LETTER

Discrete Spherical Laplacian Operator

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SUMMARY Laplacian operator is a basic tool for image processing. For an image with regular pixels, the Laplacian operator can be represented as a stencil in which constant weights are arranged spatially to indicate which picture cells they apply to. However, in a discrete spherical image the image pixels are irregular; thus, a stencil with constant weights is not suitable. In this paper a spherical Laplacian operator is derived from Gauss's theorem; which is suitable to images with irregular pixels. The effectiveness of the proposed discrete spherical Laplacian operator is shown by the experimental results.

key words: spherical camera model, discrete spherical image, Laplacian operator

1. Introduction

Omnidirectional images have wide field of view (FOV) and are widely used for mobile robots, visual surveillance and so on. Omnidirectional images are usually captured by a fisheye camera [1] or a catadioptric camera [2]. While the content in spherical image is invariant for camera rotation, the distortion relative to view direction in the captured omnidirectional image is changeable according to the projection types of omnidirectional cameras. A full-view image sensor, which consists of a pair of fisheye cameras pointing to the opposite, is shown in Fig. 1 (a); the pair of fisheye images captured by the sensor are shown in Fig. 1 (b).

In the fisheye images, while the space vertical lines passing through the center of the fisheye images appear as straight lines, those approaching to the boundary of the FOV appear as curves. This implies that directly computing features from the input planar fisheye image, such as using the conventional Laplacian operator shown in Fig. 2 (a) [8], will obtain results that vary with camera rotation. How to compute features being invariant for camera rotation from omnidirectional images is a basic problem for omnidirectional image processing.

Conventionally, an omnidirectional image is processed via a spherical model because a perspective image cannot represent a wider FOV than a hemisphere [3]–[6]. The Laplacian of spherical image is computed at a spherical polar coordinate system as

Fig. 1 (a) Full-view image sensor which consists of a pair of fisheye cameras. (b) A pair of fisheye images captured by the full-view image sensor.

Fig. 2 The Laplacian operators for a regular rectangular grid (a), and for a regular hexagonal grid (b). ε is the grid spacing.

$$\nabla^2 I_s(\theta, \varphi) = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial I_s(\theta, \varphi)}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 I_s(\theta, \varphi)}{\partial \varphi^2} \quad (1)$$

where $I_s(\theta, \varphi)$ is a spherical image on a unit sphere and (θ, φ) are the polar angle and azimuth angle, respectively. However, in Eq. (1) a singular point exists for $\theta = 0$, or, π ; it is called the pole problem in this paper.

An alternative method for omnidirectional image processing is first to map it to a spherical image, and then to process the spherical image directly. During the mapping to a spherical image, the distortion of the input omnidirectional image is normalized. A Gaussian image with quasi-uniform cells, which is obtained by dividing an icosahedron iteratively, has been proposed to represent a spherical image [7]. In the Gaussian image, each vertex has six neighboring vertices, except for twelve vertices that have five neighboring vertices; these twelve vertices correspond to that of the original icosahedron. This means that almost all the cells (i.e., pixels) at the vertices are hexagonal in shape.

For regular hexagonal grids, the Laplacian operator can also be well approximated by the stencil with constant weights, as shown in Fig. 2 (b) [8]. However, it is known that a sphere cannot be tessellated by uniform cells; that is, the pixels of a discrete spherical image are *irregular*. This

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Fig. 3 Properties of Voronoi cells.

implies that applying the Laplacian operator shown in Fig. 2 to a discrete spherical image will give rise to computational errors.

A natural method is to modify the Laplacian operator so that it is suitable for *irregular* grids in the Gaussian image. Here, a Voronoi cell is associated with each grid point [9]. Given a set of N grid points $\{P_1, P_2, \dots, P_N\}$ on the unit sphere S , the Voronoi cell k associated with P_k is defined by

$$cell_k = \{p - P_k \leq |p - P_l| \quad \forall p \in S \quad \text{and} \\ l \in \{1, 2, \dots, k-1, k+1, \dots, N\}\} \quad (2)$$

where $|x_1 - x_2|$ is the distance between points x_1, x_2 measured along the surface of the sphere.

A Voronoi cell has the following properties [10].

Let P_0 and P_1 be neighboring grid points, as shown in Fig. 3. The cell wall shared by P_0 and P_1 is the perpendicular bisector of the circular arc between the two grid points.

A cell wall is a segment of a great circle. The end points of a wall are the Voronoi corners, the points where three cell walls intersect.

The position of a corner is computed using three grid points. For points P_0, P_1 and P_2 , as shown in Fig. 3, in terms of the definition of a Voronoi cell, the Voronoi corner C is computed as

$$C = \frac{(P_2 - P_0) \times (P_1 - P_0)}{|(P_2 - P_0) \times (P_1 - P_0)|} \quad (3)$$

In this paper, a Gaussian image with the above Voronoi cells is called a Spherical Centroidal Voronoi Tessellation (SCVT) image. In a previous work [11], a data structure for the representation of SCVT image has been proposed; the algorithms of finding the neighboring pixels given a pixel of a spherical image, and mapping between spherical coordinate and spherical image pixel are given also.

In this paper, we realize the Laplacian operator in a SCVT image, and name the modified Laplacian operator as spherical Laplacian operator. The most distinctive feature of the proposed spherical Laplacian operator is that it is not necessary to assume that the shape of pixels is regular polygon.

The remainder of this paper is organized as follows. We explain the spherical Laplacian filter in Sect. 2. After showing the experimental results in Sect. 3, we present the conclusions in Sect. 4.

Fig. 4 Example of cell with six neighbors.

2. Discrete Spherical Laplacian Filter

Given a SCVT image I , a pixel, that is, a Voronoi cell, V_i is associated with a three-dimensional Cartesian coordinate $P_i(x_i, y_i, z_i)$, an area A_i , neighboring pixels $\{P_{i,j}\}$, and Voronoi corner points $\{C_{i,j}\}$. Figure 4 shows an example of cell with six neighbors.

For region V_i , in terms of the Gauss' theorem we have

$$\int_{V_i} \nabla^2 I dA = \oint_{\partial V_i} \text{grad} I ds \quad (4)$$

where ∂V_i denotes the boundary of V_i , and $\nabla^2 I$ denotes the Laplacian of the cell P_i .

Let $L = \nabla^2 I$. We have

$$\int_{V_i} L dA = \oint_{\partial V_i} \text{grad} I ds = \oint_{\partial V_i} \frac{\partial I}{\partial n} ds \quad (5)$$

Equation (5) can be approximated by

$$L(P_i)A_i = \sum_{j=1}^{K_{P_i}} \left\{ \frac{I(P_{i,j}) - I(P_i)}{l_j} \right\} a_j \quad (6)$$

where $L(P_i)$ is the Laplacian at point P_i ; l_j denotes the length between $P_{i,j}$ and P_i ; a_j denotes the length of the edge joining the Voronoi corners $C_{i,j-1}$ and $C_{i,j}$, respectively; K_{P_i} is the number of neighbors of P_i ; for pixels with six neighbors, $K_{P_i} = 6$; for pixels with five neighbors, $K_{P_i} = 5$; $C_{i,0} = C_{i,K_{P_i}}$.

Finally, we acquire the Laplacian at point P_i as follows.

$$L_s(P_i) = \frac{1}{A_i} \sum_{j=1}^{K_{P_i}} \{I_s(P_{i,j}) - I_s(P_i)\} \frac{a_j}{l_j} \quad (7)$$

Note that in the above computation we don't assume that the shape of cells is a regular polygon.

3. Experiments

In this section, experiments are carried out to evaluate the performance on the computational accuracy of the proposed spherical Laplacian operator for a synthesized image.

The performance of the proposed spherical Laplacian

Fig. 5 Synthesized spherical image and the computed Laplacian image. (a) Synthesized image. (b) The ground truth of the Laplacian of the synthesized image. (c) The Laplacian computed by the proposed SLO. (d) The Laplacian computed by the HLO.

operator (SLO) is compared with that of the conventional regular rectangular Laplacian operator (RLO), as shown in Fig. 2 (a), which is applied to an input fisheye image, and the regular hexagonal Laplacian operator (HLO), as shown in Fig. 2 (b), which is applied to a discrete spherical image, based on a synthesized image with known ground truth.

At first, we define a function on a sphere as

$$I_S(\theta, \phi) = 127(\sin \theta \sin \phi + 1) \quad (8)$$

The Laplacian of the function on the sphere can be computed easily as

$$\nabla^2 I_S(\theta, \phi) = -254 \sin \theta \cos \phi \quad (9)$$

Figure 5 (a) shows the synthesized spherical image $I_S(\theta, \phi)$ represented by Eq. (8), where the two circular images correspond to the two hemispheres of the spherical image and the rectangular images on the right is the corresponding SCVT image represented in terms of the format of [11]. Figure 5 (b) shows the Laplacian, $\nabla^2 I_S(\theta, \phi)$, of the synthesized spherical image computed by Eq. (9) (i.e., the ground truth), where the two circular images correspond to the two hemispheres of the Laplacian image and the rectangular images is the corresponding SCVT Laplacian image represented by the format of [11]. Figure 5 (c) shows the Laplacian image computed by the SLO in terms of Eq. (7), and Fig. 5 (d) shows that computed by the HLO assuming that the cells are regular hexagon. Comparing Fig. 5 (c) and Fig. 5 (d), we can visually confirm that approximating the spherical cells as regular hexagon gives rise to greater computational errors. Note that both the brightness function represented by Eq. (8) and its Laplacian represented by Eq. (9) are even; therefore, the two hemispheres shown in Fig. 5 are the same.

Next, we present the result in the case of omnidirectional image being processed by the RLO. As an example,

Fig. 6 (a) Fisheye image with hemispherical FOV. (b) The corresponding Laplacian image computed by the RLO.

Fig. 7 Images of absolute errors of the SLO, the HLO and the RLO are shown in (a), (b) and (c), respectively.

Table 1 Comparison of absolute errors

	Minimum	Maximum	Average	Variance
SLO	0.000001	0.609926	0.236119	0.031152
HLO	0.000282	11682.434540	141.324132	539401.825061
RLO	0.003807	15.221245	4.469827	14.431126

the hemispherical image shown in Fig. 5 (a) is mapped to a fisheye image under equidistance projection [1], as shown in Fig. 6 (a). The corresponding Laplacian image computed by the RLO is shown in Fig. 6 (b). It is difficult to visually find the difference between Fig. 5 (c) and Fig. 6 (b).

In the computed Laplacian images above, shown in Fig. 5 and Fig. 6, plus values are indicated in yellow, minus in blue and zero in black.

Next, we present the quantitative evaluation for the SLO, the HLO and the RLO by considering the absolute errors in comparison with the ground truth. The images of absolute errors of the SLO, the HLO and the RLO for a hemispherical FOV are shown in Fig. 7 (a), (b) and (c), respectively. The brighter pixels correspond to the greater errors. Table 1 shows the maximum, minimum and average of the absolute errors. Obviously, the SLO outperformed both the HLO and the RLO in the computational accuracy.

The relative errors for the SLO, the HLO and the RLO are also investigated. Figure 8 (a), (b) and (c) show the image of the relative errors of the SLO, the HLO and the RLO for a hemispherical FOV, respectively. The brighter pixels correspond to the greater errors. The great relative errors of the SLO appear in the places where the ground truth values are zero, as shown in Fig. 8 (a). The relative errors are great all over for the HLO, as shown in Fig. 8 (b). As expected, the relative errors become greater as approaching to the boundary of the fisheye image due to great distortions, as shown

Fig. 8 Images of relative errors of the SLO, the HLO and the RLO are shown in (a), (b) and (c), respectively.

Table 2 Comparison of relative errors

	Minimum	Maximum	Average	Variance
SLO	0.000000	0.617547	0.004789	0.000652
HLO	0.000004	16685.879636	3.558920	7887.478967
RLO	0.006182	0.060005	0.030626	0.000238

in Fig. 8 (c). Table 2 shows the maximum, minimum and average of the relative errors.

4. Conclusions

In this paper we realized the Laplacian operator in a discrete spherical image. The proposed method is called spherical Laplacian operator. In comparison with the conventional methods, the distinctive characteristic of the proposed spherical Laplacian operator is that the assumption of a regular size of pixels is not necessary. Therefore, it is suitable for processing images with irregular pixels, such as a discrete spherical image. As shown in the experimental results, the proposed spherical Laplacian operator outperforms the

conventional methods in the computational accuracy.

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