $[l, u]$ Partition for *lf-tree*

To solve the p - $[l, u]$ partition problem of a *lf-tree*, we present some new definitions in advance.

Definition 4 (*near k - $[l, u]$ partition*). A k -partition for tree T_v is called *near k - $[l, u]$ partition*, denoted by $\tilde{P}_{T_v}^k$, if

Fig. 2 (a) Tree T_v^i , and (b) an illustration for the handling of T_v^i from T_v^{i-1} and T_{v_i} in case 1 and case 2.

$W_v(\tilde{P}_{T_v}^k) \leq u$ and $l \leq W_x(\tilde{P}_{T_v}^k) \leq u$ for each $x \in \tilde{P}_{T_v}^k - \{v\}$.

There may exist J near k - $[l, u]$ partitions for a tree T_v , we can denote each of them by $\tilde{P}_{T_v}^k(j)$, $1 \leq j \leq J$. A $\tilde{P}_{T_v}^k(j)$ is called *fitting leanest* if the weight of *pivot tree* for it is minimum among all near k - $[l, u]$ partitions with *pivot tree* weighted in range $[l, u]$. According to *definition 4*, we can define a set $S(T_v, k)$ as the set of all integers w , called by *Pivot Weight Set (PWS)*, where w is the weight of *pivot tree* of each near k - $[l, u]$ partition of T_v . $S(T_v, k)$ can be formally presented as follows:

$$S(T_v, k) = \{w \mid w = W_v(\tilde{P}_{T_v}^k(j)) \text{ for each } \tilde{P}_{T_v}^k(j) \text{ of } T_v\}$$

Here, an element of $S(T_v, k)$ is called *fitting* if its value is in $[l, u]$. Based on $S(T_v, k)$, we define a compression operation $CP_l()$ for $S(T_v, k)$ and a new set $S'(T_v, k)$ as follows:

$$S'(T_v, k) = CP_l(S(T_v, k)) = \{w \mid w \in S(T_v, k), w < l\} \cup \{\min\{w \mid w \in S(T_v, k), w \geq l\}\}$$

Clearly, the compression operation $CP_l()$ is to obtain a set $S'(T_v, k)$ such that only the minimum *fitting* element is kept if there have *fitting* elements in $S(T_v, k)$, by deleting all *fitting* elements of $S(T_v, k)$ except the minimum *fitting* one.

Obviously, $S(T, p) = \{w \mid w = W_r(\tilde{P}_T^p(j)) \text{ for each } \tilde{P}_T^p(j) \text{ of } T\}$, where r is the root node of T . Then, if $S(T, p)$ contains an element w , $l \leq w \leq u$, T 's p - $[l, u]$ partition is *feasible*. In other word, T has an p - $[l, u]$ partition if there is a near p - $[l, u]$ partition $\tilde{P}_T^p$ with the weight of its *pivot tree* in $[l, u]$, i.e., $W_r(\tilde{P}_T^p) \in [l, u]$.

With the above definitions, we can use a bottom-up approach to combine near k - $[l, u]$ partitions for subtrees to obtain a global solution of near p - $[l, u]$ partitions for the root tree. For example, we assume a subtree T_v has m children $v_1, v_2, \dots, v_m$ from left to right, as illustrated in Fig. 2 (a). We first compute the p sets $\{S(T_{v_i}, k) \mid k = 1, \dots, p\}$ for each T_{v_i} , and then we arrive at node v and compute the p sets $\{S(T_v, k) \mid k = 1, \dots, p\}$ by means of dynamic programming similar to [1], [3], [5]. The details of computing $S(T_v, k)$ are described briefly as follows.

For a subtree T_v in Fig. 2 (a), with every edge (v, v_i) we associate a subtree T_v^i defined in the following way: it consists of node v , the i edges $(v, v_1), \dots, (v, v_i)$, the subtree T_{v_i} , and all subtrees rooted at left brothers of node v_i , i.e., $T_{v_1}, T_{v_2}, \dots, T_{v_{i-1}}$. Here, $T_v^0 = v$, $T_v^m = T_v$. Clearly, T_v^i can be obtained by adding T_{v_i} to T_v^{i-1} . Given a near k_1 - $[l, u]$ partition of T_v^{i-1} and a near k_2 - $[l, u]$ partition T_{v_i} , we obtain a near $[l, u]$ partition for T_v^i , by either adding *pivot tree* of $\tilde{P}_{T_{v_i}}^{k_2}$ to the *pivot tree* of $\tilde{P}_{T_v^{i-1}}^{k_1}$ or making

pivot tree of $\tilde{P}_{T_{v_i}}^{k_2}$ to form its own subtree as illustrated in Fig. 2(b). Hence, $S(T_v^i, k)$ can be derived from $S(T_{v_i}, k_1)$ and $S(T_{v_i}^{i-1}, k_2)$, where k_1 and k_2 range from 1 to k , with the condition that $k = k_1 + k_2$ or $k_1 + k_2 - 1$ for different two cases. Therefore, step by step, we can obtain p sets $\{S(T_v, k) \mid k = 1, \dots, p\}$ based on $\{S(T_{v_i}, k) \mid k = 1, \dots, p\}$, $i \in [1, \dots, m]$. We will analyze the computing detail in Sect. 4.2.

From *property 2*, we can derive that the *leanest fitting* partition among all k -[l, u]partitions for any subtree T_v^i is always sufficient to examine whether a global tree T has a p -[l, u]partition, except for those *near* k -[l, u]partitions with *pivot tree*'s weight less than l . Therefore, in the process of bottom-up traversing and computing, we only need to compute $\{S'(T_v^i, k) \mid k = 1, \dots, p\}$ for each node v with $i \in [1, \dots, m]$, until we arrive at root node. If $S'(T, p)$ contains an element w with $l \leq w \leq u$, T 's p -[l, u]partition is *feasible*. So we have the following lemma.

Lemma 1. Tree T has a p -[l, u]partition if and only if $S'(T, p)$ contains an integer w such that $l \leq w \leq u$.

4.2 Basic Algorithm for Minimum [l, u]Partition Problem

Now we present our algorithm to solve the *minimum* [l, u]partition problem for an arbitrary tree T . The pseudocode of the basic algorithm is described in Algorithm 2. By pre-processing sub-algorithm, we firstly translate the original tree T into *lf-trees* (line 1 of Algorithm 2). Then we can obtain the globally minimum solution by solving the *minimum* [l, u]partition problem for each *lf-tree* (line 2 to 31). Now we concentrate on *minimum* [l, u]partition problem for *lf-tree* specifically. We traverse each node in post order and compute p sets $\{S'(*, k) \mid k = 1, \dots, p\}$ along a *lf-tree* T , beginning at the leaf node level (*level 1*) and going up to the root level. Here, p could be any integer larger than the number of subtrees for *minimum* [l, u]partition. For simplicity, we choose $p = \lfloor \frac{w(T)}{l} \rfloor$.

- For any node v of *level 1* (leaf node *level*):

$S'(v, 1) = w(v)$; $S'(v, k) = \phi$ for each $k, 2 \leq k \leq p$. (line 5 to 6)

- For any node v of *level 2*:

Here, a node of *level 2* refers to a node for which all its children are leaf nodes. Since all children of node v are *fitting* leaf nodes, we have the following lemma.

Lemma 2. Let T_v be a subtree with an internal node v and m *fitting* leaf nodes (i.e., v has m children), V the set of all nodes of T_v , $HV(k)$ the set of the heaviest k sons of v . Then, (1) $S'(T_v, m+1) = \{w(v)\}$; (2) for $k = 1, \dots, m$, we have:

$$S'(T_v, k) = \begin{cases} \phi & \sum_{v_i \in V - HV(k-1)} w(v_i) > u, \\ \left\{ \sum_{v_i \in V - HV(k-1)} w(v_i) \right\} & \text{otherwise.} \end{cases}$$

Proof. Since T_v has exactly $m+1$ nodes, there is only one *near* $(m+1)$ -[l, u]partition, i.e., $\tilde{P}_{T_v}^{m+1} = V$. Hence, $S'(T_v, m+1) = \{w(v)\}$. Obviously, $S'(T_v, k) = \phi$ for

Algorithm 2 Finding minimum [l, u]partition for a tree T .

Require:

T tree;

Ensure:

minimum partition number p_{MIN} ;

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1:  $\tilde{T}S = \text{pre-processing}(T)$ ;
2: for each lf-tree  $\tilde{T}_j$  of  $\tilde{T}S$  do
3:    $T = \tilde{T}_j$ ;  $p = \lfloor \frac{w(T)}{l} \rfloor$ ;
4:   for all nodes  $v$  of  $T$  in post order do
5:      $S'(T_v^0, 1) = S(T_v^0, 1) = S(v) = w(v)$ ;
6:      $S'(T_v^0, k) = S(T_v^0, k) = \phi$ , for  $2 \leq k \leq p$ ;
7:     if  $\text{height}(T_v) = 2$  then
8:        $i = 1$ ;
9:       while  $w(T_v) > u$  do
10:         $S'(T_v, i) = \phi$ ;
11:        cut off the child with heaviest weight;
12:         $i = i + 1$ ;
13:      end while
14:      while  $w(T_v) \geq w(v)$  do
15:         $S'(T_v, i) = w(T_v)$ ;
16:        cut off the child with heaviest weight;
17:         $i = i + 1$ ;
18:      end while
19:       $S'(T_v, k) = \phi$ , for  $i \leq k \leq p$ ;
20:    end if
21:    if  $\text{height}(T_v) > 2$  then
22:      for  $i = 1$  to  $\text{childcount}(v)$  do
23:        for  $k = 1$  to  $p$  do
24:          compute  $S_1(T_v^i, k), S_2(T_v^i, k)$ ;
25:           $S'(T_v^i, k) = CP_i(S_1(T_v^{i-1}, k) \cup S_2(T_v^{i-1}, k))$ ;
26:        end for
27:      end for
28:    end if
29:  end for
30:   $p_{min}[j] = \min\{1 \leq k \leq p \mid S'(T, k) \text{ contains a fitting element}\}$ ;
31: end for
32:  $p_{MIN} = \sum p_{min}[j]$ ;

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$k > m+1$. Now we compute $S'(T_v, k)$, $k = 1, \dots, m$. To get a *near* k -[l, u]partition, we must cut off $k-1$ leaf nodes from T_v , forming a *pivot tree* having $m-k+2$ nodes. Clearly, this *pivot tree* has no less than 2 nodes: v and some leaf nodes. Since any leaf node is *fitting*, the weight of *pivot tree* must be larger than l . Therefore, if $S(T_v, k)$ has any element, it must be *fitting*. Obviously, the *leanest* k -[l, u]partition is formed by cutting off the heaviest $k-1$ leaf nodes, i.e., minimizing $\sum_{v_i \in V - HV(k-1)} w(v_i)$, where $V(k-1)$ denotes the set of any $k-1$ leaf nodes. If $\sum_{v_i \in V - HV(k-1)} w(v_i) > u$, then $\sum_{v_i \in V - V(k-1)} w(v_i) \geq \sum_{v_i \in V - HV(k-1)} w(v_i) > u$, $S'(T_v, k) = S(T_v, k) = \phi$. If $\sum_{v_i \in V - HV(k-1)} w(v_i) \leq u$, then $S'(T_v, k) = \{\min\{w \mid w \in S(T_v, k), w \geq l\}\} = \{\sum_{v_i \in V - HV(k-1)} w(v_i)\}$. We thus have the proof. $\square$

Based on *lemma 2*, we use a simply method to compute $S'(T_v, k)$ (line 7 to 19). We successively cut off i heaviest children of v , until $w(T_v) \leq u$. At this phase, clearly, $S'(T_v, k) = \phi$, $k = 1, \dots, i$; $S'(T_v, i+1) = w(T_v)$. Later, we continue cutting off the heaviest child of v and computing $S'(T_v, i+2), S'(T_v, i+3), \dots$, one by one. The process stops till we arrive at the last child. At last $S'(T_v, m+1) = w(v)$.

Obviously, $S'(T_v, m+2) = \dots = S'(T_v, p) = \phi$.

- For any node v of level higher than 2 (line 21 to 27): We first present a new definition and a new property.

Definition 5. Let S_a and S_b be two sets of integers. We denote by $S_a \oplus S_b$ a set of integers defined as follows:

$$S_a \oplus S_b = \{a + b \mid a \in S_a, b \in S_b, a + b \leq u\}.$$

Property 5. Let S_a, S_b, S'_a and S'_b be four sets of integers satisfying $S'_a = CP_I(S_a)$ and $S'_b = CP_I(S_b)$. Then $CP_I(S_a \oplus S_b) = CP_I(S'_a \oplus S'_b)$, $CP_I(S_a \cup S_b) = CP_I(S'_a \cup S'_b)$.

Based on *definition 5* and *property 5*, we use the following dynamic programming approach to compute $\{S'(T_v, k) \mid k = 1, \dots, p\}$. Firstly, we compute $\{S'(T_v^0, k) \mid k = 1, \dots, p\}$:

$$S'(T_v^0, 1) = w(v); S'(T_v^0, k) = \phi \text{ for each } k, 2 \leq k \leq p.$$

Then, we compute $\{S'(T_v^i, k) \mid k = 1, \dots, p\}$, $i = 1, \dots, m$, by merging a $S'(T_{v_i}, k_2)$ with $S'(T_v^{i-1}, k_1)$, where $k = k_1 + k_2 - 1$ or $k_1 + k_2$ for different two cases:

case 1: Pivot tree of $\tilde{P}_{T_{v_i}}^{k_2}$ merging with pivot tree of $\tilde{P}_{T_v^{i-1}}^{k_1}$, $k_1 + k_2 = k + 1$.

$$S_1(T_v^i, k) = \bigcup_{k_1=1}^k S'(T_v^{i-1}, k_1) \oplus S'(T_{v_i}, k + 1 - k_1).$$

case 2: Pivot tree of $\tilde{P}_{T_{v_i}}^{k_2}$ staying alone as an independent subtree, $k_1 + k_2 = k$.

$$S_2(T_v^i, k) = \bigcup S'(T_v^{i-1}, k_1),$$

where the union is taken over all integers k_1 , $1 \leq k_1 \leq k - 1$, such that $S'(T_{v_i}, k - k_1)$ has a *fitting* element.

From the two sets $S_1(T_v^i, k)$ and $S_2(T_v^i, k)$ above, one can compute $S'(T_v^i, k)$ as follows:

$$S'(T_v^i, k) = CP_I(S_1(T_v^i, k) \cup S_2(T_v^i, k)).$$

In the dynamic programming formulation above, the set $S'(T_v^i, k)$ is computed bottom-up and from left-to-right. $\{S'(T_{v_i}, k) \mid k = 1, \dots, p\}$ and $\{S'(T_v^{i-1}, k) \mid k = 1, \dots, p\}$ have already been treated when we compute $\{S'(T_v^i, k) \mid k = 1, \dots, p\}$. So $\{S'(T_v, k) \mid k = 1, \dots, p\}$ can be derived level by level, until we arrive at root node r , where we can compute $\{S'(T, k) \mid k = 1, \dots, p\}$. Then, one can compute the minimum number p_{min} for $P_T^{p_{min}}$, as follows.

$p_{min} = \min\{1 \leq k \leq p \mid S'(T, k) \text{ contains a fitting element}\}.$

In this basic algorithm, a simple optimization method is adopted to compute $A \oplus B$ for two sets A and B . We start from the first element of A (denoted by $A[1]$) and compute the sum of $A[1]$ and $B[i]$ one by one, $1 \leq i \leq |B|$. Once $A[1] + B[i] \geq l$, we stop the computing of $A[1] + B[i + 1]$, and come to $A[2]$ for new iteration, until we finish the computing at the last element of A .

It is easy to specify the time complexity of this algorithm: Any $S'(T_v^i, k)$ has at most one element larger than l , hence $|S'(T_v^i, k)| \leq l + 1 = O(l)$. The $\oplus$ calculating of any two integer sets can be executed in time $O(l^2)$. Then, one can compute the set $S_1(T_v^i, k)$ and the set $S_2(T_v^i, k)$ in time $O(pl^2)$ and $O(pl)$ respectively. Therefore the set $S'(T_v^i, k)$ is computed in time $O(pl^2)$. Last, one can recursively compute the p sets $\{S'(T_v, k) \mid k = 1, \dots, p\}$ for n nodes of T in time $O(p^2 l^2 n)$, since there are at most $2n - 1$ pairs of a node v and

an integer i in T . Thus the basic algorithm has a worst-case runtime of $O(p^2 l^2 n)$.

Obviously, Algorithm 2 can be slightly modified to actually obtain the $P_T^{p_{min}}$ which tells the specific positions of partitioning. As we know, each $w \in S'(T_v^i, k)$ corresponds to a $\tilde{P}_{T_v^i}^k$ of T_v^i . If $w \in S^1(T_v^i, k)$, the edge (v, v_i) is not deleted for $\tilde{P}_{T_v^i}^k$, then we attach a tuple $(1, w^1, k')$ to the w , where $w = w^1 + w^2$, $w^1 \in S_1(T_v^{i-1}, k')$ and $w^2 \in S_2(T_v, k - k' + 1)$. If $w \in S_2(T_v^i, k)$, then the edge (v, v_i) is deleted for $\tilde{P}_{T_v^i}^k$; we attach a tuple $(2, k')$ to the w , where $w \in S'(T_v^{i-1}, k')$. Using these tuples, one can use a backtracking process to identify those nodes whose parent edges are deleted in the *minimum* $[l, u]$ partition of T . By this way, we actually find the *minimum* $[l, u]$ partition.

4.3 Algorithm Optimization

From Sect.4.2, we know the time complexity of Algorithm 2 is affected by the size of dynamic programming set $S'(T_v^i, k)$. An opportunity for algorithm optimization is to simplify the representation and computation of $S'(T_v^i, k)$. An approach used in [3] is adopted here. The main idea is to transform $S'(T_v^i, k)$ to a set of intervals (i.e., a pair of integers), where each interval represents a “maximal d -consecutive subset” derived from $S'(T_v^i, k)$. For the soundness of this paper, we present the optimization approach briefly, omitting the proof of algorithm optimization. One can refer to [3] for relative theoretical details. Now we first present some definitions.

An ordered set of integers is called *d-consecutive* if the value difference of any two consecutive elements is at most d , where $d = u - l$. For a non-empty set A of integers and a non-empty subset $A' \subseteq A$, A' is said to be a *maximal d-consecutive* subset of A if A' is *d-consecutive* and $A' \subsetneq A'' \subseteq A$ doesn't hold for any other *d-consecutive* subset A'' of A .

Now we define a set of intervals $I(A)$, based on a set of integers A :

$$I(A) = \{[a, a'] \mid [a, a'] \text{ is the interval of a maximal } d\text{-consecutive subset of } A\}$$

For two sets of intervals $I(A)$ and $I(B)$, we define $I(A) \oplus I(B)$ as a set of intervals as follows:

$$I(A) \oplus I(B) = \{[a + b, a' + b'] \mid [a, a'] \in I(A), [b, b'] \in I(B), a + b \leq u\}.$$

Without loss of generality we assume four integers a, a' and b, b' , with $a \leq b, a' > a, b' > b$. The interval $[a, a']$ is said to be *intersected* with interval $[b, b']$ if $b - a' \leq 0$. The interval $[a, a']$ is said to be *d-interfering* with $[b, b']$ if $b - a' \leq d$. We define a *merge* operation $\oplus$ for two *d-interfering* intervals as $[a, a'] \oplus [b, b'] = [a, \max\{a', b'\}]$.

For a set of intervals I , if we repeatedly *merge d-interfering* intervals of I until no any two intervals are *d-interfering*, then we can get a new set with no intervals *d-interfering*. We denote the operation to set I by $M()$ and the new set by $M(I)$.

From the above definitions, one can have following *properties*:

Property 6. Let $\bar{A}$ and $\bar{B}$ be two d -consecutive integer sets with all elements no more than u , and let $a_{\min}$ ($b_{\min}$), $a_{\max}$ ($b_{\max}$) be the minimum and maximum elements of $\bar{A}$ ($\bar{B}$) respectively. If the interval $[a_{\min} + b_{\min}, a_{\max} + b_{\max}]$ is intersected with $[l, u]$, there must exist an element $a \in \bar{A}$ and an element $b \in \bar{B}$ such that $l \leq a + b \leq u$, and vice versa.

Property 7. Let I be a set of intervals with some intervals d -interfering with each other. There must exist an interval of $M(I)$ intersected with $[l, u]$, if an interval of I intersects with $[l, u]$.

With the above definitions and properties, the optimized algorithm computes the set $I(S'(T_v, k))$ in place of $S'(T_v, k)$. With the same approach of dynamic programming as Algorithm 2, we recursively compute interval sets $\{I(S'(T_v^i, k)) \mid k = 1, \dots, p\}$ for each node v of T from the leaves to the root, and hence computes $\{I(S'(T, k)) \mid k = 1, \dots, p\}$. We denote $I(S'(T_v^i, k))$ by $I(T_v^i, k)$, one can have the following lemma:

Lemma 3. Tree T has a p - $[l, u]$ partition if and only if $I(T, p)$ contains an interval $[a, a']$ such that $[a, a']$ intersects with $[l, u]$.

Hence, we can recursively compute interval sets $\{I(T_v^i, k) \mid k = 1, \dots, p\}$ as follows.

- For any node v of level 1:
 $I(v, 1) = \{w(v), w(v)\}; I(v, k) = \emptyset, 2 \leq k \leq p.$
- For any node v of level 2:

Each $S'(T_v, k)$ has at most one element, therefore, for $1 \leq k \leq p$,

$$I(T_v, k) = \begin{cases} \{[w, w] \mid w \in S'(T_v, k)\} & S'(T_v, k) \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

- For any node v of level higher than 2:

Firstly, $I(T_v^0, 1) = \{[w(v), w(v)]\}; I(T_v^0, k) = \emptyset, 2 \leq k \leq p.$

Then, we compute the p sets $I(T_v^i, k)$ incrementally, for $i = 1$ to m , distinguishing two cases:

case 1: Two *pivot trees* merging together

$$I_1(T_v^i, k) = \bigcup_{k_1=1}^k I(T_v^{i-1}, k_1) \oplus I(T_{v_i}, k - k_1 + 1).$$

case 2: *Pivot tree* of T_{v_i} staying as independent subtree

$I_2(T_v^i, k) = \bigcup I(T_v^{i-1}, k_1)$, where the union is taken over all $k_1, 1 \leq k_1 \leq k - 1$, such that $I(T_{v_i}, k - k_1)$ has an interval intersected with $[l, u]$.

By uniting the above two sets and doing *merge* operation, we have:

$$I(T_v^i, k) = M(I_1(T_v^i, k) \cup I_2(T_v^i, k)).$$

By the above dynamic programming, whether T has a p - $[l, u]$ partition can be checked from $I(T, p)$. By adopting a method like Sect. 4.2 (see also [3] for the detail), one can actually obtain $P_T^{P_{\min}}$.

We now turn to the runtime complexity of the optimized algorithm. Any $I(T_v^i, k)$ has at most $\lfloor \frac{u}{d+2} \rfloor + 1$ elements, hence $|I(T_v^i, k)| = O(\lfloor \frac{u}{d+2} \rfloor)$. One can compute the $\oplus$ of any two interval sets in time $O(\lfloor \frac{u}{d+2} \rfloor^2)$. Then, $I_1(T_v^i, k)$ and $I_2(T_v^i, k)$ can be computed in time $O(p \lfloor \frac{u}{d+2} \rfloor^2)$ and $O(p \lfloor \frac{u}{d+2} \rfloor)$ respectively. Therefore one can compute the

set $I(T_v^i, k)$ in time $O(p \lfloor \frac{u}{d+2} \rfloor^2)$. Lastly, we can recursively compute the p sets $\{I(T_v, k) \mid k = 1, \dots, p\}$ for n nodes of T in time $O(p^2 \lfloor \frac{u}{d+2} \rfloor^2 n)$. Thus the basic algorithm has a worst-case runtime of $O(p^2 \lfloor \frac{u}{d+2} \rfloor^2 n)$. The optimized algorithm can be slightly modified to actually obtain a $P_T^{P_{\min}}$, analogously as the method in Sect. 4.2.

5. p - $[l, u]$ Partition Problem for Arbitrary Tree

We only need to modify our aforementioned algorithms slightly, to solve the p - $[l, u]$ partition problem for arbitrary tree. Similarly, we first translate tree T into *lf-trees*. We assume T is translated into N *lf-trees* after pre-processing. For each *lf-tree* denoted by $\dot{T}(i)$, we compute and obtain $\{I(\dot{T}(i), k) \mid k = 1, \dots, p\}$ by Algorithm 2. Then the p - $[l, u]$ partition problem can be presented as follows: given $\{I(\dot{T}(i), k) \mid k = 1, \dots, p\}$ for each *lf-tree* $\dot{T}(i)$, find an integer k_i and corresponding k_i - $[l, u]$ partition $P_{\dot{T}(i)}^{k_i}$ for each $\dot{T}(i)$,

such that $\sum_{i=1}^N k_i = p$. We use a simple dynamic programming scheme to get the solution, as follows.

For any $\dot{T}(i)$, we define a set Y_i of size p , whose's element is computed as follows:

$$Y_i[k] = \begin{cases} 1 & I(\dot{T}(i), k) \text{ contains an interval intersected} \\ & \text{with } [l, u], \\ 0 & \text{otherwise.} \end{cases}$$

We denote a set of *lf-trees* $\{\dot{T}(1), \dots, \dot{T}(i)\}$ by $\dot{T}(1 \triangleright i)$. Based on $\dot{T}(1 \triangleright i)$, we define a set $Y_{1 \triangleright i}$ of size p , whose's element is computed as follows:

$$Y_{1 \triangleright i}[k] = \bigcup_{k_1=1}^{k-1} (Y_{1 \triangleright i-1}[k_1] \& Y_i[k - k_1])$$

where $\&$ indicates logical AND operator and $\bigcup$ indicates logical OR operator.

We incrementally compute the set $Y_{1 \triangleright i}$ from $i = 1$ to N , until we get $Y_{1 \triangleright N}[p] = 1$, T 's p - $[l, u]$ partition is *feasible*. Then, we can obtain k_i and k_i - $[l, u]$ partition $P_{\dot{T}(i)}^{k_i}$

for each $\dot{T}(i)$ satisfying $\sum_{i=1}^N k_i = p$, through a backtracking process of iterative computing. Then $\bigcup_{i=1}^N P_{\dot{T}(i)}^{k_i}$ is our solution.

6. Experimental Results

In this section, we evaluate the performance of our algorithm. In the simulations, trees are synthetically generated, specified mainly by three parameters: the tree depth (denoted by *treeDepth*), the range of a node's degree (denoted by *degreeRange*), and the range of a node's weight (denoted by *weightRange*). The number of nodes for a generated tree is called the *size* of the tree. We compare the performance of our algorithm against the algorithm by Ito et al. [3] in executing time. These algorithms are coded in C++ programming language and executed on a Pentium IV PC with 2.6 GHz CPU.

Fig. 3 (a) Size of different trees before and after pre-processing with fixed $[l, u]$; (b) Size of trees after pre-processing with different value of l .

6.1 Impact of Pre-processing Algorithm on Tree Size

In this experiment, we compare the *size* of original trees with those after pre-processing. We randomly generate a group of trees, each of which is generated with $\text{treeDepth} = 6$, $\text{weightRange} = [1, 10]$, $\text{degreeRange} = [2, 6]$. Firstly, we choose the fixed value of $[l, u]$ for these trees, i.e., $[l, u] = [20, 80]$. Figure 3(a) shows the *size* of 10 trees before and after pre-processing. Secondly, we set a constant value of d and choose different l for these trees. Here, $d = 60$, l is varied from 20 to 180. Figure 3(b) shows the *size* of these trees after pre-processing with different $[l, u]$. It is clear that pre-processing algorithm always decreases the *size* of a tree, as more significant as l is larger compared with the average weight of each node. Note that we also generated other groups of trees with different parameters for the above experiments and got the same comparison results as Fig. 3.

6.2 Executing Time of Finding *Minimum* $[l, u]$ Partition

Here, we compare the performance of our algorithm against algorithm by Ito et al. in executing time. Firstly, we use the same trees generated in Sect. 6.1, and $[l, u]$ is set as $[120, 180]$ for these trees. We run both algorithms to find *minimum* $[l, u]$ partition of these trees. It's clear that our algorithm works much faster than the algorithm by Ito et al. Secondly, We randomly generate two new trees $T(A)$ and $T(B)$ with the same parameters as Sect. 6.1. The *size* of the two trees are 1265 and 1810 respectively. We run algorithms of finding *minimum* $[l, u]$ partition for $T(A)$ and $T(B)$. Here d is set as a constant value ($d = 60$) and l is varied from 80 to 180. Figure 4(b) shows the executing time of both algorithms with different value of l . We can see that the running time for both algorithms decreases along with the increasing of the value of l . Obviously, our algorithm works much faster than algorithm by Ito et al. in any case. Note that we also compare our algorithm against algorithm by Ito et al. for the executing time of finding *p*- $[l, u]$ partition, we have the same comparison results.

Fig. 4 (a) Executing time of finding *minimum* $[l, u]$ partition for trees with fixed $[l, u]$; (b) Executing time of finding *minimum* $[l, u]$ partition for trees with different value of l .

7. Conclusions

We propose new algorithms of solving *minimum* $[l, u]$ partition and *p*- $[l, u]$ partition problems of a tree. To reduce the executing time of the algorithms, we adopt a pre-processing approach to cut down the scale of the original tree, and use a few pruning methods to speed up the computing process. Experiments show that our algorithm work efficiently and outperform the algorithm proposed by Ito et al. greatly.

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