1 Introduction

Theoretical understanding and practical application of thin film lubrication by means of a tribochemically generated surface layer has become increasingly vital for further reduction of friction and wear in automobile engines, fine and precision machines, and micro-mechanisms [1, 2]. An increase in the film thickness of the elastohydrodynamic lubrication (EHL) regime below 10 nm due to increased viscosity was first measured by Guangteng and Spikes, using a steel ball rolling on a glass disk lubricated with several synthetic base fluids [3]. Recently, Itoh et al. [4] showed that continuous transition from bulk viscosity to confined viscoelasticity and its dependence on shear rate and temperature could be measured using the fiber wobbling method. They found that the effective viscosity of polyalpaholefins (PAOs) increases to about three times the bulk viscosity as the confined film thickness decreases from 40 nm (PAO4) to zero. In contrast, Shibazaki et al. [5] evaluated the rheological properties of various engine oils, with and without additives, using the resonance shear measurement method. They showed that when only the viscosity modifier of polymethacrylate was used, the viscosity of the mineral base oil increased to more than 100 times the bulk viscosity as the confined film thickness decreased from more than 100 nm to zero. Other engine oils, including anti-wear additives and friction modifiers, showed rapid increases in viscosity (of more than 100 times) above a 180 nm film thickness. This significant increase in viscosity, with a decrease in film thickness from more than 100 nm, seems to contribute to fluid lubrication in the otherwise mixed lubrication regimes near dead-center positions, even when a low-viscosity base oil is used [6].

Over the past few decades, extensive studies have been carried out on texture lubrication technologies that can increase the load capacity and reduce the friction coefficient by changing the boundary/mixed lubrication regimes to the fluid lubrication regime. Among these, Adach and Kato et al. found that a super-low friction coefficient, on the order of 0.001 or less, could be obtained by employing a running-in process of water lubrication between SiC–SiC plates [7–10]. They suggested that the tribochemical reaction product SiO₂ and its hydride are dissolved in water to act as a lubricant and noted that the contact surfaces become very smooth due to tribochemical wear.
Despite the number of experimental studies on thin film lubrication regimes, few theoretical studies have been carried out to quantitatively evaluate lubrication characteristics based on continuum mechanics [11, 12]. Tichy formulated a modified Reynolds equation considering heterogeneous viscosity across the film thickness [12], showing correction factors for the modified Reynolds equation which must be numerically calculated from a given local viscosity function. For a simple viscosity model of stepwise change in the adsorbed layer, a closed form of pressure and shear stress for a one-dimensional flow equation incorporating a variable viscosity function was derived. It is a fundamental principle of fluid lubrication theory that the dominant factor governing the pressure generated in a thin lubricant film due to wedge and squeeze actions is the viscous effect, whereas the influence of the inertia effect is negligible. The shear stress acting on the local fluid layer at \( z \) is assumed to be a product of the viscosity and shear rate at \( z \) as a Newtonian fluid. The pressure in the film does not change in the \( z \) direction. From the force balance of pressure and shear stress acting on a small rectangular element of \( dx \times dz \times dy \), we could easily obtain the equilibrium equation in the \( x \) direction for an infinitely small element:

\[
\frac{d}{dx}\left( \mu \frac{du}{dx} \right) + \frac{dp}{dx} = 0.
\]

(1)

Next, we derived the lubrication equation with the viscosity expressed as a function of \( z \) in a thin surface layer. It is well known that the viscosity at high pressure is pressure-dependent. However, this effect is ignored here because, expressed as a Barus equation, it can be treated independently in the Reynolds equation for a variable viscosity.

The contemporary approach to the Reynolds equation is to solve it numerically. To make the modified Reynolds equation useful, however, it was important to express the viscosity using a simple function of \( z \), and thereby to obtain a Reynolds equation whose modified factors could be expressed as a closed form function. For this reason, we used a viscosity model of the form:

\[
\mu_z = \mu_0 + \frac{z^2 (\mu_0 - \mu_b)}{(z + z_c)^2}.
\]

(2)

Here, \( \mu_0 \) is the bulk viscosity and \( \mu_b \) is the higher viscosity at a solid surface. Figure 2(a) shows the variation of the normalized viscosity of \( \mu_z/\mu_0 \) as a function of \( z/z_c \), which is expressed in Eq. (2). The local viscosity \( \mu_z \) increases from the bulk viscosity \( \mu_b \) to \( \mu_0 \) in inverse proportion to a quadratic function: the viscosity increment is \((\mu_0 - \mu_b)/9 \) at \( z = 2z_c \), and is \((\mu_0 - \mu_b)/4 \) at \( z = z_c \).

The viscosity function in Eq. (2) was selected because the measured effective viscosity can be approximately expressed in this form by choosing three parameters for \( \mu_0, \mu_b, \) and \( z \). The experimental effective viscosities of PAO lubricants as functions of the confined film thickness \( h \) from [4] are presented in Fig. 2(b), while Fig. 2(c) shows Eq. (2) as a functions of \( z \). The effective viscosity versus film thickness can be properly expressed in this form by using the parameters \( \mu_0, \mu_b, \) and \( z \). However, because a high viscosity surface layer usually forms on both solid surfaces, there remains the question of whether Eq. (2) can be applied to only one side of the mating surfaces. As described below, the total effect of the high viscosity layers on both of the solid surfaces can be equivalently expressed by the one-side-surface layer model, via the judicious selection of \( z \).

2 Derivation of modified Reynolds equation for thin film lubrication with variable viscosity surface layer

2.1 Analytical model

Figure 1 shows the cross section of a thin lubricant film of thickness \( h \), where the lower solid surface is moving at velocity \( U \) relative to the upper stationary surface. The absolute coordinate system \( O-xyz \) is fixed on the same plane as the moving surface, and the \( x \) and \( z \) axes are aligned with the moving direction and film thickness direction, respectively. Although the intended outcome was to obtain a Reynolds equation for two-dimensional fluid flow, a one-dimensional flow equation incorporating a variable viscosity function was considered first for better understanding.

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the continuity condition is given by:

\[ \mu_z \frac{\partial z}{\partial x} + \frac{\partial}{\partial y} \left( \frac{\partial Q_y}{\partial x} \right) = - \frac{\partial}{\partial y} \left( \frac{\partial Q_y}{\partial y} \right) - \frac{\partial 
abla}{\partial y} - V \int_0^1 dR_z d \Omega \] 

where \( f_0, f_2, g_2, \) and \( g_3 \) are given in Appendix 1. Next, the flow volume \( q_x \) at \( x \) in the bearing gap is written as:

\[ q_x = \int_0^1 f_2 \, dx = - \frac{1}{\mu_z \, dx} \frac{\partial Q_x}{\partial x} \, U. \]  

Here, \( Q_1 \) and \( Q_2 \) are given by:

\[ Q_1 = \frac{1}{2} \left( \frac{h^2}{\mu_z} + \frac{h^2}{\mu_z} g_2 (h) + f_2 (h) \right), \] 

\[ Q_2 = f_2 (h) \left( h - g_2 (h) \right) + \frac{h^2}{\mu_z} + g_2 (h) \frac{h^2}{\mu_z} \left( r - 1 \right) g_2 (h) \frac{h^2}{\mu_z} \left( r - 1 \right). \] 

The quantities \( I_1 \) and \( I_2 \) are given in Appendix 1.

Therefore, under the continuity condition that \( \frac{dQ_y}{dx} = 0 \), the lubrication equation for a one-dimensional flow model considering only wedge action is given by:

\[ \frac{1}{\mu_z} \frac{d}{dx} \left[ Q_1 (h) \frac{\partial Q_y}{\partial x} \right] = \frac{\partial}{\partial y} \left( \frac{\partial Q_y}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial Q_y}{\partial y} \right) - \frac{\partial \nabla}{\partial y} \] 

In Fig. 1, if we consider fluid flow \( v \) in the \( y \) direction and the squeeze film effect due to time variation of the bearing gap, the continuity condition is given by:

\[ \frac{\partial}{\partial y} \left( \frac{\partial Q_y}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial Q_y}{\partial y} \right) = - \frac{\partial}{\partial y} \left( \frac{\partial Q_y}{\partial y} \right) - \frac{\partial \nabla}{\partial y} \] 

If we align the \( x \) axis in the direction of the sliding velocity \( U \), the flow in the \( y \) direction is just the Poiseuille flow expressed in the same form as in the \( x \) direction. Therefore, the generalized two-dimensional lubrication equation for pressure \( p \) in the bearing gap \( h \), including the variable viscosity in the surface layer on moving surface, can be expressed as:

\[ \frac{\partial}{\partial x} \left( \frac{\partial Q_y}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial Q_y}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial Q_y}{\partial y} \right) - \frac{\partial \nabla}{\partial y} = - \frac{\partial}{\partial y} \left( \frac{\partial Q_y}{\partial y} \right) - \frac{\partial \nabla}{\partial y}. \] 

Although Eqs. (6) and (7) are complex, it is confirmed that

Eqs. (8) and (9) become the ordinary Reynolds equations for the constant viscosity model when \( r_1 = 1 \).
friction testing of a sapphire hemisphere on a CNx thin film disk in PAO4 lubricant and found that the initial composite root-mean-square (rms) roughness firstly decreases from 11.1 to 7.6 nm, accompanying a friction coefficient decrease from 0.052 (boundary lubrication regime) to 0.028 (mixed lubrication regime close to lubrication regime). The film thickness increases concomitantly from 13.5 to 19.7 nm. They suggested that the decrease in the friction coefficient was attributed to the generation of an absorbed layer of PAO4 lubricant in the transformed layer of CNx. A gradual decrease of the friction coefficient, observed thereafter, was also ascribed to the continuous generation of a tribochemical adsorbed layer. In this sapphire–CNx interface it is speculated that the increase of the viscosity in the surface layer is greater than three times as the rheological change in the confined PAO4 film in the glass ball–Si wafer interface [4].

In contrast, Norihsa et al. [14, 15] reported that a recent slide-way guide in machine tools was operated under a fluid lubrication regime with a low friction coefficient at a very low sub-mm/s sliding speed, using a machine oil with long-chain alkyl acid phosphate as an additive. Demonstrations using a pin-on-ring friction test apparatus showed that the boundary friction regime could be changed to a velocity-dependent friction regime after a few-hour running-in process. The results also revealed that after the running-in process, peak parts of the surface roughness were removed, suggesting that the remaining flat and valley areas were covered with enough high viscosity adsorbed film to accommodate thin film lubrication.

Based on such experimental evidence, low-friction-coefficient texture lubrication can be considered to be elucidated by theoretical analysis of tapered/tapered land bearings, as shown in Figs. 3(a) and (b), taking into account a high viscosity surface layer. The bearing performance of various micro-textures has been theoretically analyzed using tapered and step bearing models or more sophisticated and strict models [8, 16–26]. However, no study has treated a texture model by considering the effect of a high viscosity surface layer. To address the paucity of research in this area, we calculated the lubrication characteristics of small-scale tapered/tapered land bearings with a high viscosity surface layer, using the derived modified Reynolds equation.

Figure 3(a) depicts a small tapered bearing model for the converging wedge part of rough surface or fabricated textures, whereas Fig. 3(b) depicts a small tapered land bearing model for the converging wedge and land parts of a rough truncated surface through running-in. Isotropic surface roughness has a converging wedge region (CWR) and a diverging wedge region (DWR) with the same probabilities. If we calculate the pressure distribution of the repeated bearing configuration of the CWR and DWR symmetrically, positive and negative pressures appear symmetrically. However, it is known that negative pressure cannot be fully generated due to cavitation, particularly in a staved lubricant condition, as that in parallel slider bearings. As the positive pressure increases compared with the ambient pressure, this negative pressure effect becomes negligible. In addition, because a tribochemical layer can be generated by shearing action under high positive pressure, an adsorbed molecular layer will hardly be generated under negative pressure in the DWR. Accordingly, it is reasonable to ignore the negative pressure in the DWR and to consider only the CWR and flat surface regions of tapered/tapered land regions bearings, as shown in Fig. 3. Although the widths of actual textures are often smaller than their lengths, the side flow effect is considered negligible as it does not qualitatively affect the lubrication characteristics and decreases significantly as the bearing gap decreases [21]. Therefore, the total load capacity of a textured parallel slider should be evaluated by considering the reduction due to the side flow effect and the areal density of the tapered bearing textures.

The pressure distribution $p$ of a tapered bearing having length $L$, leading gap $h_L$, and trailing gap $h_T$, as shown in Fig. 3, was obtained by solving the one-dimensional modified Reynolds equation, Eq. (8). The bearing load capacity, $W$, mean load capacity $W_m$, friction force on a moving surface $F_0$ ($>0$), and friction force on a stationary surface $F_s$ ($>0$), are calculated from the formulæ:

$$W = \int_0^L p dx, \quad W_m = W/L, \quad F_0 = -\int_0^L \tau dx, \quad F_s = -\int_0^L \tau dx.$$

Then, the friction coefficients on moving and stationary surfaces, $f_{c0}$ and $f_{c0}$, are calculated from $f_{c0} = F_0/W$ and $f_{c0} = F_s/W$, respectively.

3.2 Lubrication characteristics of a small tapered bearing

The lubricant thickness of the tapered bearing in Fig. 3(a) is determined by the taper angle $\theta = (h_T - h_L)/L$ and the trailing gap $h_T$. In the numerical calculation, the $x$-coordinate of the bearing length was divided into more than 400 equally spaced discrete positions, and the pressures at these representative positions were calculated from the discretized Reynolds equation (8) using the pressure boundary condition $p(x = 0) = p_{0w}$ ($x = L = 0$). In the following calculations, we chose a bearing length $L = 10 \mu$m, taper angle $\theta = (h_T - h_L)/L = 0.001$ rad ($h_T - h_L = 10$ nm), sliding velocity $U = 0.1$ m/s, and bulk viscosity $\mu_b = 0.01$ Pa·s as the standard bearing parameter values, unless otherwise noted.

![Fig. 3](image-url) Models for textures. (a) Tapered bearing model for the converging wedge part of a rough surface or fabricated texture, (b) Tapered land bearing model for the converging wedge and land parts of a rough truncated surface.

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To determine the critical thickness of the surface layer $z_c$ consistently with typical measured viscosity characteristics, we first calculated the effective viscosity $\mu_e = h F_c (U \ell)$ for a uniform film thickness of $h (\theta = 0)$ by using viscosity function $\mu_z$ when $z_i = 10$ for both cases of $r_{\mu} = 10$ and 100. Figure 4 plots the viscosity function $\mu_z$ and the calculated effective viscosity function $\mu_{e_z}$ together with the viscosity function $\mu_z$ with $z_i = 20$ nm. It can be seen from Fig. 4 that the viscosity $\mu_z$ versus the uniform film thickness $h$ calculated from the viscosity model with $z_i = 10$ is fairly close to the viscosity model function with $z_i = 20$ nm for both cases of $r_{\mu} = 10$ and 100. According to the measured data [4, PAO4 undergoes a 25% increase in viscosity at ~20 nm. Thus, we chose $z_i = 10$ nm as a standard value for the viscosity model function. The effects of the viscosity ratio $r_{\mu} = \mu_z / \mu_{e_z}$, trailing bearing gap $h_c$, slider length $L$, and taper angle $\theta$ on the load capacity $W$ and friction coefficients, $f_{ch}$ and $f_{c0}$, were investigated numerically.

Figures 5(a), (b), and (c) show the load capacity $W$, friction coefficient on a moving surface $f_{ch}$, and friction coefficient on a stationary surface $f_{c0}$, respectively, as functions of the trailing bearing gap $h_c$ for various viscosity ratios $r_{\mu}$ of 1, 10, and 100. From Fig. 5(a) we note that the increased ratio of $W$ due to the increase in $r_{\mu}$ from 10 to 100 is ~10, almost proportional to $r_{\mu}$ for $h_c < \sim 20$ nm, but the increased ratio of $W$ decreases to 5 around $h_c = 60$ nm. The increased ratio of $W$ due to the increase in $r_{\mu}$ from 1 to 10 is almost proportional to $r_{\mu}$ in a small $h_c$ region below 10 nm, but the increased ratio of $W$ decreases to 2 at $h_c = 60$ nm.

As can be seen in Figs. 5(b) and (c), $f_{ch}$ and $f_{c0}$ decrease with increasing $r_{\mu}$, but the decreasing rate is not remarkable and becomes negligibly small at a small $h_c$ of less than 10 nm. The notable feature is that $f_{ch}$ and $f_{c0}$ decrease significantly as the trailing gap $h_c$ decreases and become less than 0.01 at $h_c = 20$ nm if $r_{\mu} > 10$. When $h_c = 4$ nm, $f_{ch} = 0.002$ and $f_{c0} = 0.001$. In general, the load capacity generated by incompressible hydrodynamic lubrication is proportional to $(L / h_c)^2$, while the friction force is proportional to $L / h_c$. Thus, the friction coefficient becomes proportional to $h_c / L$. Therefore, the super-low friction coefficient of 0.001 experimentally observed in texture lubrication for nanometer-scale lubricating film can be largely ascribed to the running-in process for creating a smooth surface. However, to realize a thin film lubrication regime in a parallel slider bearing, it is not sufficient to only reduce the rms value of the surface roughness height to the level of the minimum bearing gap. A high pressure must also be generated in the bearing texture, high enough to deform the bearing surface and to reduce the height difference $h_c$ of all the textures on the mating slider surfaces, similar to the action of the EHL mechanism.

To confirm the texture height leveling effects due to generated bearing pressure, we assume a case where a load capacity of $W = 10^5$ N/m is generated to resist the applied load in one taper bearing texture. From the right-hand axis in Fig. 5(a), $W = 10^5$ N/m corresponds to the mean pressure of $W_p = 10^5$ N/m$^2$. Considering the texture areal rate and reduction effect due to finite width from the taper bearing model, we assume here that 1% of $W_p$ is used for the average load capacity of the parallel slider bearing. Then, the mean load capacity of the slider bearing becomes 1 MPa or 10 atm. From Fig. 5(a), we note that $W = 10^5$ N/m can be generated at $h_c < 10$ nm for $r_{\mu} = 10$ and $h_c = 20$ nm for $r_{\mu} = 100$.

Therefore, the generated pressures at $h_c = 10$ nm for $r_{\mu} = 1$ and 10 and those at $h_c = 20$ nm for $r_{\mu} = 10$ and 100 are depicted in Figs. 6(a) and (b). To show the effects of $z_i$ on the pressure and load capacity, the pressures for $z_i = 5$ and 20 nm are given for comparative purposes. From Fig. 6(a), we note that when $h_c = 10$ nm, the maximum pressure $p_{\text{max}}$ is ~0.15 GPa for $r_{\mu} = 10$ and $z_i = 10$ nm, while $p_{\text{max}} = 0.25$ GPa for $r_{\mu} = 1$. When $h_c = 20$ nm, $p_{\text{max}} = 0.22$ GPa for $r_{\mu} = 10$ and $p_{\text{max}} = 0.18$ GPa for $r_{\mu} = 100$ and $z_i = 10$ nm, as seen from Fig. 6(b). Here, if we estimate the deformation of the bearing surface using the Hertz contact model for contact between the sphere and the flat surface, the
relationship between the elastic deformation $\delta$ and the contact radius $a$ is given by $\delta = (\pi p_{\text{max}}/2E^*)a$, where $E^*$ is the composite Young's modulus. If we assume that the pressure radius $a = L/2 = 5 \mu\text{m}$, and $E^* = -200 \text{ GPa}$, then we can estimate the elastic deformations near the maximum pressure of $p_{\text{max}} = 0.15, 0.025, 0.022,$ and $0.18 \text{ GPa}$ to be $\delta = 5.9, 1.0, 1.0, 7.1 \text{ nm}$, respectively. Therefore, if we suppose that the tapered bearing textures $zc = 10 \text{ nm}$ is generated, the texture surface can deform by up to 6 nm if $h_T$ decreases to 10 nm. This means that the several-nm-irregularity of texture height is removed by the generated pressures; almost all of the textures can attain noncontact lubrication condition, even if the texture heights have irregularities of several nm and the maximum surface roughness height is reduced to less than 10 nm by the running-in process. If the high viscosity boundary layer does not form ($zc = 10$ and $z_c = 10 \text{ nm}$), the texture surface can deform by up to 6 nm if $h_T$ decreases to 10 nm. It can be said that a small estimation $zc = 10 \text{ nm}$ is generated, a similarly high load capacity can be achieved at a higher film thickness of 20 nm, although $f_{c_0}$ and $f_3$ tend to $-0.01$.

As can be seen from Figs. 6(a) and (b), a twofold increase of $z_c$ results in a 50% increase in load capacity at $h_T = 20 \text{ nm}$, but a 35% increase at $h_T = 10 \text{ nm}$. It can be said that a small estimation error of $z_c$ does not notably affect the bearing performance.

Because stiffness is very important practical aspect of bearing performance, the stiffness $-dW/dh_T$ calculated from the W versus $h_T$ relationship in Fig. 5(a) is shown in Fig. 7. Note that $-dW/dh_T$, becomes $2 \times 10^{11} \text{ N/m}^3$ at $h_T = 10 \text{ nm}$ and 20 nm when $r_\mu = 10$ and 100, respectively. As the bearing stiffness per unit area is $-dW/dh_T/L$, the total stiffness $K$ of a parallel slider with an area of $S$ is given by $K = -\eta S \times dW/dh_T/L$, where $\eta$ is the efficiency of the taper bearing to the parallel slider. If we assume $\eta = 0.01$, the stiffness of a parallel slider of $S = 5 \text{ cm}^2$ becomes $K = -\eta S \times dW/dh_T/L = 10^8 \text{ N/m}$.

As stated above, if the tapered bearing length $L$ is increased, it is estimated that the load capacity increases, whereas the friction coefficients decrease. Therefore, the effects of a bearing lengths $L$ of 10, 50, and 250 $\mu\text{m}$ on $W_L$, $f_{c_0}$, and $f_3$, are shown in Figs. 8(a), (b), and (c), respectively, for the cases of $r_\mu = 1$ and 10. The thin and thick lines indicate the cases for $r_\mu = 1$ and 10, respectively. From Fig. 8(a), it can be seen that $W_L$ can be increased dramatically by increasing slider length particularly in the higher $h_T$ region even when $r_\mu = 1$. When $r_\mu = 1$, the ratio $W_{L,10 \mu\text{m}} /W_{L,10 \mu\text{m}}$ is 55, 46, 25, and 14 and $W_{L,250 \mu\text{m}} /W_{L,10 \mu\text{m}}$ is 17, 11, 6, and 4 at $h_T = 60, 40, 20$, and 10 $\mu\text{m}$, respectively. In contrast, when $r_\mu = 10$, $W_{L,10 \mu\text{m}} /W_{L,10 \mu\text{m}}$ is 52, 35, 18, and 9, and $W_{L,250 \mu\text{m}} /W_{L,10 \mu\text{m}}$ is 12, 8, 4, and 2.4 at $h_T = 60, 40, 20$, and 10 $\mu\text{m}$, respectively. $W_{L,50 \mu\text{m}} /W_{L,10 \mu\text{m}}$ increases more than the square of the increase of $L$ in the higher $h_T$ region because the maximum pressure tends to occur near the central region of the bearing, resulting in an increase in the efficiency of generating the load capacity. As $h_T$ decreases, however, the maximum pressure position shifts to the trailing edge and the leading region cannot be effectively used for pressure generation. This is the reason why $W_{L,50 \mu\text{m}} /W_{L,10 \mu\text{m}}$ and $W_{L,250 \mu\text{m}} /W_{L,50 \mu\text{m}}$ decrease as $h_T$ decreases.
Fig. 8 Effects of the taper bearing length $L$ on the load capacity and friction coefficients when $\theta = 0.001$ rad, $U = 0.1$ m/s, $\mu_b = 0.01$ Pa-s, and $z_c = 10$ nm (Thick line: $r_\mu = 10$, thin line: $r_\mu = 1$). (a) Load capacity $W$ vs. trailing gap $h_T$ (b) Friction coefficient $f_{c0}$ on a moving surface vs. $h_T$, (c) Friction coefficient $f_c$ on a stationary surface vs. $h_T$

When $L$ is increased from 10 to 50 and 250 µm, the load capacity $W$ increases by more than 10 and 50 times, respectively, at a $h_T$ greater than 20 nm. Therefore, a 1 MPa mean load capacity can be achieved, and the compensation effect of the surface height irregularity can be generated at a higher $h_T$. Moreover, the lubrication regime can be achieved at a higher bearing gap $h_T$ if the bearing length $L$ is increased beyond $L = 10$ µm. When $L = 250$ µm and $r_\mu = 10$, it can be seen from Fig. 8(a) that $W = 1.1 \times 10^5$ N/m at $h_T = 20$ nm and $W = 5.0 \times 10^4$ N/m at $h_T = 40$ nm. Thus, the mean bearing capacity becomes $W_m (= W/L) = 0.44 \times 10^4$ N/m at $h_T = 20$ nm, and $W_m = 0.2 \times 10^4$ N/m at $h_T = 40$ nm. These mean load capacities decrease by 0.44 and 0.2 from the cases where $L = 10$ µm and $h_T = 10$ nm for $r_\mu = 10$ and $h_T = 20$ nm for $r_\mu = 100$, as can be seen from Fig. 5(a). Examining the generated pressure when $L = 250$ µm at $h_T = 20$ nm and 40 nm reveals that $p_{max} = -0.16$ GPa at $h_T = 20$ nm and $p_{max} = -0.05$ GPa at $h_T = 40$ nm. If we consider that the effective Hertz pressure radius $a$ is the bearing region where generated pressure is decreased to half of $p_{max}$, then $a = 25$ and $40$ µm at $h_T = 20$ and 40 nm, respectively. If we assume that the bearing pressure is similar to the Hertzian pressure with a radius $a$, then the elastic deformation $b$ of the bearing becomes 31 and 16 nm at $h_T = 20$ and 40 nm, respectively. Therefore, although the mean load capacity is decreased from the case where $L = 10$ µm and $h_T = 10$ nm, the leveling effect due to elastic deformation is much stronger, and, hence, a noncontact hydrodynamic lubrication condition can be achieved at a greater bearing gap when the bearing length increases.

From Figs. 8(b) and (c), it can be seen that $f_{c0}$ and $f_c$ can be decreased dramatically by increasing the bearing length $L$. This notable result is likely mainly caused by an increase in load capacity, although other factors will also contribute. It should be noted that with an increase in $L$, $f_{c0}$ decreases significantly in the higher $h_T$ region. In particular, when $L = 250$ µm, $f_{c0}$ takes an almost constant value of -0.001 at any $h_T$ less than 60 nm regardless of the value of $r_\mu$. In contrast, increasing $L$ from 10 to 50 µm causes $f_{c0}$ to decrease by more than one order of magnitude, in the region of $h_T = 20$–60 nm. Increasing $L$ from 50 to 250 µm further decreases $f_{c0}$ by more than one order of magnitude, and it becomes -0.0004 at $h_T = 60$ nm, even when $r_\mu = 1$. When $L = 250$, $f_{c0}$ decreases to zero and then exhibits a small negative value as $h_T$ decreases to less than 20 nm. This result is not changed by increasing the discretized number of the bearing length up to 1000 in the numerical calculation.

Although this is not easy to understand, it can be inferred that because the generated high-pressure region becomes concentrated on the trailing edge area as $h_T$ decreases to zero, $f_{c0}$ becomes negative due to the pressure gradient term in Eq. (11). It should be noted that although increasing $r_\mu$ does not result in a noticeable decrease in either $f_{c0}$ or $f_c$, increasing $r_\mu$ can significantly contribute to increasing $h_T$ which makes noncontact fluid lubrication possible, as described above.

From these calculated results, it can be said that if we fabricate longer tapered bearing textures on a parallel slider bearing, we can achieve a lower friction coefficient condition at a higher bearing gap. These calculated results enable us to understand the excellent experimental results in water lubrication for the SiC–SiC sliding system reported by the Kato and Adachi group [8–10]. In [8], Wang et al. showed that the friction coefficient can decrease to -0.001 after the running-in process is applied for surface roughness reduction. Despite a circular pitted texture with a pit depth of a few µm, it seems that the load capacity reaches its maximum when the pit diameter is approximately 300–400 µm, and at an even smaller pit depth. The viscosity of water is one order of magnitude less than the 0.01 Pa·s used in this calculation, but the sliding velocity seems to be more than a few times larger than 0.1 m/s. Assuming that effective value of $r_\mu$ in the tribochemical adsorbed layer is more than 10, and considering that the stationary friction coefficient $f_c$ was measured instead of $f_{c0}$, these experimental results seem to be consistent with our theoretical results. In [9], Adachi et al. showed that the friction coefficient was decreased to less than 0.0005 and the critical load was increased to more than 4000 N by 40 µm × 40 µm square pit textures. The surface roughness was decreased from 22.9 nm Ra to 4.5 nm Ra after the running-in process. From the $f_{c0}$ for $L = 50$ µm and $r_\mu = 10$ in Fig. 8(c), we note that $f_{c0}$ decreases to 0.0005 at $h_T = 16$ nm and further decreases to 0.0001 at $h_T = 5$ nm. From the $W$ for $L = 50$ µm and $r_\mu = 10$ shown in Fig. 8(a), we note that the mean load capacity of $W_m = 10^4$ N/m² can be obtained at $h_T = 14$ nm; $W_m$ increases twofold and fourfold as $h_T$ decreases to 8.8 and 5 nm, respectively. Therefore, it seems that these calculated results are compatible with the experimental results in [9].

The lubrication condition of a sphere-on-disk sliding interface with PAO4 lubricant after the running-in process reported in [13] was evaluated using a single tapered bearing model whose length was assumed to be 34 µm, similar to the Hertz contact diameter, under an applied load of 0.2 N. The
The tapered bearing characteristics shown in Figs. 6 to 8 are the calculated results when the taper angle \( \theta = 0.001 \text{ rad} \). This taper angle was selected because the load capacity almost reaches its maximum in the region of \( h_T \). This taper angle was selected because the load capacity almost reaches its maximum in the region of \( h_T \). Therefore, a mechanism of self-optimization of the taper angle initially set to 0.001 would tend to decrease to 0.0005. From Fig. 9(c), it can be seen that \( f_{c_h} \) decreases to less than 20 nm when \( \theta \) increases to 0.005. Although not shown here, it was found from numerical calculation that \( f_{c_h} \) takes a small negative value in the region of an \( h_T \) less than 40 nm when \( \theta = 0.01 \text{ rad} \). Although the reason for the small negative value of \( f_{c_h} \) is not clear, the near-zero value of \( f_{c_h} \) is consistent with the experimental results [9, 10] if the pit depth-to-length ratio can be considered to correspond to the taper angle.

Although not shown in Fig. 9, the peculiar tendencies of \( f_{c_f} \) and \( f_{c_{ch}} \) to decrease to negative values can be obtained when \( \tau_r \) ~ 1 in the uniform viscosity conditions, similarly to the case in Fig. 8. To confirm the validity of the numerical solutions of the modified Reynolds equation, Eq. (8), the analytical solutions of the conventional one-dimensional Reynolds equation for a uniform viscosity film were investigated. As described in Appendix 2, the consistency of the numerical solutions presented in this paper with the conventional analytical solutions was confirmed. Moreover, the general tendencies of \( W \), \( F_0 \), \( F_h \), \( f_{ch} \), and \( f_{c_h} \) with respect to the design parameters \( L \), \( \theta \), and \( h_T \) shown in Figs. 8 and 9 could be explained clearly.

3.3 Lubrication characteristics of small tapered land bearing

When a flat parallel slider exhibits lubrication characteristics after the running-in process, the top area of the surface roughness is worn out and the truncated surface can be modeled as an aggregation of tapered land bearings [14, 15]. Therefore, the lubrication characteristics of tapered land bearings shown in Fig. 9(b) were investigated under the same standard lubrication conditions with a high peak pressure of more than 0.1 GPa, elastic deformation causes \( \theta \) to decrease, so, for example, a taper angle initially set to 0.001 would tend to decrease to 0.005. Therefore, a mechanism of self-optimization of \( \theta \) seems to operate as \( h_T \) decreases.

To understand the opposite trend between \( f_{c_f} \) and \( f_{c_{ch}} \), the friction forces \( F_0 \) and \( F_h \) are shown in Fig. 9(b). The thick and thin lines correspond to \( F_0 \) and \( F_h \), respectively. Note that both \( F_0 \) and \( F_h \) decrease with an increase in \( \theta \), because the leading area of the bearing becomes ineffective for generating friction force. However, the decreasing rate of \( F_h \) is much greater than \( F_0 \), particularly in a smaller region of \( h_T \). Interestingly, when \( \theta = 0.005 \text{ rad} \), \( F_0 \) becomes zero, dropping to a small negative value when \( h_T \) < 20 nm, similarly to the case where \( L = 250 \mu \text{m} \) and \( \theta = 0.001 \text{ rad} \) (see Fig. 8(c)). From Fig. 9(c), it can be seen that \( f_{c_h} \) tends to increase with increasing \( \theta \) in a smaller \( h_T \) region, whereas \( f_{c_0} \) decreases with increasing \( \theta \) in the entire region of \( h_T \).

These opposite tendencies with respect to \( \theta \) can be understood if the pit depth-to-length ratio can be considered to correspond to the taper angle.

Fig. 9 Effects of taper angle \( \theta \) on (a) load capacity \( W \), (b) friction forces \( F_0 \) and \( F_h \) and (c) friction coefficient \( f_{c_0} \) and \( f_{c_h} \) when \( L = 50 \mu \text{m} \), \( U = 0.1 \text{ m/s} \), \( \mu_s = 0.01 \text{ Pa·s} \), \( r_s = 10 \), and \( z_c = 10 \text{ nm} \).
In Fig. 10, the solid line indicates the results for a tapered bearing (LLR = 0) shown in Fig. 5, while the dashed line is for the case where LLR = 1/5 (Lz = 8 µm, Ll = 2 µm), and the dotted line is for the case where LLR = 2/5 (Lz = 6 µm, Ll = 4 µm). As can be seen from Fig. 10(a), the relationships of W with hT and rµ remain almost unchanged as LLR increases to 2/5 for both cases of rµ = 1 and 10. More precisely, compared with the taper bearing case, W increases slightly in the region of hT = 2–30 nm when LLR = 1/5, as LLR increases to 2/5, W tends to decrease slightly from the tapered bearing in the region hT > 15 nm.

From Figs. 10(b) and (c), we note that fc0 and fc1 do not change from their values for the tapered bearing until the land part increases to LLR = 1/5. However, fc0 and fc1 increase by 20–40% when LLR = 2/5 for rµ = 1 and 10. This is because the friction force on the flat surface increases. The increased rate of fc0 and fc1 at LLR = 2/5 for rµ = 1 can be cancelled by increasing the surface layer viscosity to rµ = 10. From these calculated results, it can be concluded that the bearing performance does not change markedly from that of a taper bearing until 40% of the converging taper length is removed by the running-in process. Considering these results in light of the thin film lubrication mechanism caused by a high viscosity surface layer enables us to understand why the boundary lubrication regime can shift to a lubrication regime after the running-in process, as reported by Norihisa et al. [14, 15], although the detailed parameter values of the surface texture and adsorbed surface layer remain unknown.

4 Conclusions

A modified Reynolds equation was developed for use in the theoretical analysis of thin film lubrication including an increased viscosity effect in a tribochemical surface layer. The modified factors for the Reynolds equation were expressed as closed form functions for a one-sided viscosity model function determined from the measured effective viscosity for a confined lubricant. Then, to elucidate the mechanism of a super-low friction coefficient with an increased critical load capacity for a textured surface and the shift phenomena from the boundary lubrication regime to the lubrication regime observed experimentally after the running-in process, the performance of small tapered/tapered land bearings as models of textures and surface roughness was investigated numerically by solving the derived one-dimensional Reynolds equation. From the parametric study of small bearing models where the bearing length was changed from 10 to 250 µm, the increased viscosity ratio changed from 1 to 100, the taper angle changed from 0.0002 to 0.005 rad, and land length ratio changed from zero to 2/5 under conditions of 0.1 m/s sliding velocity, 0.01 Pa-s bulk viscosity, and 20 nm (2z0) surface layer thickness, typical high load capacity and lower friction coefficient in a thin film of less than several ten-nanometer-trailing gaps were clarified. The consistency between the calculated results and the various experimental results was discussed. The important findings of this study are summarized as follows:

(1) The increased load capacity and lower friction coefficient found in textured parallel slider bearings after the running-in process are mainly caused by the decreased bearing gap in the lubrication regime, rather than by the increased viscosity of the surface layer. This is because the load capacity increases proportionally to the square of the bearing gap, whereas the friction force increases proportionally to the bearing gap. However, the shift from the boundary lubrication regime to mixed-lubrication and further noncontact lubrication regimes below an approximately ten-nanometer-trailing gap hT can be realized by a high local pressure in each texture bearing, owing to the increased viscosity in a tribochemical surface layer. The local high pressures generated by the small texture bearings can compensate for surface height irregularity of up to ten nanometers in the parallel sliders and achieve noncontact hydrodynamic lubrication when the maximum surface roughness height is decreased to less than the bearing gap by means of the running-in process.

(2) When the tapered bearing length L is 10 µm, the viscosity at the solid surface increases to ten times the bulk viscosity, and the taper angle θ is 0.001 rad, the mean load capacity of the tapered bearing becomes ~90 MPa at 10 nm-hT. If 1% of this mean load capacity is effectively used for a parallel slider, the parallel slider bearing can have ~1 MPa mean load capacity at a 10 nm gap. If the viscosity at the solid surface increases to one hundred times the bulk viscosity, a 1 MPa mean load capacity can be achieved at ~20 nm-hT. At these bearing gaps, the maximum pressure on the tapered bearing becomes ~0.15 GPa and can generate a maximum elastic deformation of about 6 nm. This local deformation can contribute to compensating for the irregularity of the parallel slider surface and can induce noncontact lubrication when the surface roughness height is reduced to less than the bearing gap by means of the running-in process. The friction coefficients on both the moving and stationary
surfaces decrease to less than 0.01 when the bearing gap is reduced to less than 20 nm. If the bearing gap under the noncontact lubrication condition can be reduced to 10 nm, the friction coefficients at moving and stationary surfaces will be reduced to less than 0.004 and 0.003, respectively.

(3) When \( L \) is increased from 10 to 50 and 250 \( \mu \)m, the load capacity \( W \) increases by more than 10 and 50 times, respectively, at an \( h_t \) greater than 20 nm. Therefore, a 1 MPa mean load capacity can be obtained at a higher \( h_t \) and the compensation effect of the surface height irregularity can be generated at a higher \( h_t \). In addition, increasing \( L \) will decrease the friction coefficients remarkably. When \( L \approx 250 \mu \)m, the friction coefficient on the moving surface \( f_c \) decreases to \(-0.001\) over a wide \( h_t \) range below 60 nm. The friction coefficient on the stationary surface \( f_c \) decreases in almost inverse proportion to \( L \), and becomes 0.0004 at 60 nm-\( h_t \), dropping right down to zero and further to a small negative value below 20 nm-\( h_t \).

(4) As the taper angle \( \theta \) is increased when \( L \approx 50 \mu \)m, \( W \) decreases and \( f_c \) increases rapidly, whereas \( f_c \) decreases to zero and becomes a small negative value when \( h_t < 20 \) nm, whereas \( f_c \) becomes \(-0.005\) in a wide range of \( h_t < 40 \) nm. Although the mechanism causing the negative values of \( F_t \) and \( f_c \) is not clear yet, this super-low friction coefficient at the stationary surface \( f_c \) derived under the conditions of a large slider length and taper angle seems to be consistent with experimental results in [9, 10] if the taper angle is considered to correspond to the pit depth-to-length ratio. This result suggests that the energy consumption of a parallel slider bearing should be evaluated using the friction coefficient on a moving surface \( f_{c0} \) rather than \( f_c \).

(5) From the calculated results for a tapered land bearing as a model of truncated surface roughness after the running-in process, it was found that the load capacity does not change from that of a tapered bearing until the land length ratio \( LLR \) is increased to 2/5, \( f_c \) and \( f_{c0} \) hardly change until \( LLR \approx 1/5 \), but increase to some extent when \( LLR > 2/5 \). These results and the description in item (2) enable understanding of the shift from boundary lubrication regime to the mixed lubrication regimes after the running-in process, as reported in [14, 15].

(6) To validate the numerical analysis for a tapered bearing, it was confirmed that the analytically obtained load capacity and friction coefficients of a tapered bearing for a uniform viscosity lubricant film could yield the same values as those when \( r_s = 1 \). It was found that the friction force on a stationary surface became negative as \( \theta L/h_t \) increases to 12.14.

(7) As the calculated results are qualitatively consistent with the various noteworthy experimental results reported in the literature, it can be concluded that optimized texture-design in a parallel slider can be theoretically evaluated using the modified Reynolds equation developed in this study. Moreover, a proper curve fitting method for the viscosity model function with the measured viscosity variation in the surface layer was presented.

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Nomenclature

\[ F_s : \text{Friction force on moving surface} \ [N] \]
\[ F_t : \text{Friction force on stationary surface} \ [N] \]
\[ LLR : L/L_t, \text{Ratio of land length to bearing length} \]
\[ L : \text{Length of tapered/tapered land bearings} \ [\text{m}] \]
\[ L_t : \text{Length of land part} \ [\text{m}] \]
\[ L_p : \text{Length of taper part} \ [\text{m}] \]
\[ O-xyz : \text{Absolute rectangular coordinate system} \]
\[ U : \text{Sliding velocity} \ [\text{m/s}] \]
\[ W : \text{Load capacity of taper bearing per unit bearing width} \ [\text{N/m}] \]
\[ W_m : \text{Mean load capacity of taper bearing} \ [\text{N/m}^2] \]
\[ f_{c0} : \text{Friction coefficient on moving surface} \]
\[ f_c : \text{Friction coefficient on stationary surface} \]
\[ h : \text{Lubricant thickness} \ [\text{m}] \]
\[ h_t : \text{Leading bearing gap} \ [\text{m}] \]
\[ h_L : \text{Trailing bearing gap} \ [\text{m}] \]
\[ k : L/L_t \]
\[ p : \text{Pressure in lubricant film} \ [\text{Pa}] \]
\[ r_s : \text{Ratio of viscosity on solid surface to bulk viscosity} \]
\[ u : \text{Local flow velocity of lubricant in } x\text{-direction} \]
\[ v : \text{Local flow velocity of lubricant in } y\text{-direction} \]
\[ z : \text{Effective thickness of surface layer} \ [\text{m}] \]
\[ \theta : (h_t - h_{L})/L, \text{or} \ (h_t - h_{L})/L_T \]
\[ \tau_s : \text{Shear stress on moving surface} \ [\text{Pa}] \]
\[ \tau_t : \text{Shear stress on stationary surface} \ [\text{Pa}] \]
\[ \mu_b : \text{Bulk viscosity of lubricant} \ [\text{Pa} \cdot \text{s}] \]
\[ \mu_s : \text{Local viscosity of lubricant at } z \ [\text{Pa} \cdot \text{s}] \]
\[ \mu_0 : \text{Viscosity of lubricant on solid surface} \ [\text{Pa} \cdot \text{s}] \]

References

Appendix 1

\[
f_1(h) = \left[ \frac{h^2}{2} \frac{z^2(\mu - 1)}{h^2 + 2h + r_z \mu} + z^2(\mu - 1)^{1/2} \arctan \frac{h + z}{z(\mu - 1)^{1/2}} \right]
\]

\[
f_2(h) = \left[ \frac{h^2}{2} \frac{z^2(\mu - 1)}{h^2 + 2h + r_z \mu} + z^2(\mu - 1)^{1/2} \left[ \arctan \frac{h + z}{z(\mu - 1)^{1/2}} - \arctan \frac{1}{(\mu - 1)^{1/2}} \right] \right]
\]

\[
f_3(h) = \left[ h - z(\mu - 1)^{1/2} \arctan \frac{h + z}{z(\mu - 1)^{1/2}} \right]^{-1}
\]

\[
g_1(h) = \left[ \frac{h^2}{2} \frac{z^2(\mu - 1)}{h^2 + 2h + r_z \mu} + z^2(\mu - 1)^{1/2} \left[ \arctan \frac{h + z}{z(\mu - 1)^{1/2}} - \arctan \frac{1}{(\mu - 1)^{1/2}} \right] \right]^{-1}
\]

\[
g_2(h) = \left[ h - z(\mu - 1)^{1/2} \arctan \frac{h + z}{z(\mu - 1)^{1/2}} - \arctan \frac{1}{(\mu - 1)^{1/2}} \right]^{-1}
\]

\[
l_1 = (h + z) \arctan \frac{h + z}{z(\mu - 1)^{1/2}} - z \arctan \frac{z}{z(\mu - 1)^{1/2}} - \frac{z(\mu - 1)^{1/2}}{2} \ln \left[ z^2(\mu - 1) + (h + z)^2 \right]
\]

\[
l_2 = (h + z) \ln \left[ (h + z)^2 + (\mu - 1)^2 \right] - 2(h + z) + 2(\mu - 1)^{1/2} z \arctan \frac{h + z}{z(\mu - 1)^{1/2}}
\]

\[-z \ln \left[ (h + z)^2 + (\mu - 1)^2 \right] + 2z - 2(\mu - 1)^{1/2} \cdot z \cdot \arctan \frac{z}{z(\mu - 1)^{1/2}}
\]

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Appendix 2

To validate the numerical solutions obtained from the modified Reynolds equation, Eq. (8), the analytical solutions from the conventional Reynolds equation for a uniform viscosity lubricant film are investigated. In lubrication handbooks, the load capacity $W$ is usually written as follows, where $m = h_i/h_T$,

$$W = 6\mu L \left( \frac{1}{h_T} \right)^2 \frac{1}{(m - 1)^2} \left\{ \ln m - 2\frac{(m - 1)}{(m + 1)} \right\}$$

This expression is not appropriate for evaluating the $W$ versus $h_i$ relationship for given design parameters of $L$ and $\theta$. Using the more appropriate dimensionless parameter $k = L \theta / h_T$ (=$m - 1$), the various quantities discussed in this paper can be expressed as follow:

$$W = 6\mu L \left( \frac{1}{\theta} \right)^2 \left\{ \ln(k + 1) - 2\frac{k}{k + 2} \right\} \quad (12)$$

$$F_a = \mu L \int \frac{dx}{h} + \frac{1}{2} L \theta \frac{\partial p}{\partial x} \int dx = \mu L \left( \frac{1}{\theta} \right) \left\{ \ln(k + 1) + 3 \ln(k + 1) - 2\frac{2k}{(k + 1)} \right\} \quad (13)$$

$$F_b = \mu L \left( \frac{dx}{h} - \frac{1}{2} \frac{\partial p}{\partial x} \int dx \right) = \mu L \left( \frac{1}{\theta} \right) \left\{ \ln(k + 1) - 3 \ln(k + 1) - 2\frac{2k}{(k + 2)} \right\} \quad (14)$$

$$f_{c_1} = \frac{\theta (k + 2) \ln(k + 1) - 3k}{3 (k + 2) \ln(k + 1) - 2k}$$ \quad (15)$$

$$f_{c_2} = \frac{\theta [k - 2] \ln(k + 1) - 3k}{3 (k + 2) \ln(k + 1) - 2k}$$ \quad (16)

It was confirmed that these equations can give the same values as the numerical solutions for $r_c = 1$ presented in Figs. 5 and 8. As can be seen from Eqs. (13) and (14), the difference between $F_a$ and $F_b$ results from the plus and minus signs of the pressure derivative term. Since negative values of $F_1$ have never been discussed previously to the knowledge of the author, the values of $A = \ln(k + 1)$ and $B = 3[\ln(k + 1) - 2\ln(k + 1)]$ are plotted as a functions of $k$ in Fig. 11. Note that $A$ and $B$ correspond to the shear force term and pressure derivative term, respectively. As $k$ increases, the difference between $A + B$ and $A - B$ increases, and $A - B$ becomes negative above the critical value of $k = 12.14$. From this analysis, it can be understand why $f_{c_1}$ becomes negative as $h_i$ decreases when $L$ and $\theta$ increase. However, the detailed physics of negative friction remains to be studied.