Study of Belt Friction in Over-Wrapped Condition

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The property of belt friction in an over-wrapped condition was studied. A fractional expression with respect to the ratio of belt tensions was derived by this study. The Euler’s belt formula was included as a special case in the equation derived. The equation is a function of the contact angle θ and two kinds of the coefficient of friction μ and μb, where θ is wrapping angle of a belt around a cylinder, μ is the coefficient of friction between the belt and the cylinder and μb is the coefficient of friction between the belt and belt respectively. As the equation signifies the possibility of self-locking, the self-locking mechanism was investigated theoretically by the equation. It occurs when the discriminant of self-locking condition became 0 or negative. The necessary conditions for self-locking are μb<μ and a sufficient over-wrapping angle of the belt. Providing the coefficient of friction μ and the ratio of the coefficients of friction κ=μb/μ, some critical over-wrapping angles for self-locking were calculated numerically. Experiments were carried out to confirm the occurrence of self-locking predicted by the equation by using a polyethylene film to reduce friction between the belt. Self-locking occurred in the experiment with the polyethylene film wrapped around the belt. But it never occurred without the polyethylene film as predicted by the equation.

Keywords: Euler’s belt formula, belt friction, self-locking, locking condition, over-wrapping

1. Introduction

It is well known that the equation of belt friction obtained by Euler shows that the fraction of belt tension is an exponential function of a product μθ, where μ is the coefficient of friction and θ is the angle of contact between the belt and the cylinder respectively1). The Euler’s belt formula is also valid for the friction of rope. So the frictional force increases greatly with an increment of the product μθ. In practice, a belt or rope is often wrapped around itself in a winch or hoist. But as far as Euler’s belt formula is concerned, the effect of over-wrapping of belt on the belt friction is not taken into account. This study was carried out focusing the effect of over-wrapping of belt on the belt friction. Then, the self-locking mechanism and the discriminant of self-locking condition were found in the analysis.

2. Theory

Figure 1 shows a belt wrapped around a cylindrical surface. The point Pi (i=1, 2, 3) is a boundary of contact and Ti (i=1, 2, 3, 4) is tension of the belt. Symbol θi denotes the angle of point Pi. The belt was over-wrapped around the belt in the range from P1 to P2 denoted as the angle θ1. X axis is taken so as to pass through the point P2, T1 is bigger than T4. T3 is an imaginary belt tension. There is no contact area between P2 and P3 due to the thickness of the belt-end. According to the theory of belt friction1), analysis starts with the following equation.

\[ T_i = e^{μbT_i} T_3 = e^{μbT_3} \]  

(1)

where μb is the coefficient of friction between the belt and belt. Let’s denote T3 is an inner belt tension at the
point $P_1$ as shown in Fig. 1. The belt tension $T_2$ can be expressed by the belt tension $T'_2$:

$$T_2 = T'_2 = e^{\theta_1} T'_1, \quad (2)$$

where $\mu$ is the coefficient of friction between the belt and cylinder. Making use of Eqs.(1) and (2), $T_1$ can be expressed as

$$T_1 = e^{\mu \theta_1} T'_1. \quad (3)$$

The inner belt is normally pressed to the cylinder by the outer belt. The normal force to a small segment of the inner belt at angle $\theta$ denoting as $dN_b$ is

$$dN_b = e^{\theta_1} T_1 d\theta. \quad (4)$$

On the other hand, the normal force is also generated by inner belt tension itself. Let’s denote the normal force of the belt between $P_1$ and $P_2$ as $N_p$. Normal force acting to a small segment of the cylinder at angle $\theta$ is given by

$$dN_c = e^{\theta_1} T_1 d\theta. \quad (5)$$

Then, by making use of Eqs.(4) and (5), the frictional force between the inner belt and cylinder denoting as $F_{12}$ is given by

$$F_{12} = \int_0^{\theta_1} \mu dN_b + \int_0^{\theta_1} \mu dN_c$$

$$= (e^{\mu \theta_1} - 1) \mu T_1' + (e^{\mu \theta_1} - 1) T_2' \quad (6)$$

Denoting $r$ is a radius of the cylinder and neglecting the thickness of the belt, the equilibrium equation of moment denoting $r$ is a radius of the cylinder and neglecting the thickness of the belt is given as

$$T_1 \cdot r = (F_{12} + F_{13} + T_2) r. \quad (7)$$

Here, the frictional force $F_{12}$ acting on the surface between $P_1$ and $P_2$ is given by

$$F_{12} = \int_0^{\theta_1} \mu e^{\mu \theta} d\theta = (e^{\mu \theta_1} - 1) T_1' \quad (8)$$

Substituting Eqs.(6) and (8) into Eq.(7) gives

$$T_1 = \left( e^{\mu \theta_1} - 1 \right) \mu T_1' + \left( e^{\mu \theta_1} - 1 \right) T_2' + e^{\mu \theta_1} T_2 \quad (9)$$

Substituting $T_2$ and $T_1'$ in Eq.(9) as functions of $T_1$ by making use of Eqs.(1) and (3) gives

$$T_1 = \frac{e^{\theta_1}}{1 - e^{\mu \theta_1} \left( \frac{\mu}{\mu_b} - 1 \right) + e^{-\mu \theta_1} \left( \frac{\mu}{\mu_b} - 1 \right)} T_2 \quad (10)$$

This is the targeted equation that expresses the relation between $T_1$ and $T_2$.

### 3. Discussion

Equation (10) can be checked by supposing an extreme case of either $\mu=0$ or $\mu_b=0$. Substituting $\mu=0$ into Eq. (10) gives $T_1=T_2$ as a matter of course. Substitution of $\mu_b=0$ into Eq. (10) requires limiting operation.

$$\lim_{\mu_b \to 0} (1 - e^{\mu \theta_1}) \frac{\mu}{\mu_b} = -\mu \theta_1 \quad (11)$$

Making use of Eq.(11), Eq.(10) becomes Eq.(12) for the case of $\mu_b=0$.

$$T_1 = \frac{e^{\theta_1}}{1 - e^{\mu \theta_1} \left( \frac{\mu}{\mu_b} - 1 \right) + e^{-\mu \theta_1} \left( \frac{\mu}{\mu_b} - 1 \right)} T_2 \quad (12)$$

Equation (12) implies the belt may be locked firmly when the denominator of the fraction in Eq.(12) becomes 0. Substituting $\mu=0$ into Eq. (12) again as a matter of course.

Substituting $\mu=\mu_b$ into Eq. (10) gives

$$T_1 = e^{\mu \theta_1} T_2 \quad (13)$$

Equation (13) is exactly the same form as the Euler’s belt formula though was derived from the expression that took an effect of over-wrapping of belt into account. Equation (13) implies that the belt cannot be locked on the cylinder as far as the wrapping angle is finite.

Letting $\theta_1=0$ in Eq.(10) to eliminate the over-wrapping part gives

$$T_1 = e^{\mu \theta_1} T_2 \quad (14)$$

This is the well-known Euler’s belt formula. So Eq.(10) was proved to have expanded to include the Euler’s belt formula. Equation (14) can also be obtained from Eqs.(12) and (13).

Next, let’s consider some locking conditions by over-wrapping of belt. According to Eq.(10), the belt tension ratio $T_2/T_1$ can be expressed as

$$\frac{T_2}{T_1} = \frac{(1 - e^{\mu \theta_1}) \left( \frac{\mu}{\mu_b} - 1 \right) + e^{-\mu \theta_1} \left( \frac{\mu}{\mu_b} - 1 \right)}{1 - e^{\mu \theta_1} \left( \frac{\mu}{\mu_b} - 1 \right) + e^{-\mu \theta_1} \left( \frac{\mu}{\mu_b} - 1 \right)} \quad (15)$$

The locking condition is satisfied when the numerator of Eq.(15) becomes 0 meaning $T_2=0$. So, the discriminant of locking condition can be expressed as

$$\Gamma = (1 - e^{\mu \theta_1}) \left( \frac{\mu}{\mu_b} - 1 \right) + e^{-\mu \theta_1} \left( \frac{\mu}{\mu_b} - 1 \right) \quad (16)$$

Locking condition is satisfied for the case of $\Gamma=0$. Critical point is $\Gamma=0$. Here, $\kappa$ denotes a ratio of the coefficient of friction.

$$\kappa = \mu_b / \mu \quad (17)$$

As $e^{\mu \theta} \geq 1$ and $e^{-\mu \theta_1} > 0, \kappa$ should be less than unity to make the value of locking discriminant of Eq.(16) be $\Gamma<0$. As can be seen in Fig.1, the angle $\theta_1$ is smaller than $2\pi$ due to the thickness of the belt. From geometrical consideration in Fig.1, following equation is obtained.

$$\cos \alpha = \frac{r - t}{r + t} = 1 - \frac{t}{r} \quad (18)$$

Here, $t$ is thickness of the belt and $r$ is a radius of the cylinder. When angle $\alpha$ is small, cosine can be expanded in the Maclaurin series as

$$\cos \alpha = 1 - \frac{\alpha^2}{2} + \cdots \quad (19)$$

By equating Eq.(18) with Eq.(19), the angle of non-contact $\alpha$ can be roughly estimated by
\[ \alpha \approx \sqrt{\frac{2t}{r}} \]  

(20)

Figure 2 shows the comparison of analytical solution by Eq.(18) and approximate curve by Eq.(20). They agree well in the range \( t/r < 0.05 \). With an increment of \( t/r \), the angle obtained by Eq.(20) gives somewhat larger value than the theoretical value. But it is accurate enough for practical use for a range of \( t/r < 0.1 \).

Supposing the angle of non-contact is \( \alpha = 15^\circ \) meaning the inner angle of contact \( \theta_3 = 345^\circ \) and \( t/r = 0.035 \), the corresponding critical locking condition can be evaluated by solving Eq.(16). Figure 3 shows some solutions. The critical angle of belt locking \( \theta_1 \) decreases with an increment of the coefficient of friction \( \mu \). Provided the coefficient of friction is constant, the critical angle of belt locking \( \theta_1 \) increases with an increment of \( \kappa \). This fact means that the belt is likely to lock with a decrement of \( \kappa \). So the smaller coefficient of friction \( \mu_b \) is preferable for self-locking. The limiting condition for belt locking is \( \kappa = 0 \) or \( \mu_b = 0 \).

To confirm the belt locking condition is satisfied, the value of belt tension ratio \( T_4/T_1 \) in Eq.(15) and that of \( \Gamma \) in Eq.(16) were calculated along the line A-B in Fig.3. Figure 4 shows the results. Left-side corresponds to the point A and right-side corresponds to the point B. The value of \( \Gamma \) in Eq.(16) decreases almost linearly with an increment of angle \( \theta_1 \). The critical over-wrapping angle \( \theta_1 \) is \( \theta_1 = 122^\circ \). The belt tension ratio \( T_4 / T_1 \) decreases smoothly with an increment of over-wrapping angle \( \theta_1 \). It becomes negative where \( \theta_1 \) is larger than the critical angle. Negative value of the fraction of belt tension means that either belt tension becomes compressive force in order to satisfy equilibrium equation of the force. But a belt cannot bear compressive force so that the negative fraction of the belt tensions means the occurrence of belt locking.

Figure 5 shows the boundary curve of belt locking for the case of \( \mu_b = 0 \). This curve is formed with intersecting points of the curves at y-axis in Fig. 3, i.e. the angle \( \theta_1 \) at \( \kappa = 0 \) of the curves in Fig. 3. The critical angle \( \theta_1 \) decreases with an increment of the coefficient of friction \( \mu \). Provided \( \mu_b = 0 \) and \( \theta_3 = 345^\circ \), self-locking never occur in the condition bellow this curve. On the other hand, self-locking occurs in the condition above this curve.
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Figure 6 illustrates the effect of $\kappa$ on the fraction of belt tension. Fraction of belt tension $T_4/T_1$ decreases rapidly with an increment of over-wrapping angle $\theta_1$ for the case of smaller $\kappa$.

Making use of Eq.(14), the convergence point is calculated as $T_4/T_1 = \exp(-\mu \theta_3) \approx 0.164$. It is clear that the fraction of belt tension $T_4/T_1$ is greatly influenced by the magnitude of $\kappa$. The belt tension ratio $T_4/T_1$ decreases with an increment of over-wrapping angle $\theta_1$ except for the case of $\kappa=1.4$. When $\kappa \geq 1$, the fraction of belt tension is always positive that means self-locking never occurs. Providing $\theta_1=360^\circ$, $\theta_3=345^\circ$ and $\mu=0.3$, the critical ratio of the coefficient of friction $\kappa_c$ for the self-locking was calculated by using discriminant Eq.(16). It was $\kappa_c=0.735$. The corresponding line was plotted with a dashed line in Fig. 6. The magnitude of $\kappa$ should be less than $\kappa_c$ for the sake of self-locking.

Figure 7 shows the change of critical ratio of the coefficient of friction $\kappa_c$ with non contact angle $\alpha$. Smaller ratio of the coefficient of friction is required for self-locking for belt of smaller coefficient of friction.

Figure 8 shows a method by which the coefficient of friction between the belt and belt can be reduced in order to satisfy self-locking condition. Some experiments were carried out to confirm the occurrence of self-locking by using this method. When a polyethylene film was wrapped in the belt with sufficient over-wrapping angle $\theta_1$, an occurrence of self-locking was confirmed. But self-locking never occurred without the polyethylene film.

4. Conclusion

The property of belt friction in an over-wrapped condition was studied. A fractional expression with respect to the ratio of belt tensions was derived. Self-locking mechanism of belt friction was investigated by the equation. The discriminant of self-locking was derived. The necessary conditions for self-locking are the larger coefficient of friction $\mu$ combined with the smaller coefficient of friction $\mu_b$ for belt-belt friction and sufficient over-wrapping angle of the belt. Occurrence of self-locking was confirmed by the experiment with the condition predicted by the equation derived.

References
