Article

Study of Euler's Belt Formula

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Abstract

Euler's belt formula for a fixed circular drum is well known and is an important formula for mechanical engineers. However, there is little book in which belt equation for a drum of any contour is discussed precisely. This study was carried out to confirm the validity of the Euler's belt formula applied to a drum of any contour. The calculated result was verified to ascertain whether it satisfied the static equilibrium equations of the force and moment. The results perfectly satisfied the static equilibrium equations of the force. As for an equilibrium of the moment, it was found in the case of a belt wrapped around an elliptical drum that the moment from the belt tension was equal to the integrated moment caused by frictional force and normal force along wrapped length. Normal force distribution under a belt on a cylindrical surface was evaluated experimentally by a pressure sensitive film which was found to correspond qualitatively to the theory. As an example, an equation to calculate the normal pressure and tension of a belt wrapped on an elliptical drum was derived.

Keywords

belt friction, curvature, contact pressure, statics, contour integral

1 Introduction

Euler's belt formula is well known and is an important formula for mechanical engineers. It appears in all machine design text books [1,2]. However, there are few books in which a calculation method of the frictional force or normal force against a shape other than a cylinder is explained. The calculation method has to be verified so that it satisfies the static equilibrium of force and moment to an entire body. Equations were derived by which the frictional force of a belt wrapped two and three times around a cylindrical surface can be calculated [3,4]. By using these equations, the self-locking condition of belt was clarified. A one way clutch utilizing the self-locking property of belt was devised [5]. The effectiveness of such a clutch was confirmed even in an off-centered condition [6]. The belt equation for any profiled surface was examined in the previous paper [7]. The frictional force of the belt on an any profiled surface can be evaluated by integrating the curvature with respect to arc length s . However, as a mechanical system, the calculated result did not corroborate whether it satisfied the fundamental static equilibrium equations of the force and the moment. Generally, in most textbooks dealing with engineering mechanics, a normal force N from the drum to the belt is eliminated so as to obtain belt tension, therefore the profile of a normal force distribution is not clearly explained. Since the normal force N is not shown explicitly, an equilibrium equation of force and moment cannot be

calculated. In order to confirm an equilibrium of force and moment, the profile of normal force distribution is also necessary. As far as the calculation of belt friction is concerned, the equilibrium of the force and the moment were not perfectly verified especially in the case of a drum of any contour. This study was carried out to confirm the validity of the Euler's belt formula in terms of equilibrium of forces and moment when it was applied to a noncircular cylinder. Mathematical proof verified that the results perfectly satisfied the static equilibrium equations of the force. As far as equilibrium equation of the moment is concerned, it was verified against an elliptical surface. Therefore this proved that the moment due to the frictional force and the normal force obtained by Euler's equation is equal to the moment by the belt tensions.

2 Theory

Figure 1 shows a flexible belt wrapped around a noncircular cylinder. T_1 and T_2 denote tension of the belt. The belt tension T_2 is assumed to be larger than T_1 , i. e. $T_2 > T_1$. The profile of the cylinder is uniquely determined by a position vector $\mathbf{r}$. The position vector $\mathbf{r}$ is a vector function of a scalar variable t .

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} \quad (1)$$

An element of arc denoted by ds and a tangent vector $\mathbf{u}$ and a principal normal $\mathbf{m}$ to the curve $\mathbf{r}$ are given by [8],

Fig. 1 A belt wrapped around a noncircular shaft. ρ is the radius of curvature and an element of arc is subtended by the angle $d\theta$

$$ds = |\dot{\mathbf{r}}| dt, \quad \mathbf{u} = \frac{d\mathbf{r}}{ds} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}, \quad \mathbf{m} = \frac{\mathbf{u}'}{|\mathbf{u}'|} \quad (2)$$

Derivatives with respect to the arc length s are always denoted by primes such as $\mathbf{u}'$ while derivatives with respect to the parameter t are denoted by dots such as $\dot{\mathbf{r}}$. An outward normal vector $\mathbf{n}$ is opposite direction to $\mathbf{m}$.

$$\mathbf{n} = -\mathbf{m} \quad (3)$$

The coefficient of the friction between the belt and the cylinder is μ . N denotes the normal force distribution per unit length as in the text book [2]. The equilibrium requirement in normal direction yields

$$N ds = N \rho d\theta = T d\theta \quad (4)$$

Making use of Eq. (4), Normal load N can be expressed in terms of the curvature of the curve denoted by κ and the tension of belt T .

$$N = T / \rho = \kappa T \quad (5)$$

The equilibrium requirement in the tangential direction yields

$$\mu N ds = \mu \kappa T ds = dT, \quad \mu \kappa ds = dT / T \quad (6)$$

According to the vector analysis [8], the curvature κ is given by

$$\kappa = \frac{|\mathbf{du}|}{ds} = \frac{\sqrt{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})(\ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{3/2}} \quad (7)$$

Making use of Eqs. (6) and (7), belt tension T at any point can be calculated. Substituting Eq. (1) into Eqs. (2) and (3) gives

$$\left. \begin{aligned} ds &= \sqrt{\dot{x}^2 + \dot{y}^2} dt, \quad \mathbf{u} = \frac{\dot{x}\mathbf{i} + \dot{y}\mathbf{j}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \\ \mathbf{m} &= \frac{-\dot{y}\mathbf{i} + \dot{x}\mathbf{j}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \quad \mathbf{n} = \frac{\dot{y}\mathbf{i} - \dot{x}\mathbf{j}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \end{aligned} \right\} \quad (8)$$

An angle between x axis and an outward unit normal vector $\mathbf{n}$ is denoted by θ .

$$\left. \begin{aligned} \theta &= \tan^{-1} \left(\frac{-\dot{x}}{\dot{y}} \right), \quad \tan \theta = \frac{-\dot{x}}{\dot{y}} \\ \sin \theta &= \frac{-\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \quad \cos \theta = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \end{aligned} \right\} \quad (9)$$

Differentiating the angle θ with respect to parameter t yields,

$$\dot{\theta} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \quad (10)$$

On the other hand, substituting Eq.(1) into (7) yields,

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \quad (11)$$

Taking an absolute value of the numerator in Eq. (10) and making

Table 1 Value of variable at contact boundary

belt tension, T	T_1	T_2
position vector, $\mathbf{r}$	$\mathbf{r}_1$	$\mathbf{r}_2$
parameter, t	0	t_2
contacting arc length, s	0	L
angle of normal vector $\mathbf{n}$, θ	θ_1	θ_2

Fig. 2 Analytical model of a belt wrapped around an ellipse with major radius a and minor radius b .

use of Eq. (8), the product κds in Eq. (6) can be transformed into an function of θ .

$$\mu \kappa ds = \mu \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} dt = \mu \dot{\theta} dt \quad (12)$$

For simplicity, $t = 0$ at a boundary with $\theta = \theta_1$ and $T = T_1$, the tension of belt T and normal force N at any point can be calculated by using Eqs. (5), (6) and (12).

$$T = T_1 \exp \left(\mu \int_0^t \kappa ds \right) = T_1 \exp \{ \mu (\theta - \theta_1) \} = T_1 \exp (\mu \phi) \quad (13)$$

where, $\phi = \theta - \theta_1$

The values of an each variable defined at the contact boundary are summarized in Table 1. The corresponding coordinate system is shown in Fig. 2.

2.1 Verification of equilibrium of force in x and y direction

The summation of the normal force distribution $N ds$ and accompanied frictional force $\mu N ds$ should be balanced to the belt tension T_1 and T_2 . The summation of forces caused by $N ds$ in the y direction can be calculated by

$$\Gamma_{ny} = \int_0^L N \mathbf{n} \cdot \mathbf{j} ds \quad (14)$$

Substituting Eqs. (5), (8) and (11) into Eq. (14) gives

$$\Gamma_{ny} = -T_1 \int_0^{t_2} \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \cdot \dot{x} \exp \left(\mu \int_0^t \kappa ds \right) dt \quad (15)$$

Making use of Eqs. (9) and (10), Eq. (15) can be expressed in terms of θ .

$$\Gamma_{ny} = T_1 \int_{\theta_1}^{\theta_2} \sin \theta \exp \{ \mu (\theta - \theta_1) \} d\theta \quad (16)$$

Equation (16) can be integrated to be

$$\Gamma_{ny} = T_1 \left\{ \frac{\cos \theta_1 - \mu \sin \theta_1 + e^{\mu(\theta_2 - \theta_1)} (\mu \sin \theta_2 - \cos \theta_2)}{1 + \mu^2} \right\} \quad (17)$$

In the same manner, the summation of forces caused by the frictional force in the y direction can be determined by

$$\Gamma_{uy} = \int_0^L \mu N(-\mathbf{u}) \cdot \mathbf{j} ds \quad (18)$$

Substituting Eqs. (5), (8) and (11) into Eq. (18) gives

$$\Gamma_{uy} = -\mu T_1 \int_0^{\theta_2} \frac{\dot{x}\dot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \dot{y} \exp\{\mu(\theta - \theta_1)\} d\theta \quad (19)$$

Making use of Eqs. (9) and (10), Eq. (19) can be expressed in terms of θ .

$$\Gamma_{uy} = -\mu T_1 \int_{\theta_1}^{\theta_2} \cos\theta \exp\{\mu(\theta - \theta_1)\} d\theta \quad (20)$$

Equation (20) can be integrated to be

$$\Gamma_{uy} = \mu T_1 \left\{ \frac{\sin\theta_1 + \mu \cos\theta_1 - e^{\mu(\theta_2 - \theta_1)} (\sin\theta_2 + \mu \cos\theta_2)}{1 + \mu^2} \right\} \quad (21)$$

Note that $T_1 \exp\{\mu(\theta_2 - \theta_1)\} = T_2$, making use of Eqs.(17) and (21), y directional component of the force by the contact force yields

$$\Gamma_y = \Gamma_{ny} + \Gamma_{uy} = T_1 \cos\theta_1 - T_2 \cos\theta_2 \quad (22)$$

From a geometrical consideration, the belt tension vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ are expressed as

$$\mathbf{T}_1 = T_1 (\sin\theta_1 \mathbf{i} - \cos\theta_1 \mathbf{j}), \quad \mathbf{T}_2 = T_2 (-\sin\theta_2 \mathbf{i} + \cos\theta_2 \mathbf{j}) \quad (23)$$

Making use of Eqs. (22) and (23), the equilibrium equation of force in y direction yield

$$\Gamma_y + (\mathbf{T}_1 + \mathbf{T}_2) \cdot \mathbf{j} = 0 \quad (24)$$

In the same manner, the summation of force caused by frictional force in the x direction can be determined by

$$\Gamma_{nx} = \int_0^L N \mathbf{n} \cdot \mathbf{i} ds \quad (25)$$

Substituting Eqs. (5), (8) and (11) into Eq. (25) gives

$$\Gamma_{nx} = T_1 \int_0^{\theta_2} \frac{\dot{x}\dot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \cdot \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \exp\{\mu(\theta - \theta_1)\} d\theta \quad (26)$$

Making use of Eqs. (9) and (10), Eq. (26) can be expressed in terms of θ .

$$\Gamma_{nx} = T_1 \int_{\theta_1}^{\theta_2} \cos\theta \exp\{\mu(\theta - \theta_1)\} d\theta \quad (27)$$

Equation (27) can be integrated to be

$$\Gamma_{nx} = -T_1 \left\{ \frac{\sin\theta_1 + \mu \cos\theta_1 - e^{\mu(\theta_2 - \theta_1)} (\sin\theta_2 + \mu \cos\theta_2)}{1 + \mu^2} \right\} \quad (28)$$

Next, the summation of the force caused by friction in the x direction can be calculated by

$$\Gamma_{ux} = \int_0^L \mu N(-\mathbf{u}) \cdot \mathbf{i} ds \quad (29)$$

Substituting Eqs. (5), (8) and (11) into Eq. (29) gives

$$\Gamma_{ux} = -\mu T_1 \int_0^{\theta_2} \frac{\dot{x}\dot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \exp\{\mu(\theta - \theta_1)\} \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} d\theta \quad (30)$$

Making use of Eqs. (9) and (10), Eq. (30) can be expressed in terms of θ .

$$\Gamma_{ux} = \mu T_1 \int_{\theta_1}^{\theta_2} \sin\theta \exp\{\mu(\theta - \theta_1)\} d\theta \quad (31)$$

Equation (31) can be integrated to be

$$\Gamma_{ux} = \mu T_1 \left\{ \frac{\cos\theta_1 - \mu \sin\theta_1 + e^{\mu(\theta_2 - \theta_1)} (\mu \sin\theta_2 - \cos\theta_2)}{1 + \mu^2} \right\} \quad (32)$$

Making use of Eqs. (28) and (32) and noting $T_1 \exp\{\mu(\theta_2 - \theta_1)\} = T_2$, the x directional component of the force caused by contact force yield

$$\Gamma_x = \Gamma_{nx} + \Gamma_{ux} = -T_1 \sin\theta_1 + T_2 \sin\theta_2 \quad (33)$$

Making use of Eqs. (23) and (33), the equilibrium equation of force in x direction yield

$$\Gamma_x + (\mathbf{T}_1 + \mathbf{T}_2) \cdot \mathbf{i} = 0 \quad (34)$$

Eqs. (24) and (34) prove that contact force obtained by the Euler's belt formula perfectly satisfies the static equilibrium equation.

2.2 Verification of equilibrium of moment

An equation of equilibrium of moment is

$$\mathbf{r}_1 \times \mathbf{T}_1 + \mathbf{r}_2 \times \mathbf{T}_2 + \int_{t_1}^{t_2} \{\mathbf{r} \times N \mathbf{n} + \mathbf{r} \times \mu N(-\mathbf{u})\} ds = 0 \quad (35)$$

The first two terms in Eq. (35) are of the moment caused by belt tension T_1 and T_2 and is denoted by M_B . It is expressed in components as

$$M_B = T_2 (x_2 \cos\theta_2 + y_2 \sin\theta_2) - T_1 (x_1 \cos\theta_1 + y_1 \sin\theta_1) \quad (36)$$

The integrand of the last term in Eq. (35) is a small moment generated by contact force working on a small arc ds . Let denote this moment by dM_C .

$$dM_C = \mathbf{r} \times N \mathbf{n} ds + \mathbf{r} \times \mu N(-\mathbf{u}) ds \quad (37)$$

Substituting Eqs. (1) and (8) in Eq. (37) yield

$$dM_C = -N \{ x \dot{x} + y \dot{y} + \mu (x \dot{y} - \dot{x} y) \} dt \quad (38)$$

If the position vector $\mathbf{r}$ is defined by Eq. (1), Eq. (38) can be integrated with respect to parameter t .

However, the equilibrium Eq. (35) cannot be calculated at this stage. Therefore, as an example, an elliptical drum was chosen to calculate Eqs. (35) and (38).

2.3 Verification of equilibrium of moment to elliptical surface

An elliptical profile is defined by

$$\mathbf{r} = x(t) \mathbf{i} + y(t) \mathbf{j} = a \cos t \mathbf{i} + b \sin t \mathbf{j} \quad (39)$$

Substituting Eq. (39) into Eq. (8) gives

$$\left. \begin{aligned} ds &= \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt, \quad \mathbf{u} = \frac{-a \sin t \mathbf{i} + b \cos t \mathbf{j}}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \\ \mathbf{m} &= -\frac{b \cos t \mathbf{i} + a \sin t \mathbf{j}}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}, \quad \mathbf{n} = \frac{b \cos t \mathbf{i} + a \sin t \mathbf{j}}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \end{aligned} \right\} \quad (40)$$

Substituting Eq. (39) into Eq. (11) gives

$$\kappa = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \quad (41)$$

Curvilinear integral of the curvature κ on an ellipse gives

$$\int_{t_1}^t \kappa ds = \tan^{-1} \left(\frac{a}{b} \tan t \right) - \tan^{-1} \left(\frac{a}{b} \tan t_1 \right) = \phi \quad (42)$$

Paying attention to the components of normal vector $\mathbf{n}$ in Eq. (40), Eq. (42) means an intersection angle between vectors $\mathbf{n}$ and $\mathbf{n}_1$. Let us denote this angle by ϕ as in Eq. (42). Substituting Eq. (39) into Eq. (38) gives

$$dM_C = -N \{ (b^2 - a^2) \sin t \cos t + \mu ab \} dt \quad (43)$$

As can be seen in Eqs. (5), (6), (41) and (42), the normal load distribution N in Eq. (43) yields

$$N = \kappa T = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} T_1 \exp(\mu \phi) \quad (44)$$

Substituting Eq. (44) into Eq. (43) gives

$$dM_C = -\frac{ab T_1 \{ (b^2 - a^2) \sin t \cos t + \mu ab \}}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \exp(\mu \phi) dt \quad (45)$$

Equation (45) is too complex to integrate with respect to t . Therefore, the moment M_B in Eq. (36) is calculated. Making use of Eq. (9) and (39), following results were obtained.

$$\left. \begin{aligned} \sin \theta &= \frac{a \sin t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \\ \cos \theta &= \frac{b \cos t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \end{aligned} \right\} \quad (46)$$

Substituting Eqs. (39) and (46) into Eq. (36) gives

$$M_B = T_1 ab \left(\frac{\exp(\mu \phi_2)}{\sqrt{a^2 \sin^2 t_2 + b^2 \cos^2 t_2}} - \frac{1}{\sqrt{a^2 \sin^2 t_1 + b^2 \cos^2 t_1}} \right) \quad (47)$$

where $\phi_2 = \tan^{-1} \left(\frac{a}{b} \tan t_2 \right) - \tan^{-1} \left(\frac{a}{b} \tan t_1 \right)$

Here, letting a function $f(t)$ as

$$f(t) = \frac{\exp(\mu \phi)}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \quad (48)$$

Making use of Eq. (42), differentiating the angle ϕ in the numerator of Eq. (48) with respect to t yields

$$\frac{d\phi}{dt} = \frac{ab}{a^2 \sin^2 t + b^2 \cos^2 t} \quad (49)$$

On the other hand, differentiating the denominator of Eq. (48) with respect to t yields

$$\frac{d}{dt} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} = \frac{-(b^2 - a^2) \sin t \cos t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \quad (50)$$

Making use of Eqs. (49) and (50), the differentiation of $f(t)$ in Eq. (48) with respect to t yields

$$\frac{d}{dt} f(t) = \exp(\mu \phi) \frac{\{(b^2 - a^2) \sin t \cos t + \mu ab\}}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \quad (51)$$

Here, comparing Eq. (51) with Eq. (45), it is noted that Eq. (45) can be written as

$$dM_C = -abT_1 \frac{d}{dt} f(t) dt \quad (52)$$

Integrating Eq. (52) with respect to t from t_1 to t_2 gives

$$\begin{aligned} M_C &= -abT_1 f(t) \Big|_{t_1}^{t_2} \\ &= -abT_1 \left(\frac{\exp(\mu \phi_2)}{\sqrt{a^2 \sin^2 t_2 + b^2 \cos^2 t_2}} - \frac{1}{\sqrt{a^2 \sin^2 t_1 + b^2 \cos^2 t_1}} \right) \end{aligned} \quad (53)$$

The equilibrium equation of moment is the sum of Eqs. (47) and (53). It becomes

$$M_B + M_C = 0 \quad (54)$$

Equation (54) means that the equilibrium of moment can be perfectly satisfied by the Euler's belt formula. Letting $a = b = r$ in Eqs. (47) and (53), they become familiar results.

$$M_B = T_1 r [\exp\{\mu(t_2 - t_1)\} - 1], \quad M_C = -M_B \quad (55)$$

Fig. 3 Model of thin-walled cylindrical pressure vessel

3 Discussion

The normal load N caused by a taut belt wrapped around a curved surface with curvature κ is calculated by means of Eq. (5). The same equation can also be derived by considering a force balance between the hoop stress and the internal pressure p of a thin-walled pressure vessel as shown in Fig. 3. According to the theory of a thin-walled cylindrical pressure vessel with radius r , height h and thickness t , the hoop stress σ can be obtained by [9]

$$\sigma = \frac{pr}{t} \quad (56)$$

Equation (56) can be deformed to

$$\frac{\sigma ht}{r} = \kappa T = ph = N \quad (57)$$

In Eq. (57), the sum of the hoop stress σ over the cross sectional area ht is interpreted as belt tension T and the sum of the normal force caused by an internal pressure p per unit length in the circumferential direction is interpreted as N . Here, in the case of a noncircular cylinder, the same relation is applied by using the radius of an osculating circle or the circle of the curvature of the curve as the radius of pressure vessel. Similar consideration was also done by Lubarda [10].

In a usual application of the Euler's belt formula, a profile of the normal load N is not seriously considered. However, it is important to know the magnitude of contact force N between wires in a wire rope to calculate the internal friction of the wire rope. Provided the tension of the wire and its curvature κ are known, the corresponding normal force N can easily be calculated by using Eq. (5). As far as mechanical analysis of wire rope is concerned, three dimensional equations of equilibrium are used [11]. However, they are not so easy to solve. On the contrary, by neglecting the effect of bending rigidity of wire, Eq. (5) might be useful for estimating contact force from the tension of wire because the curvature of wire is easily obtained. To confirm the validity of Eq. (5), a simple experiment was carried out. The test device is shown in Fig. 4. A belt and pressure sensitive film (Fuji

Fig. 4 Test device to evaluate normal load distribution on cylinder

Fig. 5 Pressure pattern obtained with tension $T = 500$ N

Fig. 6 Image of normal pressure distribution of a belt wrapped halfway around cylinder and the same tension T applied at both ends

Film PRESSCALE LLW) were wrapped halfway around a steel cylinder with radius 30 mm as shown in Fig. 4. The same tension T was applied to both ends of the belt. The corresponding normal pressure N seems to become the maximum at the top of the cylinder. However, neglecting the effect of friction, Eq. (5) indicates that the magnitude of normal pressure N is constant because the curvature of the cylinder is constant. A sample of the test results is shown in Fig. 5. The texture of the belt was footprinted on a film. It should be noted that the color becomes gradually fainter as it approaches the middle of the footprint. This fact may be attributed to the effect of friction. Providing the coefficient of friction $\mu = 0, 0.1, 0.2$ and 0.3 , profiles of the corresponding normal pressure distribution are calculated. Figure 6 shows the result. The magnitude of the contact pressure decreases with each increment of the coefficient of friction. It also decreases gradually as it approaches the top of the cylinder. The pressure pattern shown in Fig. 5 and the calculated results shown in Fig. 6 correspond well qualitatively.

Making use of Eqs. (42) and (44), the normal force N can be calculated for the case where a belt wrapped halfway around an ellipse and the same tension T_1 applied at both ends.

$$N = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} T_1 \exp \left\{ -\mu \tan^{-1} \left(\frac{a}{b} \tan t \right) \right\} \quad (58)$$

To make the normal force N in dimensionless form, it was divided by an apparent normal pressure $N_m = T_1/a$,

$$\frac{N}{N_m} = \frac{a^2 b}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \exp \left\{ -\mu \tan^{-1} \left(\frac{a}{b} \tan t \right) \right\} \quad (59)$$

Substituting $t = 0$ into Eq. (59), the ratio N/N_m at the end of an ellipse can be obtained.

$$\frac{N_{\text{end}}}{N_m} = \left(\frac{a}{b} \right)^2 \quad (60)$$

On the contrary, substituting $t = \pi/2$ into Eq. (59), the ratio N/N_m at the top of the ellipse can be obtained.

$$\frac{N_{\text{top}}}{N_m} = \frac{b}{a} \exp \left(-\mu \frac{\pi}{2} \right) \quad (61)$$

So, making use of Eqs. (60) and (61), the ratio of normal force N at the end to top of an ellipse is given by

$$\frac{N_{\text{end}}}{N_{\text{top}}} = \left(\frac{a}{b} \right)^3 \exp \left(\mu \frac{\pi}{2} \right) \quad (62)$$

According to Eq. (60), it is evident that the normal force N at the end of the ellipse becomes infinity as the minor axis b approaches

Fig. 7 Profiles of non-dimensional normal pressure distribution of a belt wrapped halfway around ellipse and the same tension T_1 applied at both ends

Fig. 8 Change of belt tension ratio T/T_1 with ratio x/a

0. It should be noted that the dimension of normal force N is N/m. An effective loading length on the ellipse decreases with a decrement of minor axis b , which makes the normal force N even greater. However, the equilibrium of force is maintained. Provided that the coefficient of friction $\mu = 0, 0.3$, $b/a = 0.4, 0.7$ and 1.0 , some examples are calculated by using Eq. (59). Figure 7 shows the results. It is clearly seen that the dimensionless normal pressure N/N_m in the vicinity of both ends of the ellipse increases with an decrement of b/a . According to Eq. (61), it can be seen that the dimensionless normal pressure N/N_m at the top of the ellipse is proportional to the value of ellipticity b/a . From these results, it should be noted that normal pressure N increases greatly with a decrement of a radius of the curvature at the contact boundary of the belt. This characteristic is important to avoid damage caused by belt fastening which sometimes occurs in logistics. To avoid damage caused by belt fastening, the curvature at the contact boundary should be larger.

When considering the practical use of belts in logistics, the change in belt tension due to friction is also important. Assuming that something is fastened to a truck with a belt or rope by applying tension T_1 , the change of non-dimensional belt tension T/T_1 on an ellipse is considered. Figure 8 shows some results. The coefficient of friction is $\mu = 0.3$. There was a great difference in normal pressure N in accordance with ellipticity. However, there is not a great deal of difference in belt tension by ellipticity. Since the ratio of belt tension is determined by an angle between the normal vectors on the ellipse under consideration, the non-dimensional belt tension T/T_1 at the top, i. e. $x/a = 0$ and the left end of an ellipse, i. e. $x/a = -1$ become $\exp(-\mu\pi/2) = 0.624$ and $\exp(-\mu\pi) = 0.390$ respectively regardless of ellipticity. Except for these two points, the curve of T/T_1 changes in accordance with the directional change of normal vector on the ellipse.

4 Conclusion

Based upon the Euler's belt formula, the frictional property of a belt was examined for the case of a noncircular cylinder. The calculated result was verified to determine whether it satisfied the static equilibrium equations of the force and moment. Evidence from this study showed that the results perfectly satisfied the static equilibrium equations of the force. As for an equilibrium of the moment, it was found in the case of a belt wrapped around an elliptical surface that the moment from the belt tension is equal to the integrated moment caused by frictional force and normal force along wrapped length. Normal force distribution under a belt on a cylindrical surface was evaluated experimentally by means of a pressure sensitive film. This was found to correspond qualitatively to the theory. An equation to calculate normal pressure and tension of a belt wrapped on an elliptical drum was derived.

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