SUMMARY Curve extension is a useful function in shape modeling for cyberworlds, while a Ball B-Spline Curve (BBSC) has its advantages in representing freeform tubular objects. In this paper, an extension algorithm for the BBSC with $G^2$-continuity is investigated. We apply the extending method of B-Spline curves to the skeleton of the BBSC through generalizing a minimal strain energy method from 2D to 3D. And the initial value of the $G^2$-continuity parameter for the skeleton is selected by minimizing the approximate energy function which is a problem with $O(1)$ time complexity. The corresponding radius function of the control ball points is determined through applying the $G^2$-continuity conditions for the skeleton to the scalar function. In order to ensure the radii of the control ball points are positive, we make a decision about the range of the $G^2$-continuity parameter for the radius and then determine it by minimizing the strain energy in the affected area. Some experiments comparing our method with other methods are given. And at the same time, we present the advantage of our method in modeling flexibility from the aspects of the skeleton and radius. The results illustrate our method for extending the BBSC is effective.

key words: ball B-spline, curve extension, $G^2$-continuity, minimal energy

1. Introduction

Ball B-Spline Curve (BBSC) is a skeleton based solid parametric representation of 3D freeform objects, which represents its centreline/skeleton directly as well as every point of 3D objects. As it generalizes the B-Spline curve through an additional radii function, BBSC represents a curve with widths. Therefore it is suitable for representing shapes like vascular in cyberworlds. In 2005, Seah et al. first proposed Ball B-Spline Curves through extending Disk Bézier curves [1]. In 2007, Wu solided the theory of the BBSC and its fundamental properties and algorithms [2]. After that, the BBSC is applied to different fields like modeling tubular objects [3], and modeling plant stem [4], leaf [5] and root [6]. These plant models are also used in computer game [7] and computer animation [8]. BBSC is also used for representing blood vascular [9]. As the BBSC is a parametric representation, it is flexible for interactive modeling. Therefore the BBSC is employed to human modeling [10] and plant modeling [11] by sketch.

Extension is a useful function in geometric modeling. Many approaches on extension algorithms are discussed. As B-Spline is a popular representation in CAD and geometric modeling, most of papers on extension are about B-Spline extension algorithms. Recently, Hu proposed the DeBoor algorithm for extending B-Spline Curves [12]. With their method, it needs many times to readjust some control points when extending to multiple target points. Minimal strain energy model is applied to adjust the shape of the extended curve by some researchers [13]–[15]. They chose the approximate objective function to determine the degrees of freedom. After that, Zhou established the objective functions based on exact energy and exact curvature variation [16].

In BBSC, extension is also a useful function for BBSC modeling. In this paper, we investigate the extension problem on the BBSC. As a BBSC can be regarded as two parts [2]: the skeleton (a 3D B-Spline curve) and the radius function (a B-Spline scalar function), the extension algorithm of the BBSC can be divided into two parts: the extension of the skeleton, a 3D B-Spline curve and the calculation of the radii function. In this paper, we generalize Zhou’s extending algorithm proposed in [16] to 3D case to solve the first part. And the initial value of the $G^2$-continuity parameter is selected by minimizing the approximate strain energy. For the second part, radius function is achieved through applying $G^2$-continuity conditions to the scalar function. And a discussion about the range of the $G^2$-continuity parameter is made to ensure the radii of the control ball points are positive.

This paper is organized as follows. In Sect. 2, the definition of the BBSC and the $G^2$-continuity conditions between BBSCs are investigated. Sections 3–5 present the BBSC extension algorithm with $G^2$-continuity, which includes two parts: the extension of the skeleton and the radius function separately. Section 6 generalizes the above extension algorithm to the condition with multiple target ball points. Some experiment examples and comparisons on our method with other methods are depicted in Sect. 7. Finally, Sect. 8 provides the conclusion and discusses the future work.

2. Fundamentals

2.1 Ball B-Spline Curves

Define a ball as $\langle P; r \rangle = \{ x \in \mathbb{R}^3 \mid |x - P| \leq r, P \in \mathbb{R}^3, r \in \mathbb{R}^+ \}$. So a $p$-degree BBSC is

$$\langle B \rangle (t) = \sum_{i=0}^{n} N_i(t) \langle P_i; r_i \rangle, \quad 0 \leq t \leq 1.$$
JIANG et al.: G 2-CONTINUITY EXTENSION ALGORITHM OF BALL B-SPLINE CURVES

B´ezier curve: point. The extended part of this BBSC is a cubic ball

\( p_i \) is called control point, and \( r_i \) is called control radius. \( N_{i,p}(t) \) is the \( i \)-th B-Spline basis of degree \( p \).

\[
\langle B \rangle (t) = \sum_{i=0}^{n} N_{i,p}(t) \langle P_i \rangle = \sum_{i=0}^{n} \langle N_{i,p}(t)P_i; N_{i,p}(t)r_i \rangle
\]

\[
= \left( \sum_{i=0}^{n} N_{i,p}(t)P_i; \sum_{i=0}^{n} N_{i,p}(t)r_i \right)
\]

a BBSC can be regarded as two parts: the skeleton: \( c(t) = \sum_{i=0}^{n} N_{i,p}(t)P_i \), a 3D B-Spline curve and the radius function:

\[
r(t) = \sum_{i=0}^{n} N_{i,p}(t)r_i, \text{ a B-Spline scalar function.}
\]

For the properties of BBSC (see Fig. 1), refer to [2]. Figure 2 presents the simulation results of the BBSC for tubular objects.

Fig. 2  The rendered BBSC (d) The BBSC after transformation

\[\text{Fig. 1  A BBSC created by interpolation.}\]

2.2 G 2-Continuity Conditions

Given a \( p \)-degree BBSC \( \langle B \rangle (t) \), \( \langle R; r_R \rangle \) is the extending ball point. The extended part of this BBSC is a cubic ball Bézier curve: \( \langle Q \rangle (u) = \sum_{i=0}^{3} B_i(u)Q_i; r_Q_i \), where \( B_i(u) = \binom{3}{i}u^i(1-u)^{3-i} \) \( (i = 0, 1, 2, 3) \) are Bernstein basis functions on \([0, 1]\). G 2-continuity is satisfied at the joint ball point between \( \langle Q \rangle (u) \) and \( \langle B \rangle (t) \). The conditions are as follows:

\[
\begin{align*}
\langle Q \rangle (0) &= \langle B \rangle (1) \\
\langle Q \rangle'(0) &= \alpha \langle B \rangle'(1) \\
\langle Q \rangle''(0) &= \alpha^2 \langle B \rangle''(1) + \beta \langle B \rangle'(1)
\end{align*}
\]

(1) is viewed as two parts: the G 2-continuity conditions for the skeleton (2) and for the radius (3).

\[
\begin{align*}
\beta &= 1 \\
q &= c(1) \\
q' &= \alpha_1 c'(1) \\
q'' &= \alpha_1^2 c''(1) + \beta_1 c'(1) \\
r_Q(0) &= r(1) \\
r_Q'(0) &= \alpha_2 r'(1) \\
r_Q''(0) &= \alpha_2^2 r''(1) + \beta_2 r'(1)
\end{align*}
\]

where \( \alpha_1 > 0, \alpha_2 > 0, \beta_1 \) and \( \beta_2 \) are arbitrary real numbers. For simplicity, we make \( \beta_1 = 0, \beta_2 = 0 \). And \( c(t) = \sum_{i=0}^{n} N_{i,p}(t)P_i \), the skeleton of \( \langle B \rangle (t) \).

Substitute derivatives of \( c(t) \) and \( r(t) \) into Eqs. (2) and (3), the control ball points of \( \langle Q \rangle (u) \) are determined by

\[
\begin{align*}
Q_0 &= P_n \\
Q_1 &= P_n + \frac{P_n - P_{n-1}}{t_{n+3} - t_n} \\
Q_2 &= P_n + 2\alpha_1 \frac{P_n - P_{n-1}}{t_{n+3} - t_n} \\
&+ \frac{P_n - P_{n-1}}{t_{n+3} - t_n}(t_{n+3} - t_n) - \frac{P_{n-1} - P_{n-2}}{t_{n+2} - t_{n-1}}(t_{n+2} - t_n) \\
Q_3 &= R
\end{align*}
\]

At the same time, the radii of the control ball points are

\[
\begin{align*}
r_Q(0) &= r_n \\
r_Q(1) &= r_n + \frac{r_n - r_{n-1}}{t_{n+3} - t_n} \\
r_Q(2) &= r_n + 2\alpha_2 r_n - r_{n-1} \\
&+ \frac{r_n - r_{n-1}}{t_{n+3} - t_n}(t_{n+3} - t_n) - \frac{r_{n-1} - r_{n-2}}{t_{n+2} - t_{n-1}}(t_{n+2} - t_n) \\
r_Q(3) &= r_R
\end{align*}
\]

3. Extension of the Skeleton with G 2-Continuity

3.1 Connection Optimization by Fairness

The degree of freedom \( \alpha_1 \) is determined by minimizing the strain energy of the skeleton. In [13]–[15], the objective functions of energy are approximate. In this paper, we apply the objective functions based on exact energy variation to solve \( \alpha_1 \):

\[
\text{...}
\]
$E_{\text{energy}} = \int k^2(s)ds = \int_0^1 \frac{h_1}{h_2}du,$  \hspace{1cm} (6)

where
\begin{align*}
h_1 &= (z''(u)y'(u) - z'(u)y''(u))^2 \\
&+ (x''(u)x'(u) - x'(u)x''(u))^2 \\
&+ (x''(u)y'(u) - x'(u)y''(u))^2,
\end{align*}
\begin{align*}
h_2 &= \left(\sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2}\right)^5,
\end{align*}
d$s$ is the differential of curve arc length, $\kappa(s)$ is the curvature defined as: $\kappa(s) = \frac{|q'(u) \times q''(u)|}{|q'(u)|^3}$. Equation (6) is a strong non-linear problem. We transfer it into a non-linear least squares problem and then apply the Gauss-Newton-NL2SOL method [17] to solve it. The following is the solution method in detail.

Firstly, decompose Eq. (6) into sum of square form by composite trapezoidal rule:
\[
\int_0^{t_n} f(t)dt = \frac{t_n - t_0}{n} \sum_{k=1}^{n-1} f(t_0 + k\frac{t_n - t_0}{n}) + \frac{f(t_0) + f(t_n)}{2},
\]
where $n$ is the number of subintervals.

Secondly, let $g(u) = \sqrt{\frac{h_1}{h_2}}$. Substitute $g(u)$ into objective function (6):
\[
\arg \min_{\alpha > 0} E_{\text{energy}}(\alpha) = \frac{1}{n} \left( \left( \frac{g(0)}{\sqrt{2}} \right)^2 + \left( \frac{g(1)}{\sqrt{2}} \right)^2 + \sum_{k=1}^{n-1} \frac{g(k)}{n} \right). \hspace{1cm} (7)
\]
Then, let
\[
A(\alpha) = \left( \frac{g(0)}{\sqrt{2}}, \frac{g(1)}{\sqrt{2}}, \frac{1}{n}, \ldots \frac{1}{n} \right). \hspace{1cm} (8)
\]
Therefore minimizing objective function (6) is converted into a non-linear least squares problem, i.e. minimizing Eq. (8). GaussNewton-NL2SOL method is applied to solve it. The detailed process is shown in Algorithm 1.

### 3.2 Defining the Initial \( \alpha_1 \)

The convergence of Algorithm 1 depends on the given initial value \( \alpha_{1(0)} \). It should be chosen close to one of the local minimum resolutions in the domain of \( \alpha_1 > 0 \) to ensure the convergence of the algorithm. In our method, \( \alpha_{1(0)} \) is given by minimizing the approximate energy function, that is,
\[
\arg \min_{\alpha > 0} E_G(\alpha_1) = \int_0^1 \left\| q''(t) \right\|^2 dt.
\]

\( E_G \) is differentiated with respect to \( \alpha_1 \):
\[
\frac{dE_G}{d\alpha_1} = 0, \quad \alpha_1 > 0.
\]

Solve the above algebraic equation to get a positive \( \alpha_{1(0)} \) [13]. The given initial value \( \alpha_{1(0)} \) is close to one local minimum resolution. So Algorithm 1 is convergent. And more importantly, the initial value chosen by this method is related to the input BBSC data, avoiding the trouble to choose the initial value according to different conditions.

Figure 3 is an example for different initial values \( \alpha_{1(0)} \) under the condition of different input BBSCs. At the same time, Algorithm 1 also takes into account the probable existence of small residual and large residual problems, ensuring the convergence rate.

### 4. Extension of the Radius Function with \( G^2 \)-Continuity

#### 4.1 Discussing the Range of \( \alpha^2 \)

In order to make the radius function of the extended part reasonable, \( r_{Q,1} \) and \( r_{Q,2} \) must be greater than 0. And for
we get the following constraints inequalities.

\[
\begin{align*}
\begin{cases}
 r_{Q,1} &= r_n + \frac{B}{2} > 0 \\
r_{Q,2} &= r_n + B\alpha_2 + A\alpha_2^2 > 0 \\
|r_{Q,1} - r_n| &= \frac{\alpha_2}{2}|B| < \|Q_1 - P_n\| \\
|r_{Q,1} - r_n| &= r_n - r_R + \frac{B}{2}\alpha_2 < \|Q_1 - R\| \\
|r_{Q,2} - r_{Q,1}| &= \frac{B}{2}\alpha_2 + A\alpha_2^2 < \|Q_2 - Q_1\| \\
|r_{Q,2} - r_n| &= r_n - r_R + B\alpha_2 + A\alpha_2^2 < \|Q_2 - R\| \\
|r_{Q,2} - r_n| &= B\alpha_2 + A\alpha_2^2 < \|Q_2 - P_n\| \\
\end{cases}
\end{align*}
\]

where \( A = \frac{r_n - r_{n-1}}{(t_{n+3} - t_n)(t_{n+2} - t_n)} - \frac{r_{n-1} - r_{n-2}}{(t_{n+2} - t_{n-1})(t_{n+2} - t_n)}, \)
\( B = 2\frac{r_{n-1} - r_{n-2}}{t_{n+3} - t_n}. \)

Solve the above inequalities of one unknown, and we can get the range of \( \alpha_2. \) According to whether \( A \) is equal to zero, these seven inequalities are divided into two parts. One part contains the following three inequalities.

\[
\begin{align*}
\begin{cases}
 r_{Q,1} &= r_n + \frac{B}{2} > 0 \\
r_{Q,1} - r_n &= \frac{\alpha_2}{2}|B| < \|Q_1 - P_n\| \\
r_{Q,1} - r_R &= |r_n - r_R + \frac{B}{2}\alpha_2 < \|Q_1 - R\| \\
\end{cases}
\end{align*}
\]

And the other part concludes the remaining four ones.

\[
\begin{align*}
\begin{cases}
 r_{Q,2} &= r_n + B\alpha_2 + A\alpha_2^2 > 0 \\
r_{Q,2} - r_{Q,1} &= \frac{B}{2}\alpha_2 + A\alpha_2^2 < \|Q_2 - Q_1\| \\
r_{Q,2} - r_n &= r_n - r_R + B\alpha_2 + A\alpha_2^2 < \|Q_2 - R\| \\
r_{Q,2} - r_n &= |B\alpha_2 + A\alpha_2^2 < \|Q_2 - P_n\| \\
\end{cases}
\end{align*}
\]

It is easy to get the range of \( \alpha_2 \) by solving Eq. (10). Consider three conditions:

1. If \( B > 0 \), the range of \( \alpha_2 \) are \( \left\{ 0 < \alpha_2 < \frac{2\|Q_1 - P_n\|}{|B|}, \right\} \)
   \( \left\{ 0 < \alpha_2 < \frac{2\|Q_1 - R\|}{|B|} \right\} \) and \( \{ \alpha_2 > 0 \} \) respectively for the three inequalities. Compare the boundary values of these intervals to get the intersection of these sets.

2. If \( B < 0 \), the range of \( \alpha_2 \) is the intersection of \( \left\{ 0 < \alpha_2 < \frac{2r_n}{|B|} \right\} \) and \( \left\{ 0 < \alpha_2 < \frac{2\|Q_1 - P_n\|}{|B|} \right\}. \)

3. If \( B = 0 \), for these three inequalities, \( \alpha_2 > 0. \) Furthermore, \( |r_n - r_R| < \|Q_1 - R\| \) is required to ensure \( |r_{Q,1} - r_R| < \|Q_1 - R\| \) reasonable. This condition depends on the input.

It is not easy to solve Eq. (11) because of the probable existence of quadratic term. Consider three conditions:

1. If \( A = 0 \), the quadratic inequalities are degenerated into inequalities in one. We use the way of discussion for Eq.(10) to solve it, i.e. calculate the intersection of \( \{ \alpha_2 > 0 \} \) \( (B > 0) \) or \( \{ 0 < \alpha_2 < -\frac{r_n}{|B|} \} \)
   \( \left\{ 0 < \alpha_2 < \frac{2\|Q_2 - Q_1\|}{|B|} \right\} \) \( (B < 0) \) and \( \left\{ \frac{(r_R - r_n - \|Q_2 - R\|)}{|B|} \right\} \) \( \left\{ \frac{(r_R - r_n - \|Q_2 - R\|)}{|B|} \right\} \) \( (B = 0) \) the same problem for \( B = 0 \) that \( |r_n - r_R| < \|Q_2 - R\| \) is required to ensure \( |r_{Q,2} - r_R| < \|Q_2 - R\| \) reasonable. This condition also depends on the input.

2. If \( A > 0 \), the quadratic equations open up, discuss the solution according to the relationship between the corresponding \( \Delta \) and zero.

3. If \( A < 0 \), the quadratic equations open down, discuss the solution according to the relationship between the corresponding \( \Delta \) and zero.
4.2 Defining $\alpha_2$

The degree of freedom $\alpha_2$ is determined by minimizing the strain energy of $r_\alpha(u)$. For simplicity, we select the approximate energy function, that is,

$$
\text{arg min}_{\alpha_2 \in \mathcal{H}} (R_{\text{energy}}(\alpha_2)) = \int_0^1 ||p''(t)||^2 dt.
$$

$\mathcal{H}$ is the range of $\alpha_2$. The strain energy function $R_{\text{energy}}$ is a quartic equation with respect to $\alpha_2$, so we use a direct method to solve the extremum problem of this single-variable function. $R_{\text{energy}}(\alpha_2)$ is differentiated with respect to $\alpha_2$:

$$
\frac{d(R_{\text{energy}}(\alpha_2))}{d\alpha_2} = 0, \quad \alpha_2 \in \mathcal{H},
$$

that is,

$$
2(r_\alpha''(1) \times r_\alpha''(1))\alpha_2^3 + 9(r_\alpha''(1) \times r_\alpha''(1))\alpha_2^2
+ 6(r_\alpha''(1) \times r_\alpha''(1) + 2r_\alpha'(1) \times r_\alpha'(1) - r_\alpha''(1) \times r_\alpha)\alpha_2
+ 12(r_\alpha''(1) \times r_\alpha'(1) - r_\alpha'(1) \times r_\alpha) = 0. \quad (12)
$$

Solve Eq. (12) to get at most three extreme points. According to the distribution of these extreme points, it is easy to determine the monotonicity of the quartic function $R_{\text{energy}}$ in the range of $\alpha_2$ discussed above. And the minimum energy is got finally, so as the corresponding $\alpha_2$.

5. Re-Computing Control Ball Points of the BBSC

After reparameterization, the given BBSC and the extending Ball Bézier curve can achieve $C^2$ continuity. The parameter transformation for $\langle B\rangle(t)$ and $\langle Q\rangle(u)$ is $u = (t - 1)/\alpha_1$, and then $\langle Q\rangle(u)$ is reparameterized as $\langle Q\rangle(t) = \langle Q\rangle \left( \frac{t - 1}{\alpha_1} \right), \quad t \in [1, 1 + \alpha_1]$. A new BBSC $\langle B\rangle(t)$ defined on $[0, 1 + \alpha_1]$ is constructed

$$
\langle B\rangle(t) = \begin{cases} 
\langle B\rangle(t), & t \in [0, 1] \\
\langle Q\rangle(t), & t \in (1, 1 + \alpha_1].
\end{cases} \quad (13)
$$

Its knot vector is

$$
T_2 = \{0, \ldots, 0, t_{p+1}, \ldots, t_n, 1, 1, 1, 1 + \alpha_1, \ldots, 1 + \alpha_1\}_{p+1}
$$

with control ball points $\langle \{P_0; r_0\}, \langle P_1; r_1\rangle, \ldots, \langle P_n; r_n\rangle, \langle Q_1; r_{Q1}\rangle, \langle Q_2; r_{Q2}\rangle, (R; r_R)\rangle$. We apply the knot-removal algorithm [13] to remove ‘1’ twice from $T_2$ and compute the new control ball points $\langle \tilde{P}_i; \tilde{r}_i\rangle$. The computing process in detail is provided in the following Algorithm 2.

Algorithm 2

Step 1: $\langle \tilde{P}_i; \tilde{r}_i\rangle = \langle P_i; r_i\rangle, \quad (i = n - 2, n - 1, n);$

Step 2:

$$
\left\{ \begin{array}{l}
\langle \bar{P}_i; \bar{r}_i\rangle = \langle P_i; r_i\rangle, \quad i = n - r, \ldots, n - 1 \\
\langle \bar{P}_i; \bar{r}_i\rangle = \langle P_{i-1}; r_{i-1}\rangle - \lambda \langle P_i; r_i\rangle, \quad i = n + r, \ldots, n \\
\lambda = \frac{1 - t_i}{t_{i+3-r} - t_i}, \quad r = 2, 1;
\end{array} \right.
$$

Step 3: $\langle \tilde{P}_i; \tilde{r}_i\rangle = \langle P_i; r_i\rangle, \quad i = n - 2, n - 1, n$

Using Algorithm 2 we get the new control ball points of $\langle \bar{B}\rangle(t)$ after removing multiple 1. The knot vector $T_2$ can be standardized as

$$
T_3 = \left\{ \begin{array}{c}
t_{p+1}, \ldots, t_n, 1 + \alpha_1, \ldots, 1 + \alpha_1, \frac{1}{1 + \alpha_1 + \alpha_2}, \ldots, 1 \end{array} \right\}_{p+1}.
$$

6. Extension with Multiple Target Ball Points

This method can be applied to extend a BBSC to multiple target ball points, two ball points, for example. Suppose the two extending ball points be $\langle R_1; r_1\rangle$ and $\langle R_2; r_2\rangle$. Using Algorithm 1 to extend $\langle B\rangle(t)$ to $\langle R_1; r_1\rangle$, we get the corresponding knot $1 + \alpha_1$. In the same way, for the second ball point $\langle R_2; r_2\rangle$, we get the corresponding knot of $\langle R_2; r_2\rangle$, $1 + \alpha_1 + \alpha_2$. Then the knot vector of final extended curve $\langle \bar{B}\rangle(t)$ is:

$$
T_4 = \left\{ \begin{array}{c}
0, \ldots, 0, t_{p+1}, \ldots, t_n, \frac{1}{1 + \alpha_1 + \alpha_2}, \ldots, 1 + \alpha_1 + \alpha_2, \frac{1}{1 + \alpha_1 + \alpha_2}, \ldots, 1 \end{array} \right\}_{p+1}
$$

7. Experiment Results

In this section, the experiments are given for comparing our method with the curve unclamping method in [12] and the minimal strain energy method in [13]. Figure 4 shows a cubic BBSC and the target ball. The coordinates of five control ball points of the BBSC are $\langle (-6.5, 1.5, 0); 0.2, 1 \rangle$. (a) The BBSC with control ball points (b) The rendered result

![Fig. 4](image-url)
Fig. 5  Extension of a cubic BBSC to one target ball.

(a) The extending curve by unclamping method  
(b) The extending curve by minimal strain energy method  
(c) The extending curve by our method

![Extension of a cubic BBSC to one target ball](image)

Table 1  Numerical comparison of the extending results.

<table>
<thead>
<tr>
<th></th>
<th>Curve unclamping method</th>
<th>Minimal strain energy method</th>
<th>Minimal exact strain energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy of the skeleton</td>
<td>2.62393</td>
<td>3.02987</td>
<td>2.57336</td>
</tr>
<tr>
<td>Rot of the skeleton</td>
<td>0.719564</td>
<td>0.773361</td>
<td>0.670418</td>
</tr>
</tbody>
</table>

\[
\langle -5.7, 3.8, 0 \rangle; 0.2, \langle -3.1, 3.2, 0 \rangle; 0.2, \langle -2.6, 1, 0 \rangle; 0.2, \\
\langle -2, 0.5, 0 \rangle; 0.2. \text{ The coordinate of the target ball point is} \\
\langle 0.9, 2.3, 0 \rangle; 0.4. \text{ Figure 5 (a) is the extending curve constructed by the unclamping method in [12], and Fig. 5 (b) is} \\
\text{the curve constructed by the minimal strain energy method in [13], Fig. 5 (c) is the extending result by our method.} \\

We choose the strain energy and the rotation number [18] as two indexes for the experiment results. Rotation number is an important index for analyzing geometric properties in differential geometry. For a planar curve \( p(t), \) \( 0 \leq t \leq 1, \) with \( k(t) \) as its curvature, its rotation number is defined as:

\[
\text{Rot} = \frac{1}{2\pi} \int_0^1 |k(t)||p'(t)|dt.
\]

As shown in Table 1, the extending curve generated by our method has less strain energy and less rotation number than the unclamping method and the minimal strain energy method. The optimal \( \alpha_1 \) is found to be 0.65, and the fairness of the curve is greatly improved. Some portions of the minimization curve of \( \alpha_1 \) are shown in Fig. 6. Figure 7 is an example for extending the cubic BBSC to two target balls.

From Table 1, the extended curve generated by the unclamping method can also have approximate minimal energy, while the advantage for our method compared with the unclamping method is the modeling flexibility. The extended curve created by the unclamping method is unique. And it can’t be adjusted manually. In CAD systems, we hope for more modeling results. Our method provide two parameters, \( \alpha_1 \) for the skeleton and \( \alpha_2 \) for the radii, to adjust the shape of the extended curve under \( G^2 \)-continuity conditions. Figure 8 shows the extension results of tubular objects with different \( \alpha_1 \) and \( \alpha_2 \).
Experiment results show that our extending method is another advantage over other methods. These extensions are provided. And the modeling flexibility of our method and multiple target balls are implemented. In the end, some remarks are made for future papers.

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References

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