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Completely Independent Spanning Trees on Some Interconnection Networks

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SUMMARY Let \( T_1, T_2, \ldots, T_k \) be spanning trees in a graph \( G \). If, for any two vertices \( u, v \) of \( G \), the paths joining \( u \) and \( v \) on the \( k \) trees are mutually vertex-disjoint, then \( T_1, T_2, \ldots, T_k \) are called completely independent spanning trees (CISTs for short) of \( G \). The construction of CISTs can be applied in fault-tolerant broadcasting and secure message distribution on interconnection networks. Hasunuma (2001) first introduced the concept of CISTs and conjectured that there are \( k \) CISTs in any \( 2k \)-connected graph. Unfortunately, this conjecture was disproved by P´eterfalvi recently. In this note, we give a necessary condition for \( k \)-connected \( k \)-regular graphs with \([4/2]\) CISTs. Based on this condition, we provide more counterexamples for Hasunuma’s conjecture. By contrast, we show that there are two CISTs in 4-regular chordal rings \( CR(N, d) \) with \( N = k(d - 1) + j \) under the condition that \( k \geq 4 \) is even and \( 0 \leq j < 4 \). In particular, the diameter of each constructed CIST is derived.

key words: completely independent spanning trees, interconnection networks, chordal rings

1. Introduction

Interconnection networks take a key role in multiprocessor systems and can be represented by a graph, where vertices and edges represent processors and communication links between processors, respectively. Let \( G \) be a simple undirected graph with vertex set \( V(G) \) and edge set \( E(G) \). The connectivity of a graph \( G \), denoted by \( \kappa(G) \), is the minimum cardinality of a set \( F \subset V(G) \) such that \( G - F \) is disconnected or trivial, where \( G - F \) denotes the graph that removes all vertices of \( F \) in \( G \). A graph \( G \) is \( k \)-connected if and only if \( \kappa(G) \geq k \).

Let \( G \) be a graph. For \( x, y \in V(G) \), two paths joining \( x \) and \( y \) are vertex-disjoint if they have no common vertex except \( x \) and \( y \), and edge-disjoint if they share no common edge. A set of \( k \) spanning trees of \( G \) rooted at a vertex \( r \) is said to be independent spanning trees (ISTs for short) if the paths from any vertex \( v \neq r \) to \( r \) on the \( k \) trees are mutually vertex-disjoint. A set of \( k \) spanning trees of \( G \) is said to be completely independent spanning trees (CISTs for short) if for any two vertices \( x, y \in V(G) \), the paths joining \( x \) and \( y \) on the \( k \) trees are mutually vertex-disjoint. Also, we say that two spanning trees \( T_1 \) and \( T_2 \) are edge-disjoint if \( E(T_1) \cap E(T_2) = \emptyset \).

Constructing CISTs has many applications on interconnection networks such as fault-tolerant broadcasting and secure message distribution [3]. Hasunuma [8] conjectured that there are \( k \) CISTs in any \( 2k \)-connected graph and showed the NP-completeness for determining the existence of two CISTs in an arbitrary graph \( G \). So far the study related to CISTs has received less attention except for [1], [8]–[10]. Araki [1] showed that a graph \( G \) of \( n \) vertices has two CISTs if the minimum degree of \( G \) is at least \( n/2 \), and the square of a 2-connected graph has two CISTs. Hasunuma showed that there are \( k \) CISTs in the underlying graph of any \( k \)-connected line digraph \( L(G) \) [8], there are two CISTs in any 4-connected maximal planar graph [9], and there are two CISTs in the Cartesian product of any 2-connected graphs [10]. Recently, P´eterfalvi [15] gave counterexamples to disprove Hasunuma’s conjecture. He showed that there exists a \( k \)-connected graph which does not contain two CISTs for each \( k \geq 2 \).

In this note, we carry on the research of CISTs on some families of interconnection networks. Section 2 gives a necessary condition for \( k \)-connected \( k \)-regular graphs with \([k/2]\) CISTs. According to this condition, more interconnection networks are emerged in the negative to Hasunuma’s conjecture. Section 3 presents construction schemes of CISTs on 4-regular chordal rings under some restricted conditions. Finally, concluding remarks are given in the last section.

2. A Necessary Condition of CISTs for \( k \)-Connected \( k \)-Regular Graphs

Hasunuma [8] provided the following characterization of CISTs.

Theorem 1: [8] \( T_1, T_2, \ldots, T_k \) are CISTs in a graph \( G \) if and only if they are edge-disjoint and for any vertex \( v \) of \( G \), there is at most one spanning tree \( T_i \) such that \( v \) is not a leaf in \( T_i \).

Theorem 2: Let \( G \) be a \( k \)-connected \( k \)-regular graph for \( k \geq 4 \). If \( G \) admits \([k/2]\) CISTs, then the following condition holds:

\[
\left| \left\lfloor \frac{|V(G)| - 2}{[k/2]} \right\rfloor \right| \leq \left\lfloor \frac{|V(G)|}{[k/2]} \right\rfloor.
\]
Proof: Suppose that there are \([k/2]\) CISTs in \(G\) and let \(T\) be one with the maximum number of leaves. By Theorem 1, every non-leaf of \(T\) must be a leaf in any other \([k/2] - 1\) CISTs, and thus it consumes exactly \([k/2] - 1\) incident edges for those CISTs in \(G\). Since \(G\) is \(k\)-regular, the number of incident edges is at most \(k - [k/2] + 1\) for every non-leaf of \(T\). Clearly, the spanning tree \(T\) has \(|V| = |V(G)|\) vertices, \(|V| - 1\) edges and \(2|V| - 2\) incidences (i.e., every edge offers two incidences). Let \(I\) be the set of non-leaves in \(T\). Since every leaf joining to \(T\) is by way of an incidence, the total number of incidences associated with vertices in \(I\) is \((2|V| - 2) - (|V| - |I|)\). Thus, we have \((2|V| - 2) - (|V| - |I|) \leq |I|(k - [k/2] + 1)\). This implies that

\[
\left(\frac{|V| - 2}{k}\right) - \left(\frac{[k/2]}{k}\right) \leq |I|.
\]

(2)

Again, by Theorem 1, we know that every vertex could be served as a non-leaf at most once for all CISTs in \(G\). Since \(T\) has the maximum number of leaves, it contains at most \(|V|/\lceil k/2 \rceil\) non-leaves. Thus,

\[
|I| \leq \frac{|V|}{\left\lceil k/2 \right\rceil}.
\]

(3)

By (2) and (3), the proof is done. \(\square\)

Note that if a \(k\)-connected \(k\)-regular graph \(G\) fulfills the condition of inequality (1), it does not assure that \(G\) admits \([k/2]\) CISTs.

The \(n\)-dimensional hypercube \(Q_n\) is a graph with \(2^n\) vertices in which each vertex corresponds to an \(n\)-tuple on the set \([0, 1]^n\) and two vertices are linked by an edge if and only if they differ in exactly one coordinate. Clearly, \(Q_n\) is an \(n\)-connected \(n\)-regular graph. To overcome some shortcomings of a pure hypercube network, variations of hypercube architecture have been proposed to improve their efficiency, e.g., crossed cubes \(CQ_n\), twisted cubes \(TQ_n\), Möbius cubes \(MQ_n\), and locally twisted cubes \(LTQ_n\). Again, all variations of hypercube are \(n\)-connected \(n\)-regular graphs.

Example 1: For \(Q_n\) with \(n\) odd, it is easy to see that the inequality (1) always holds. We now consider \(Q_{10}\) and check the inequality (1). Since \(|V(Q_{10})| = 1024\), we have \([1024 - 2]/5 = 205\) in the left-hand side and \([1024/5] = 204\) in the right-hand side. By Theorem 2, \(Q_{10}\) does not admit 5 CISTs. If we employ the inequality to check \(Q_n\), for \(2 \leq n \leq 30\), the results are negative to Hasunuma’s conjecture when \(n \in\{10, 12, 14, 20, 22, 24, 26, 28, 30\}\). In fact, there are infinitely such many hypercubes because the inequality (1) does not hold when \(n\) is even and \(n \neq 2^r\) for some positive integer \(r\).

For \(n\) fixed, since each variation (such as \(CQ_n, TQ_n, MQ_n\) and \(LTQ_n\)) has the same degree and the same number of vertices as \(Q_n\), we can obtain similar results. \(\square\)

The Cartesian product of two graphs \(G_1\) and \(G_2\), denoted by \(G_1 \times G_2\), is the graph with \(V(G_1) \times V(G_2)\) as its vertex set and an edge \((u_1, u_2), (v_1, v_2) \in E(G_1 \times G_2)\) if and only if either \((u_1, v_1) \in E(G_1)\) and \(u_2 = v_2\) or \((u_2, v_2) \in E(G_2)\) and \(u_1 = v_1\). It is obvious that \(\kappa(G_1 \times G_2) = \delta(G_1) + \delta(G_2)\), where \(\delta(G)\) denotes the minimum degree of the graph \(G\). Also, Chiu and Shieh [5] showed that \(\kappa(G_1 \times G_2) \geq \kappa(G_1) + \kappa(G_2)\). Thus, if \(G_1\) is a \(k_i\)-connected \(k_i\)-regular graph for \(i = 1, 2\), then \(\kappa(G_1 \times G_2) = k_1 + k_2\).

An \(n\)-dimensional torus network can be defined as the Cartesian product of \(n\) cycles. Hasunuma and Morisaka [10] showed that there are two CISTs in any \(2\)-dimensional torus network. Moreover, they posed the following problem: Can we construct \(n\) CISTs in any \(n\)-dimensional torus network for \(n \geq 3\)? The following example shows that there exists an instance for which an \(n\)-dimensional torus network does not admit \(n\) CISTs.

Example 2: Suppose that \(G_1\) and \(G_2\) are two \(k\)-connected \(k\)-regular graphs for \(k \geq 2\). Let \(n_1 = |V(G_1)|\) and \(n_2 = |V(G_2)|\). Clearly, \(|V(G_1 \times G_2)| = n_1n_2\) and \(G_1 \times G_2\) is a \((2k)\)-connected \((2k)\)-regular graph. Obviously, if \(k \leq 3\), then \([n_1n_2 - 2)/k) \leq \lfloor n_1n_2/k)\rfloor\). Thus, the inequality (1) always holds in this case. For \(k \geq 4\), let \(n_1n_2 = km + \ell\), where \(m \geq 1\) and \(0 \leq \ell < k\). If \(\ell \geq 3\), then

\[
\left\lfloor \frac{n_1n_2 - 2}{k} \right\rfloor = \left\lfloor \frac{(km + \ell - 2)}{k} \right\rfloor > m = \left\lfloor \frac{km + \ell}{k} \right\rfloor = \left\lfloor \frac{n_1n_2}{k} \right\rfloor.
\]

Thus, the result is negative to Hasunuma’s conjecture [8]. Otherwise, the inequality (1) still holds.

In particular, we consider two \(2\)-dimensional torus networks \(G_1 = C_3 \times C_3\) and \(G_2 = C_4 \times C_4\). Note that both \(G_1\) and \(G_2\) are \(4\)-connected \(4\)-regular graphs. Clearly, \(G_1 \times G_2\) is a \(4\)-dimensional torus network and is also a \(8\)-connected \(8\)-regular graph. Since \(|V(G_1 \times G_2)| = 135 = 4 \times 33 + 3\), it follows from Theorem 2 that \(G_1 \times G_2\) does not admit 4 CISTs. \(\square\)

A complete \(k\)-partite graph, denoted by \(K_{n_1,n_2,\ldots,n_n}\), is a graph whose vertex set \(V\) can be partitioned into \(n\) nonempty subsets \(V_1, V_2, \ldots, V_n\) such that \(|V_i| = n_i\) for \(1 \leq i \leq k\) and two vertices \(a, b\) are adjacent if and only if \(a \in V_i\) and \(b \in V_j\) for \(i, j \in \{1, 2, \ldots, k\}\) and \(i \neq j\). In particular, if \(k = 2\) (respectively, \(k = 3\)), the graph is called a complete bipartite graph (respectively, complete tripartite graph). Pai et al. [14] showed that there are \([n/2]\) CISTs in a complete graph with \(n \geq 4\) vertices, there are \([n/2]\) CISTs in a complete bipartite graph \(K_{m,n}\) with \(m \geq n \geq 4\), and there are \([\frac{n+1}{2}]\) CISTs in a complete tripartite graph \(K_{m,n,n+1}\) with \(n \geq n \geq 1\) and \(n_2 + n_1 \geq 4\).

Example 3: Consider the complete tripartite graph \(K_{n,n,n,n,n}\). Clearly, \(K_{n,n,n,n,n}\) is a \(k\)-connected \(k\)-regular graph with \(k = 2n\). Since \(|V| = 3n\), we have \([3n - 2)/n) \leq 3\). Thus, the inequality (1) always holds in this case. However, for \(K_{n,n,n,n,n}\), since \(|V| = 4n\) and it is a \((3n)\)-connected \((3n)\)-regular graph, if \(n \leq 3\), the inequality (1) still holds. On the other hand, we can find that the result is negative to Hasunuma’s conjecture when \(n \geq 4\). Similarly, for \(K_{n,n,n,n,n,n}\), since \(|V| = 5n\) and it is a \((4n)\)-connected \((4n)\)-regular graph, if \(n \leq 2\), the inequality (1) holds; otherwise, the result is negative to Hasunuma’s conjecture. \(\square\)
3. Constructions of CISTs on Chordal Rings

Chordal rings (also called distributed loop networks) are a variation of ring networks. A chordal ring \( CR(N, d) \) is a graph with vertex set \( V = \{0, 1, 2, \ldots, N-1\} \) and edge set \( E = \{(u, v) : u, v \in V \text{ and } (u - v) \equiv 1 \text{ or } d \mod N\} \). Obviously, \(|E| = 2n\). For notational convenience, all arithmetic applied to a chordal ring are taken modulo \( N \). To ensure that every vertex has degree 4, we assume \( N > 4 \) and \( d < N/2 \). Hence, the four edges \((u, u-1), (u, u+1), (u, u-d), (u, u+d)\) are said to be incident to a vertex \( u \in V \) with hop \( 1, +1, -d \) and \(+d\), respectively. Also, \((u, u+1)\) and \((u, u-d)\) are called forward edges of \( u \), and \((u, u-1)\) and \((u, u+d)\) are called backward edges of \( u \). For example, a chordal ring \( CR(16, 4) \) is shown in Fig. 1. Chordal rings were introduced to enhance the reliability and fault-tolerance of ring networks [2], [6], [7]. Previous researches for investigating algorithmic properties of chordal rings can refer to [4], [12], [13]. In particular, Iwasaki et al. [11] proposed a linear time algorithm for constructing four ISTs on a chordal ring. At a later time, Yang et al. [16] make an improvement on the construction by reducing the heights of ISTs.

Apply Theorem 1 to chordal rings, we can easily derive the following property.

**Corollary 3:** Two trees \( T_1 \) and \( T_2 \) are CISTs in a chordal ring \( CR(N, d) \) if and only if \( E(T_1) \cap E(T_2) = \emptyset \), \(|E(T_i)| = N-1\) for \( i = 1, 2 \), and there is no vertex \( v \) such that \( d_1(v) = d_2(v) = 2 \), where \( d_i(v) \) denotes the degree of \( v \) in \( T_i \).

**Theorem 4:** Let \( N = k(d-1) + j \). For each even integer \( k \geq 4 \) and \( 0 \leq j \leq 4 \), there are two CISTs in \( CR(N, d) \).

**Proof:** By \( N = k(d-1) + j \) we mean that all vertices of \( CR(N, d) \) are partitioned into \( k \) groups with \( j \) remanent vertices, where each group consists of \( d - 1 \) vertices. Let

\[
V_i = \{n : i(d-1) \leq n < i(d-1) + d - 2\}
\]

denote the set of vertices in the \( i \)-th group of \( CR(N, d) \). And, let

\[
V_i^- = \{n : i(d-1) \leq n < i(d-1) + d - 3\}.
\]

That is, \( V_i^- \) is the set that removes the last element from \( V_i \). Also, for each hop \( h = +1 \) or \( h = +d \), let \( E_i(h) = \{(n, n+h) : n \in V_i\} \) and \( E_i^- (h) = \{(n, n+h) : n \in V_i^-\} \) be the set of edges incident to vertices of \( V_i \) and \( V_i^- \) with hop \( h \), respectively.

The proof is by constructing two spanning trees, say \( T_B \) and \( T_R \) (for blue tree and red tree), in \( CR(N, d) \) for each case \( j = 0, 1, \ldots, 4 \). Since it is easy to check up with the vertices incident to edges, we only need to provide the set of edges for describing a spanning tree.

**Case** \( j = 0 \). The constructions are

\[
E(T_B) = \bigcup_{i=0,2,\ldots,k-2} E_i(+1) \cup \bigcup_{i=0,2,\ldots,k-2} E_i(+d) \cup E_{k-2}(+d)
\]

and

\[
E(T_R) = \bigcup_{i=1,3,\ldots,k-1} E_i(+1) \cup \bigcup_{i=1,3,\ldots,k-3} E_i(+d) \cup E_{k-1}(+d).
\]

In the above construction of \( T_B \) and \( T_R \), the edge incident to a vertex \( u \in V_i \) is a forward edge. Clearly, for each \( i = 0, 2, \ldots, k-2 \) (respectively, \( i = 1, 3, \ldots, k-1 \)), vertices of \( V_i \) in \( T_B \) (respectively, \( T_R \)) are connected, and vertices of \( V_{i+1} \) in \( T_B \) (respectively, \( T_R \)) are connected through a path induced by \( V_i \). Moreover, there is only one edge \((i(d-1) + d - 2, (i+2)(d-1))\) with hop \( d \) that connects the two groups of vertices \( V_i \) and \( V_{i+2} \) except for \( i = k - 1 \) and \( i = k - 2 \). Indeed, for \( (N-1) \in V_{k-1} \), we have \((N-1, N-1 + d) \notin E(T_B)\); and for \( (N-1) \in V_{k-1} \), we have \((N-1, 0) \notin E(T_B)\). This assures that \( T_B \) and \( T_R \) are connected and acyclic. Also, it is easy to verify that \(|E(T_B)| = |E(T_R)| = N-1\), \( E(T_B) \cap E(T_R) = \emptyset \) and there is no vertex of degree two in both \( T_B \) and \( T_R \). Thus, by Corollary 3, \( T_B \) and \( T_R \) are two CISTs of \( CR(N, d) \).

In the remaining cases, we explicitly describe the sets of \( E(T_B) \) and \( E(T_R) \). Also, we omit the discussion that \( T_B \) and \( T_R \) are CISTs because all arguments are similar to case \( j = 0 \).

**Case** \( j = 1 \). The constructions are

\[
E(T_B) = \bigcup_{i=0,2,\ldots,k-2} E_i(+1) \cup \bigcup_{i=0,2,\ldots,k-2} E_i(+d)
\]

and

\[
E(T_R) = \bigcup_{i=1,3,\ldots,k-1} E_i(+1) \cup \bigcup_{i=1,3,\ldots,k-1} E_i(+d).
\]

**Case** \( j = 2 \). The constructions are

\[
E(T_B) = \bigcup_{i=0,2,\ldots,k-2} E_i(+1) \cup \bigcup_{i=0,2,\ldots,k-2} E_i(+d) \cup \{(0, N-1)\}
\]

and

\[
E(T_R) = \bigcup_{i=1,3,\ldots,k-1} E_i(+1) \cup \bigcup_{i=1,3,\ldots,k-1} E_i(+d).
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\[ E(T_R) = \bigcup_{i=1,3,\ldots,k-1} E_i(+) \bigcup \bigcup_{i=1,3,\ldots,k-1} E_i(+) \bigcup \{(N-2, N-2 + d)\} \]

**Case** \(j = 3\). In this case, we adjust the last group as \(V_{k-1} = \{n: (k-1)(d-1) + 1 \leq n \leq k(d-1)\}\) and let \(E_{k-1}(h) = \{(n, n+h): n \in V_{k-1}\}\) for \(h \in \{+1, +d\}\). Then, the constructions are

\[ E(T_R) = \bigcup_{i=1,3,\ldots,k-1} E_i(+) \bigcup \bigcup_{i=1,3,\ldots,k-1} E_i(+) \bigcup \{(0, N-1), (N-2, N-2 - d)\} \]

and

\[ E(T_R) = \bigcup_{i=1,3,\ldots,k-1} E_i(+) \bigcup \bigcup_{i=1,3,\ldots,k-1} E_i(+) \bigcup \{(N-1, N-1 + d), (N-2, N-2 + d)\} \]

**Case** \(j = 4\). In this case, we adjust the last group as \(V_{k-1} = \{n: (k-1)(d-1) + 2 \leq n \leq (k-1)(d-1) + d\}\) and let \(E_{k-1}(h) = \{(n, n+h): n \in V_{k-1}\}\) for \(h \in \{+1, +d\}\). Then, the constructions are

\[ E(T_R) = \bigcup_{i=1,3,\ldots,k-1} E_i(+) \bigcup \bigcup_{i=1,3,\ldots,k-1} E_i(+) \bigcup \{(0, N-1), (N-2, N-2 - d), (N-3, N-3 - d)\} \]

and

\[ E(T_R) = \bigcup_{i=1,3,\ldots,k-1} E_i(+) \bigcup \bigcup_{i=1,3,\ldots,k-1} E_i(+) \bigcup \{(N-1, N-1 + d), (N-2, N-2 + d), (N-1 - d, N-2 - d)\} \]

According to Theorem 4, it is easy to compute the diameters of CISTS. Therefore, we have the following theorem.

**Theorem 5**: Let \(N = k(d-1) + j\). For each even integer \(k \geq 4\) and \(0 \leq j \leq 4\), the diameters of \(T_B\) and \(T_R\) constructed in Theorem 4 are as follows:

\[ \text{diam}(T_B) = \begin{cases} \frac{N}{2} & \text{for } j = 2, 4 \\ \lfloor \frac{N}{2} + 1 \rfloor & \text{for } j = 0, 1, 3 \end{cases} \]

and

\[ \text{diam}(T_R) = \begin{cases} \lfloor \frac{N}{2} + 1 \rfloor & \text{for } j = 0, 1, 2, 4 \\ \lfloor \frac{N}{2} + 2 \rfloor & \text{for } j = 3 \end{cases} \]

4. Concluding Remarks

As we have mentioned earlier, research results related to CISTs are limited. This paper gives preliminary results of CISTs on some interconnection networks. According to Examples 1 and 2, the future research can be continued to study the CIST problem on Cartesian product of \(n\)-connected graphs with large \(n\), e.g., high dimensional tori, hypercubes and their variations. Finally, we specifically state that the construction of two CISTs in 4-regular chordal rings has not been solved completely.

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