Convex Grid Drawings of Plane Graphs with Pentagonal Contours

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SUMMARY In a convex drawing of a plane graph, all edges are drawn as straight-line segments without any edge-intersection and all facial cycles are drawn as convex polygons. In a convex grid drawing, all vertices are put on grid points. A plane graph G has a convex drawing if and only if G is internally triconnected, and an internally triconnected plane graph G has a convex grid drawing on an $(n-1) \times (n-1)$ grid if either G is triconnected or the triconnected component decomposition tree $T(G)$ of G has two or three leaves, where n is the number of vertices in G . An internally triconnected plane graph G has a convex grid drawing on a $2n \times 2n$ grid if $T(G)$ has exactly four leaves. In this paper, we show that an internally triconnected plane graph G has a convex grid drawing on a $6n \times n^2$ grid if $T(G)$ has exactly five leaves. We also present an algorithm to find such a drawing in linear time. This is the first algorithm that finds a convex grid drawing of such a plane graph G in a grid of polynomial size.

key words: algorithm, convex grid drawing, graph drawing, plane graph, triconnected

1. Introduction

Recently automatic aesthetic drawing of graphs has created intense interest due to their broad applications, and as a consequence, a number of drawing methods have come out [12]. The most typical drawing of a plane graph is a *straight line drawing*, in which all edges are drawn as straight line segments without any edge-intersection. A straight line drawing is called a *convex drawing* if every facial cycle is drawn as a convex polygon. One can find a convex drawing of a plane graph G in linear time if G has one [3], [4], [12].

A straight line drawing of a plane graph is called a *grid drawing* if all vertices are put on grid points of integer coordinates. This paper deals with a *convex grid drawing* of a plane graph. Throughout the paper we assume for simplicity that every vertex of a plane graph G has degree three or more. Then G has a convex drawing if and only if G is “internally triconnected,” that is, G can be extended to a triconnected graph by adding a vertex in the outer face and joining it to all outer vertices [10], [13]. One may thus assume without loss of generality that G is internally triconnected. If either G is triconnected or the “triconnected component decomposition tree” $T(G)$ of G has two or three leaves, then G has a convex grid drawing on an $(n-1) \times (n-1)$ grid and such a drawing can be found in linear time, where n is the number of vertices of G [1], [2], [9]. Furthermore, an internally triconnected plane graph G has a convex grid

Fig. 1 (a) Plane graph G , (b) subgraphs G_c , G_l and G_r , (c) convex grid drawings D_l of G_l , D_r of G_r and D_c of G_c , and (d) convex grid drawing D of G .

drawing on a $2n \times 2n$ grid if $T(G)$ has exactly four leaves [8], [11], [15]. However, it has not been known whether G has a convex grid drawing of polynomial size if $T(G)$ has five or more leaves. Figure 1 (a) depicts an internally triconnected plane graph G , Fig. 2 (c) the triconnected component decomposition tree $T(G)$ of G , which has five leaves l_1, l_2, l_3, l_4 and l_5 .

In this paper, we show that an internally triconnected plane graph G has a convex grid drawing on a $6n \times n^2$ grid if $T(G)$ has exactly five leaves, and present an algorithm to find such a drawing in linear time. The algorithm is outlined as follows: we first divide a plane graph G into a left subgraph G_l , a right subgraph G_r and a center subgraph G_c as illustrated in Fig. 1 (b) for the graph in Fig. 1 (a); we then construct convex grid drawings with triangular contours of G_l and G_r and construct a convex grid drawing with heptagonal contour of G_c by a so-called shift method as illustrated in Fig. 1 (c); we finally combine these three drawings to a convex grid drawing with pentagonal contour of G as illustrated in Fig. 1 (d). This is the first algorithm that finds a convex grid drawing of such a plane graph G in a grid of polynomial size.

The remainder of the paper is organized as follows. In Sect. 2 we give some definitions and known lemmas. In

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Sect. 3 we explain an algorithm for G_c . In Sect. 4 we explain an algorithm for G_l and G_r . In Sect. 5 we present our convex grid drawing algorithm. Finally we conclude in Sect. 6.

2. Preliminaries

In this section, we give some definitions and known lemmas.

A $W \times H$ integer grid consists of $W + 1$ regular vertical grid lines and $H + 1$ regular horizontal grid lines, and has a rectangular contour. We call W and H the *width* and *height* of the integer grid, respectively. We denote by $W(D)$ the width of a minimum integer grid enclosing a grid drawing D of a graph, and by $H(D)$ the height of D .

We denote by $G = (V, E)$ an undirected connected simple graph with vertex set V and edge set E . We often denote the set of vertices of G by $V(G)$ and the set of edges by $E(G)$. Throughout the paper we denote by n the number of vertices in G . An edge joining vertices u and v is denoted by (u, v) . The *degree* of a vertex v in G is the number of neighbors of v in G .

A plane graph G divides the plane into connected regions, called *faces*. The infinite face is called an *outer face*, and the others are called *inner faces*. The boundary of a face is called a *facial cycle*. A cycle is represented by a clockwise sequence of the vertices in the cycle. We denote by $F_o(G)$ the outer facial cycle of G . A vertex on $F_o(G)$ is called an *outer vertex*, while a vertex not on $F_o(G)$ is called an *inner vertex*. In a convex drawing of a plane graph G , all facial cycles must be drawn as convex polygons. The convex polygonal drawing of $F_o(G)$ is called the *outer polygon*. We call a vertex of a polygon an *apex* in order to avoid the confusion with a vertex of a graph.

We call a vertex v of a connected graph G a *cut vertex* if its removal from G results in a disconnected graph, that is, $G - v$ is not connected. A connected graph G is *biconnected* if G has no cut vertex. We call a pair $\{u, v\}$ of vertices in a biconnected graph G a *separation pair* if its removal from G results in a disconnected graph, that is, $G - \{u, v\}$ is not connected. A biconnected graph G is *triconnected* if G has no separation pair. A biconnected plane graph G is *internally triconnected* if, for any separation pair $\{u, v\}$ of G , both u and v are outer vertices and each connected component of $G - \{u, v\}$ contains an outer vertex.

Let $G = (V, E)$ be a biconnected graph, and let $\{u, v\}$ be a separation pair of G . Then, G has two subgraphs $G'_1 = (V_1, E'_1)$ and $G'_2 = (V_2, E'_2)$ satisfying the following two conditions (a) and (b).

- (a) $V = V_1 \cup V_2$, $V_1 \cap V_2 = \{u, v\}$; and
- (b) $E = E'_1 \cup E'_2$, $E'_1 \cap E'_2 = \emptyset$, $|E'_1| \geq 2$, $|E'_2| \geq 2$.

For a separation pair $\{u, v\}$ of G , $G_1 = (V_1, E'_1 + (u, v))$ and $G_2 = (V_2, E'_2 + (u, v))$ are called *split graphs* of G with respect to $\{u, v\}$. The new edges (u, v) added to G_1 and G_2 are called the *virtual edges*. Even if G has no multiple edges, G_1 and G_2 may have. Dividing a graph G into two split graphs G_1 and G_2 is called *splitting*. Reassembling the two split graphs G_1 and G_2 into G is called *merging*. Merging

Fig. 2 (a) Split components of the graph G in Fig. 1 (a), (b) triconnected components of G , and (c) a decomposition tree $T(G)$.

is the inverse of splitting. Suppose that a graph G is split, the split graphs are split, and so on, until no more splits are possible, as illustrated in Fig. 2 (a) for the graph in Fig. 1 (a) where virtual edges are drawn by dotted lines. The graphs constructed in this way are called the *split components* of G . The graph in Fig. 1 (a) has ten split components as illustrated in Fig. 2 (a). The split components are of three types: a triconnected graph; a triple bond (i.e. a set of three multiple edges); and a triangle (i.e. a cycle of length three). The *triconnected components* of G are obtained from the split components of G by merging triple bonds into a bond and triangles into a ring, as far as possible, where a *bond* is a set of multiple edges and a *ring* is a cycle. Thus the triconnected components of G are of three types:

- (a) a triconnected graph;
- (b) a bond; and
- (c) a ring (which is not a triconnected graph, of course).

The split components of G are not necessarily unique, but the triconnected components of G are unique [7], [14]. Two triangles in Fig. 2 (a) are merged into a single ring, and hence the graph in Fig. 1 (a) has nine triconnected components as illustrated in Fig. 2 (b).

For a separation pair $\{u, v\}$ of G , two triconnected components H_i and H_j ($i \neq j$) are called the *triconnected components of G with respect to $\{u, v\}$* if it is possible to merge H_i with H_j at $\{u, v\}$. Let $T(G)$ be a tree such that each node corresponds to a triconnected component H_i of G and there is an edge (H_i, H_j) , $i \neq j$, in $T(G)$ if and only if H_i and H_j are triconnected components with respect to the same separation pair, as illustrated in Fig. 2 (c). We call $T(G)$ a *triconnected component decomposition tree* or simply a *decomposition tree* of G [7], [14]. We denote by $\ell(G)$ the number of leaves of $T(G)$. Then $\ell(G) = 5$ for the graph G in Fig. 1 (a). (See Fig. 2 (c).) If G is triconnected, then $T(G)$ consists of a single isolated node and hence $\ell(G) = 1$.

The following three lemmas are known.

Lemma 1: [7] A decomposition tree $T(G)$ of a graph G can be found in linear time.

Lemma 2: [10] Let G be a biconnected plane graph in which every vertex has degree three or more. Then the following three statements are equivalent to each other:

- (a) G has a convex drawing;

- (b) G is internally triconnected; and
- (c) both vertices of every separation pair are outer vertices, and a node of the decomposition tree $T(G)$ of G has degree two if it is a bond.

Lemma 3: [10] If a plane graph G has a convex drawing D , then the number of apices of the outer polygon of D is no less than $\max\{3, \ell(G)\}$, and there is a convex drawing of G whose outer polygon has exactly $\max\{3, \ell(G)\}$ apices.

Since G is an internally triconnected simple graph and every vertex of G has degree three or more, by Lemma 2 every leaf of $T(G)$ is neither a bond nor a ring but a tri-connected graph. Lemmas 2 and 3 imply that if $T(G)$ has exactly five leaves, that is, $\ell(G) = 5$ then the outer polygon of every convex drawing of G must have five or more apices. Our algorithm finds a convex grid drawing of G whose outer polygon is a pentagon and hence has exactly five apices, as illustrated in Fig. 1 (d).

In Sects. 3 and 4, we will present two algorithms to draw the center subgraph G_c , the left subgraph G_l and the right subgraph G_r . (See Fig. 1 (c).) These algorithms use the following “canonical decomposition.” Let $G = (V, E)$ be an internally triconnected plane graph, and let $V = \{v_1, v_2, \dots, v_n\}$. Let v_1, v_2 and v_n be three arbitrary outer vertices appearing counterclockwise on $F_o(G)$ in this order. We may assume that v_1 and v_2 are consecutive on $F_o(G)$; otherwise, add a virtual edge (v_1, v_2) to the original graph, and let G be the resulting graph. Let $\Pi = (U_1, U_2, \dots, U_m)$ be an ordered partition of V into nonempty subsets $U_1, U_2, \dots, U_m$, where $U_1 \cup U_2 \cup \dots \cup U_m = V$ and $U_i \cap U_j = \emptyset$ for any indices i and j , $1 \leq i < j \leq m$. We denote by G_k , $1 \leq k \leq m$, the subgraph of G induced by $U_1 \cup U_2 \cup \dots \cup U_k$, and denote by $\overline{G}_k$, $0 \leq k \leq m-1$, the subgraph of G induced by $U_{k+1} \cup U_{k+2} \cup \dots \cup U_m$. Clearly, $G_k = G - U_{k+1} \cup U_{k+2} \cup \dots \cup U_m$ and $G = G_m = \overline{G}_0$. We say that Π is a *canonical decomposition* of G (with respect to vertices v_1, v_2 and v_n) if the following three conditions (cd1)–(cd3) hold:

- (cd1) $U_m = \{v_n\}$, and U_1 consists of all the vertices on the inner facial cycle containing edge (v_1, v_2) .
- (cd2) For each index k , $1 \leq k \leq m$, G_k is internally triconnected.
- (cd3) For each index k , $2 \leq k \leq m$, all the vertices in U_k are outer vertices of G_k , and

- (a) if $|U_k| = 1$, then the vertex in U_k has two or more neighbors in G_{k-1} and has one or more neighbors in $\overline{G}_k$ when $k < m$, as illustrated in Fig. 3 (a); and
- (b) if $|U_k| \geq 2$, then each vertex in U_k has exactly two neighbors in G_k , and has one or more neighbors in $\overline{G}_k$, as illustrated in Fig. 3 (b).

Although the definition of a canonical decomposition above is slightly different from the one given in [2], they are effectively equivalent to each other. A canonical decomposition $\Pi = (U_1, U_2, \dots, U_{11})$ with respect to vertices v_1, v_2 and v_n of the graph in Fig. 4 (a) is illustrated in Fig. 4 (b).

Fig. 3 (a) Graphs G_k for which $|U_k| = 1$ and (b) $|U_k| \geq 2$.

Fig. 4 (a) An internally triconnected plane graph G , (b) a canonical decomposition Π of G , and (c) a pentagonal drawing D of G .

By the condition (cd3), one may assume that all the vertices in U_k , $1 \leq k \leq m$, consecutively appear clockwise on $F_o(G_k)$. We number all vertices of G by $1, 2, \dots, n$ so that they appear in $U_1, U_2, \dots, U_l$ in this order. We call each vertex in G by the number i , $1 \leq i \leq n$. Thus one can define an order $<$ on the vertices in G . For a vertex u , $1 \leq u \leq n-1$, we denote by $w^*(u)$ the largest neighbor of u .

The following lemma is known.

Lemma 4: [9] Assume that G is an internally triconnected plane graph and $\ell(G) \leq 3$. Then one can find a canonical decomposition Π of G in linear time if v_1, v_2 and v_n are chosen as follows.

Case 1: $\ell(G) = 3$.

In this case, from each of the three triconnected components corresponding to leaves of $T(G)$, we choose an arbitrary outer vertex of G which is not a vertex of the separation pair of the component.

Case 2: $\ell(G) = 2$.

In this case, we choose two vertices from the two leaves of $T(G)$, similarly to Case 1 above. We choose an arbitrary outer vertex of G other than them as the third one.

Case 3: $\ell(G) = 1$.

In this case, G is triconnected. We choose three arbitrary outer vertices of G .

3. Pentagonal Drawing

Let G be a plane graph having a canonical decomposition $\Pi = (U_1, U_2, \dots, U_m)$ with respect to vertices v_1, v_2 and v_n , as illustrated in Fig. 4 (b). Miura *et al.* [11] give a linear-time algorithm, called the *pentagonal drawing algorithm*, to find a convex grid drawing of G with a pentagonal outer polygon, as illustrated in Fig. 4 (c). The algorithm is based on the so-called shift methods given by Chrobak and Kant [2] and de Fraysseix *et al.* [5]. Unfortunately, this algorithm

cannot find a convex grid drawing of an internally triconnected plane graph G whose decomposition tree has four or more leaves. Thus we will modify the algorithm to find a convex grid drawing with heptagonal contour of the center subgraph G_c of G (whose decomposition tree has exactly three leaves), as illustrated in Fig. 1 (c), in Sect. 5.4.

We then outline the pentagonal drawing algorithm. Let v_l be an arbitrary vertex on the path going from v_1 to v_n clockwise on $F_0(G)$, and let $v_r(\neq v_l)$ be an arbitrary vertex on the path going from v_2 to v_n counterclockwise on $F_0(G)$, as illustrated in Fig. 4 (a). Let V_l be the set of all vertices on the path going from v_1 to v_l clockwise on $F_0(G)$, and let V_r be the set of all vertices on the path going from v_2 to v_r counterclockwise on $F_0(G)$. Our algorithm finds a convex grid drawing of G whose outer polygon is a pentagon with apices v_1, v_2, v_r, v_n and v_l , as illustrated in Fig. 4 (c).

More precisely, the pentagonal drawing algorithm finds an “inner convex” grid drawing D_k for each k , $1 \leq k \leq m$, in which all inner facial cycles are convex polygons. Let $F_0(G_{k-1}) = w_1, w_2, \dots, w_t, w_1 = v_1, w_t = v_2$. Let w_f be the vertex with the maximum index f among all the vertices w_i , $1 \leq i \leq t$, on $F_0(G_{k-1})$ that are contained in V_l . Let w_g be the vertex with the minimum index g among all the vertices w_i , $1 \leq i \leq t$, on $F_0(G_{k-1})$ that are contained in V_r . Of course, $1 \leq f < g \leq t$. We denote by $\angle w_i$ the interior angle of apex w_i of the outer polygon of D_{k-1} . We call w_i a *convex apex* of the polygon if $\angle w_i < 180^\circ$. We denote the current position of a vertex v by $P(v)$; $P(v)$ is expressed by its x - and y -coordinates of $(x(v), y(v))$. The drawing D_k of G_k satisfies the following six conditions (sh1)–(sh6).

- (sh1) $P(w_1) = (0, 0)$ and $P(w_t) = (2|V(G_{k-1})| - 2, 0)$.
- (sh2) $x(w_1) = x(w_2) = \dots = x(w_f), x(w_f) < x(w_{f+1}) < \dots < x(w_g), x(w_g) = x(w_{g+1}) = \dots = x(w_t)$, where $x(w_i)$ is the x -coordinate of w_i .
- (sh3) Every edge $(w_i, w_{i+1}), f \leq i \leq g-1$, has slope $-1, 0$, or 1 .
- (sh4) The Manhattan distance between any two grid points w_i and $w_j, f \leq i < j \leq g$, is an even number.
- (sh5) Every inner face of G_{k-1} is drawn as a convex polygon.
- (sh6) Vertex $w_i, f+1 \leq i \leq g-1$, has one or more neighbors in G_{k-1} if w_i is a convex apex.

The following lemma is known.

Lemma 5: [11] For a plane graph G having a canonical decomposition $\Pi = (U_1, U_2, \dots, U_m)$ with respect to v_1, v_2 and v_n , the pentagonal drawing algorithm yields a convex grid drawing of G on a $W \times H$ grid with $W = 2n - 2$ and $H \leq n^2 - n - 2$ in linear time.

4. Triangular Drawing

Let G be a plane graph having a canonical decomposition $\Pi = (U_1, U_2, \dots, U_m)$ with respect to vertices v_1, v_2 and v_n , as illustrated in Fig. 5 (b). In this section, we present

Fig. 5 (a) An internally triconnected plane graph G , (b) a canonical decomposition Π of G , and (c) a triangular drawing D of G .

Fig. 6 Drawings D_1 of G_1 (a), (b) for Case 1 and (c) for Case 2.

a linear-time algorithm, called a *triangular drawing algorithm*, to find a convex grid drawing of G with a triangular outer polygon. This algorithm is based on the algorithm in [2], and will be used by our convex grid drawing algorithm to draw the left subgraph G_l and the right subgraph G_r of G in Sects. 5.2 and 5.3, respectively.

Our algorithm finds a convex grid drawing of G whose outer polygon is a triangle with apices v_1, v_n and v_2 ; the side v_1v_2 is horizontal, the side v_1v_n has slope -1 , and the side v_2v_n has slope $-1/2$, respectively, as illustrated in Fig. 5 (c). Let V_l be the set of all vertices on the path going from v_1 to v_n clockwise on $F_0(G)$, as illustrated in Fig. 5 (a).

We first obtain a drawing D_1 of the subgraph G_1 of G induced by all vertices in U_1 . Let $F_0(G_1) = w_1, w_2, \dots, w_t, w_1 = v_1$, and $w_t = v_2$. We draw G_1 as illustrated in Fig. 6, depending on whether (v_1, v_2) is a real edge or not, and $w_2 \in V_l$ or not.

Initialize:

Case 1: v_1 and v_2 are adjacent in the original graph G , that is, (v_1, v_2) is a real edge (see Figs. 6 (a) and (b)).

Set $P(w_1) = (0, 0)$;

Case 1(a): $w_2 \in V_l$ (see Fig. 6 (a)).

Set $P(w_i) = (i - 3, 1)$ for each $i, 2 \leq i \leq t - 1$;

Set $P(w_t) = (t - 2, 0)$;

Case 1(b): $w_2 \notin V_l$ (see Fig. 6 (b)).

Set $P(w_i) = (i - 1, 1)$ for each $i, 2 \leq i \leq t - 1$;

Set $P(w_t) = (t, 0)$;

Case 2: Otherwise, that is, (v_1, v_2) is a virtual edge (see Fig. 6 (c)).

Set $P(w_i) = (i - 1, 0)$ for each $i, 1 \leq i \leq t$.

We then extend a drawing D_{k-1} of G_{k-1} to a drawing D_k of G_k for each index $k, 2 \leq k \leq m$. For each $k, 2 \leq k \leq m$, let $F_0(G_{k-1}) = w_1, w_2, \dots, w_t$, where $w_1 = v_1, w_t = v_2$, and $w_1, w_2, \dots, w_t$ appear clockwise on $F_0(G_{k-1})$ in this order, as illustrated in Fig. 7. Let $U_k = \{u_1, u_2, \dots, u_l\}$. Let w_p be the leftmost neighbor of u_1 , and let w_q be the rightmost neighbor of u_l . Let w_f be the vertex with the maximum index f among all the vertices $w_i, 1 \leq i \leq t$, on $F_0(G_{k-1})$ that are contained

Fig. 7 Graphs (a) G_{k-1} and (b) G_k when Case $x(w_q) - x(w_p) + \epsilon$ is an odd number, graphs (c) G_{k-1} and (d) G_k when Case $x(w_q) - x(w_p) + \epsilon$ is an even number.

in V_l .

Assume that a drawing D_{k-1} of G_{k-1} satisfies the following five conditions (sh'1)–(sh'5).

- (sh'1) $P(w_1) = (0, 0)$, $x(w_t) \leq 2|V(G_{k-1})|$ and $y(w_t) = 0$.
- (sh'2) $x(w_1) > x(w_2) > \dots > x(w_f)$, $x(w_f) \leq x(w_{f+1}) \leq \dots \leq x(w_t)$, where $x(w_i)$ is the x -coordinate of w_i .
- (sh'3) Every edge (w_i, w_{i+1}) , $1 \leq i \leq t-1$, has slope -1 , $-1/2$, 0 or $[1, +\infty]$.
- (sh'4) Every inner face of G_{k-1} is drawn as a convex polygon.
- (sh'5) Vertex w_i , $2 \leq i \leq t-1$, has one or more neighbors in $\bar{G}_{k-1}$ if w_i is a convex apex.

Indeed D_1 satisfies the five conditions above. We extend D_{k-1} to D_k so that D_k satisfies them, as follows.

In our algorithm, we wish to put the vertex u_1 of U_k on a grid point so that the edge (w_p, u_1) has slope either -1 (if $u_1 \in V_l$) or $[1, +\infty]$ (otherwise) and put the vertex u_l on a grid point so that the edge (u_l, w_q) has slope $-1/2$. Furthermore, if $|U_k| \geq 2$, then we wish to put the vertices $u_2, u_3, \dots, u_{l-1}$ so that, for each i , $1 \leq i \leq l-1$, the edge (u_i, u_{i+1}) has slope 0 and the distance between two vertices u_i and u_{i+1} is equal to 1 . For this purpose, before installing $U_k = \{u_1, u_2, \dots, u_l\}$ to D_{k-1} , we shift some vertices of G_{k-1} to the right as illustrated in Figs. 7 (a)–(d).

Let ϵ be 0 if $u_1 = w^*(w_p)$, and 1 otherwise. If $x(w_q) - x(w_p) + \epsilon$ is an odd number, as illustrated in Fig. 7 (a), then we shift $w_q, w_{q+1}, \dots, w_t$ of G_{k-1} and some inner vertices of G_k to the right by distance $|U_k|$, as illustrated in Fig. 7 (b). Otherwise $(x(w_q) - x(w_p) + \epsilon)$ is an even number, as illustrated in Fig. 7 (c), we shift $w_q, w_{q+1}, \dots, w_t$ of G_{k-1} and some inner vertices of G_k to the right by distance $|U_k| + 1$, as illustrated in Fig. 7 (d).

Then, we install U_k to D_{k-1} as follows.

Install U_k :

Case 1: $u_1 \in V_l$.

For each i , $1 \leq i \leq l$, we set

$$x(u_i) = 2x(w_p) - x(w_q) + 2y(w_p) - 2y(w_q)$$

Fig. 8 Installing U_k to D_{k-1} .

$$+ l - 2 + i,$$

and set

$$y(u_i) = -x(w_p) + x(w_q) - y(w_p) + 2y(w_q) - l + 1,$$

as illustrated in Figs. 8 (a), (b);

Case 2: Otherwise.

For each i , $1 \leq i \leq l$, we set

$$x(u_i) = x(w_p) + i - 1 + \epsilon,$$

and set

$$y(u_i) = y(w_q) - (x(w_p) - x(w_q) + l - 1 + \epsilon) \times 1/2,$$

as illustrated in Figs. 8 (c), (d) for the case $\epsilon = 0$ and in Figs. 8 (e), (f) for the case $\epsilon = 1$.

Clearly, the drawing D_k of G_k extended from D_{k-1} satisfies conditions (sh'1), (sh'2) and (sh'3). One can prove similarly as in [2] that D_k satisfies conditions (sh'4) and (sh'5). By the condition (sh'1), we have $x(w_2) \leq 2n$ in D_m of $G_m = G$ and hence one can easily show that the triangular drawing algorithm yields a convex grid drawing of G on a $W \times H$ grid with $W \leq 4n$ and $H \leq 2n$. Furthermore, one can easily show that the triangular drawing algorithm takes linear time.

We thus have the following lemma.

Lemma 6: For a plane graph G having a canonical decomposition $\Pi = (U_1, U_2, \dots, U_m)$ with respect to v_1, v_2 and v_n , the triangular drawing algorithm yields a convex grid drawing of G on a $W \times H$ grid with $W \leq 4n$ and $H \leq 2n$ in linear time.

5. Convex Grid Drawing Algorithm

In this section, we present a linear-time algorithm to find a convex grid drawing D of an internally triconnected plane

Fig. 9 Decomposition trees $T(G)$ (a) having a node of degree five, (b) having a node of degree four and a node of degree three, and (c) having three nodes of degree three.

graph G whose decomposition tree $T(G)$ has exactly five leaves. Such a graph G does not have a canonical decomposition, and hence none of the algorithms in [1], [2], [6] and [9] and the pentagonal drawing algorithm in Sect. 3 can find a convex grid drawing of G . Our algorithm draws the outer facial cycle $F_o(G)$ as a pentagon as illustrated in Fig. 1 (d). The algorithm first divides G into a left subgraph G_l , a right subgraph G_r and a center subgraph G_c as illustrated in Fig. 1 (b), then draw G_l and G_r by using the triangular drawing algorithm in Sect. 4 and draw G_c by using a modified pentagonal drawing algorithm in Sect. 5.4 as illustrated in Figs. 1 (c), and finally combine these three drawings to a convex grid drawing of G as illustrated in Fig. 1 (d).

5.1 Division

We first explain how to divide G into G_l , G_r and G_c . (See Figs. 1 (a) and (b).)

One may assume that the five leaves l_1, l_2, l_3, l_4 and l_5 of $T(G)$ appear clockwise in $T(G)$ in this order, as illustrated in Fig. 9. Clearly, there are three cases to consider.

Case a: exactly one node u_5 of $T(G)$ has degree five and each of the other non-leaf nodes has degree two as illustrated in Fig. 9 (a).

Case b: exactly one node u_4 has degree four, exactly one node u_3 has degree three and each of the other non-leaf nodes has degree two as illustrated in Fig. 9 (b).

Case c: exactly three nodes u_3, u_4 and u_5 have degree three and each of the other non-leaf nodes has degree two as illustrated in Fig. 9 (c).

We only consider Case a, because the other cases are similar.

As the five apices of the pentagonal contour of G , we choose five outer vertices $a_i, 1 \leq i \leq 5$ of G ; let a_i be an arbitrary outer vertex in the component corresponding to leaf l_i that is not a vertex of the separation pair of the component. The five vertices a_1, a_2, a_3, a_4 and a_5 appear clockwise on $F_o(G)$ in this order as illustrated in Fig. 1 (a).

Let $P(l_i), 1 \leq i \leq 5$, be a path from l_i to u_5 in $T(G)$. We choose arbitrary two nonconsecutive leaves l_i and l_{i+2} , where indices are computed as modulo 5. We then split the graphs corresponding to $P(l_i) - u_5$ and corresponding to $P(l_{i+2}) - u_5$ from G . Let G_l be the graph corresponding to $P(l_i) - u_5$, let G_r be the graph corresponding to $P(l_{i+2}) - u_5$

Fig. 10 (a) A triangular drawing of G_l , (b) a drawing D_l of G_l , (c) a triangular drawing of G_r , and (d) a drawing D_r of G_r .

and let G_c be the graph corresponding to $G - G_l - G_r$. In Fig. 1 (b), G_l is the graph corresponding to $P(l_1) - u_5$, and G_r is the graph corresponding to $P(l_3) - u_5$. Let $\{v_{l_1}, v_{l_2}\}$ be the separation pair for G_l and G_c , let $\{v_{r_1}, v_{r_2}\}$ be the separation pair for G_r and G_c and let $v_{l_1}, v_{l_2}, v_{r_2}$ and v_{r_1} appear around $F_o(G)$ in this order, as illustrated in Fig. 1 (a). If (v_{l_1}, v_{l_2}) (resp. (v_{r_1}, v_{r_2})) is a real edge of G , then let G_l (resp. G_r) and G_c be the graphs obtained by deleting a virtual edge (v_{l_1}, v_{l_2}) (resp. (v_{r_1}, v_{r_2})) from each graph defined above.

5.2 Drawing of G_l

We obtain a triangular drawing of G_l by using the triangular drawing algorithm in Sect. 4 as $v_1 = a_i, v_2 = v_{l_1}$ and $v_n = v_{l_2}$, respectively, as illustrated in Fig. 10 (a). We finally rotate the drawing by 90° clockwise and obtain a drawing D_l of G_l , as illustrated in Fig. 10 (b).

5.3 Drawing of G_r

Let G'_r be a “mirror” copy of G_r . We obtain a triangular drawing of G'_r by using the triangular drawing algorithm in Sect. 4 as $v_1 = a_{i+2}, v_2 = v_{r_1}$ and $v_n = v_{r_2}$, respectively. We then modify the drawing of G'_r using the left-right reflection and we obtain a triangular drawing of G_r , as illustrated in Fig. 10 (c). We finally rotate the drawing by 90° counter-clockwise and obtain a drawing D_r of G_r , as illustrated in Fig. 10 (d).

5.4 Drawing of G_c

In this section, we present a linear-time algorithm, called a *heptagonal drawing algorithm* to find a convex grid drawing of G_c with a heptagonal outer polygon. This algorithm finds a convex grid drawing of G_c whose outer polygon is a heptagon with apices $v_1, v_2, v_{r_1}, v_{r_2}, v_n, v_{l_2}$ and v_{l_1} . We modify

the pentagonal drawing algorithm in Sect. 3, as follows.

We use the pentagonal drawing algorithm as $v_1 = a_{i+4}$, $v_2 = a_{i+3}$, $v_n = a_{i+1}$, $v_l = v_{l_2}$ and $v_r = v_{r_2}$. Actually, the method deciding the coordinates of each vertex other than v_{l_2} and v_{r_2} is quite similar to the pentagonal drawing algorithm. Thus we will explain how to decide the coordinates of v_{l_2} and v_{r_2} . We only consider the case for v_{l_2} . The case for v_{r_2} is similar.

If $v_{l_2} \in U_k$, then v_{l_2} should be the first vertex of U_k . Let $U_k = \{v_{l_2} = u_1, u_2, \dots, u_l\}$, let w_p be the leftmost neighbor of $v_{l_2} = u_1$, and let w_q be the rightmost neighbor of u_l . Then $w_p = v_{l_1}$, of course.

We first shift $w_1, w_2, \dots, w_p$ of G_{k-1} and some inner vertices of G_k to the left by distance $|U_k|$, and then shift $w_q, w_{q+1}, \dots, w_t$ of G_{k-1} and some inner vertices of G_k to the right by distance $|U_k|$. After the operation, we shift all vertices of G_{k-1} to the right by distance $|U_k|$ so that $P(w_1) = (0, 0)$.

Let D_m be the Manhattan distance between w_p and w_q . We wish to put vertices $U_k = \{v_{l_2} = u_1, u_2, \dots, u_l\}$ so that the edge $(w_p, u_1) = (v_{l_1}, v_{l_2})$ has slope $+2$ and the edge (u_l, w_q) has slope -1 . Furthermore, if $|U_k| \geq 2$, then we wish to put vertices $u_2, u_3, \dots, u_{l-1}$ so that, for each i , $1 \leq i \leq l-1$, the edge (u_i, u_{i+1}) has slope 0 and the distance between two vertices u_i and u_{i+1} is equal to 2 . Then $D_m - 2(|U_k| - 1)$ should be a multiple of 3 , that is, $(D_m - 2(|U_k| - 1)) \bmod 3 = 0$ and hence we will do some additional shift operations if $D_m \bmod 3 \neq 0$. We have the following two cases to consider.

Case (I): $(D_m - 2(|U_k| - 1)) \bmod 3 = 1$.

In this case, we shift $w_q, w_{q+1}, \dots, w_t$ of G_{k-1} and some inner vertices of G_k to the right by distance 2 .

Case (II): $(D_m - 2(|U_k| - 1)) \bmod 3 = 2$.

In this case, we shift $w_q, w_{q+1}, \dots, w_t$ of G_{k-1} and some inner vertices of G_k to the right by distance 1 .

Then, $D_m - 2(|U_k| - 1)$ becomes a multiple of 3 and hence the straight line with slope $+2$ through $(x(w_p), y(w_p))$ and the straight line with slope -1 through $(x(w_q) - 2(|U_k| - 1), y(w_q))$ intersect at a grid point, which is denoted by P .

In Sect. 5.5, we will combine the drawing D_l of the left subgraph G_l and the drawing D_c of the center subgraph G_c so that the edge (v_{l_1}, v_{l_2}) of G_l overlaps the same one of G_c . If the length of the edge (v_{l_1}, v_{l_2}) of G_l is not equal to the length of the edge (v_{l_1}, v_{l_2}) of G_c , then we widen the narrower one by the shift operation so that both have the same length. We have the two cases to consider.

Case (i): $H(D_l) > y(P) - y(w_p)$. (See Fig. 11 (a).)

In this case, we shift $w_q, w_{q+1}, \dots, w_t$ of G_{k-1} and some inner vertices of G_k to the right by distance $(H(D_l) - (y(P) - y(w_p))) \times 3/2$. In Fig. 11 (a), $H(D_l) - (y(P) - y(w_p))$ is equal to 2 and hence we shift by distance 3 , as illustrated in Fig. 11 (b). Since $H(D_l) - (y(P) - y(w_p))$ is an even number, $(H(D_l) - (y(P) - y(w_p))) \times 3/2$ is a multiple of 3 and hence $D_m - 2(|U_k| - 1)$ is still a multiple of 3 .

Case (ii): $H(D_l) \leq y(P) - y(w_p)$. (See Fig. 11 (c).)

Fig. 11 Illustrations for (a),(b) Case (i), and (c),(d) Case (ii).

In this case, we extend the drawing D_l so that the length of the edge (v_{l_1}, v_{l_2}) of G_l is equal to the length between $P(w_p)$ and P , as illustrated in Fig. 11 (d).

We then put vertices in U_k so that the edge (v_{l_1}, v_{l_2}) has slope $+2$ and the edge (u_l, w_q) has slope -1 . Furthermore, if $|U_k| \geq 2$, then we put vertices $u_2, u_3, \dots, u_{l-1}$ so that the edges (u_i, u_{i+1}) has slope 0 and the distance between two vertices u_i and u_{i+1} is equal to 2 , for each i , $1 \leq i \leq l-1$, as illustrated in Figs. 11 (b) and (d).

One can prove similarly as in [11] that the heptagonal drawing algorithm yields a convex grid drawing of G_c . We finally consider the width $W(D_c)$ and the height $H(D_c)$ of the drawing D_c of G_c .

Let n_l be the number of vertices of G_l , let n_r be the number of vertices of G_r and let n_c be the number of vertices of G_c . Then $n_l + n_r + n_c = n + 4$, of course. Since every vertex of a plane graph G has degree three or more, each component corresponding to l_i of $T(G)$, for each i , $1 \leq i \leq 5$ has four or more vertices and hence we have $n_l, n_r \geq 4$ and $n_c \geq 10$. Since we may shift by distance $2 + H(D_l) \times 3/2$ for v_{l_2} and by distance $2 + H(D_r) \times 3/2$ for v_{r_2} , we have

$$W(D_c) \leq 2n_c + 4 + H(D_l) \times 3/2 + H(D_r) \times 3/2.$$

By Lemma 6 and the algorithms in Sects. 5.2 and 5.3, we have $H(D_l) \leq 4n_l$ and $H(D_r) \leq 4n_r$ and hence

$$W(D_c) \leq 2n_c + 4 + 6n_l + 6n_r.$$

Since $n_l + n_r + n_c = n + 4$ and $n_c \geq 10$, we have

$$W(D_c) \leq 6(n_l + n_r + n_c) + 4 - 4n_c \leq 6n.$$

Furthermore, one can prove similarly as in [11] that $H(D_c) \leq n^2$.

We thus have the following lemma.

Lemma 7: For a plane graph G_c having a canonical decomposition $\Pi = (U_1, U_2, \dots, U_m)$ with respect to v_1, v_2 and v_n , the heptagonal drawing algorithm yields a convex grid drawing of G_c on a $W \times H$ grid with $W \leq 6n$ and $H \leq n^2$ in linear time.

5.5 Drawing of G

We first arrange D_c so that $x(a_{i+4}) = 0$ and $y(a_{i+4}) = 0$. We then arrange D_l so that the edge (v_{l_1}, v_{l_2}) of G_l overlaps the same one of G_c . We also arrange D_r so that the edge (v_{r_1}, v_{r_2}) of G_r overlaps the same one of G_c . We finally remove the edges (v_{l_1}, v_{l_2}) and (v_{r_1}, v_{r_2}) if they are not original edges of G , as illustrated in Fig. 1 (d).

5.6 Validity of Drawing Algorithm

In this section, we show that the drawing D obtained above is a convex grid drawing of G . By Lemma 6, both D_l and D_r are convex grid drawings. Furthermore, by Lemma 7, D_c is also a convex grid drawing. Therefore, one can easily show that D is a convex grid drawing of G with pentagonal contour. Clearly, the size of the grid of the drawing D of G is equal to the size of D_c of G_c and hence, by Lemma 7, we have $W(D) \leq 6n$ and $H(D) \leq n^2$.

We thus have the following theorem.

Theorem 1: Assume that G is an internally triconnected plane graph, every vertex of G has degree three or more, and the triconnected component decomposition tree $T(G)$ has exactly five leaves. Then our algorithm finds a convex grid drawing of G on a $6n \times n^2$ grid in linear time.

6. Conclusions

In this paper, we showed that every internally triconnected plane graph G whose decomposition tree $T(G)$ has exactly five leaves has a convex grid drawing on a $6n \times n^2$ grid, and we present a linear-time algorithm to find such a drawing. This is the first algorithm that finds a convex grid drawing of such a graph G on a grid of polynomial size. The remaining problem is to obtain an algorithm for an internally triconnected plane graph whose decomposition tree has six or more leaves.

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