On Bound Attainability in Classes of Networks with Prescribed Node Degrees

Pavel Selin, Hitoshi Obara
Graduate School of Electrical and Electronics Engineering, Akita University

1. Introduction

Classes of undirected networks (weighted graphs) without loops with prescribed degrees of nodes are considered. For an arbitrary partition of the set of nodes into two subsets, the variable quantities are the sums of arc weights on each subset and the sum of the weights of the arcs which are incident to the two subsets. For these variables, attainable upper and lower bounds are obtained in the case of constraining the arc weights by a common constant and by the degrees of the nodes. The result is applicable in the theory of network flows if the networks are considered as directed graphs with a symmetrical matrix of the capacities of arcs.

2. Results and discussion

A network \( S = \{U(n); H; X\} \) is a set of nodes \( U(n) = \{u_1, \ldots, u_n\} \), a set of all unordered arcs \( H = \{(u_i, u_j): 1 \leq i < j \leq n\} \) and an arc capacities function \( X = \{x_{ij}\} \), where \( x_{ij} = x_{ji} \geq 0 \), \( 1 \leq i < j \leq n \). The degree of node \( u_i \) of network \( S \) is defined as the sum of the capacities of the arcs which are incident to this node: \( \deg u_i = \sum_{j=1}^{n} x_{ij}, 1 \leq i \leq n \). Let \( c \geq 0 \) is the constraint to the arc weights. We say that vector \( A \in \mathbb{R}^n \) is c-realizable in a network if there exists a network \( S = \{U(n); H; X(A)\} \) such that \( \deg u_i = a_i \), \( 1 \leq i \leq n \) and \( x_{ij} \leq c \). This network is referred to as the c-realization of vector \( A \).

A bipartite network \( S = \{U(n), V(m); H; X\} \) consists of, by definition, two sets of nodes \( U(n) = \{u_1, \ldots, u_n\} \), \( V(m) = \{v_1, \ldots, v_m\} \), a set of all unordered arcs \( H = \{(u_i, v_j): 1 \leq i \leq n, 1 \leq j \leq m\} \) and an arc capacities function \( X = \{x_{ij}\} \), where \( x_{ij} = x_{ji} \geq 0 \), \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). The degrees of nodes \( u_i \) and \( v_j \) are defined as the sums of the capacities of the arcs that are incident to these nodes: \( \deg u_i = \sum_{j=1}^{m} x_{ij}, 1 \leq i \leq n \), \( \deg v_j = \sum_{i=1}^{n} x_{ij}, 1 \leq j \leq m \). Let \( A \in \mathbb{R}^n \) is c-realizable in a bipartite network, where \( c \geq 0 \) if there exists a network \( S = \{U(n), V(m); H; X(A)\} \) such that \( \deg u_i = a_i \), \( 1 \leq i \leq n \), \( \deg v_j = b_j \), \( 1 \leq j \leq m \) and \( x_{ij} \leq c \leq v_j \). A pair of vectors \( (A, B) \) is c-realizable in a network if there exists a network \( S = \{U(n), V(m); H; X(A)\} \) such that \( \deg u_i = a_i \), \( 1 \leq i \leq n \), \( \deg v_j = b_j \), \( 1 \leq j \leq m \) and \( x_{ij} \leq c \leq v_j \).

In what follows, each network will be identified with the arc capacities function. Without loss of generality we assume that the coordinates of vector \( A \) and the individual vectors of pair \( (A, B) \) are arranged in nonincreasing order \( A \in \mathbb{R}^n_+ \) and \( (A, B) \in \mathbb{R}^{n+m}_+ \).

3. Example

We investigate the vector \( A = (8, 8, 2, 2, 2, 2, 2, 2) \) in \( \mathbb{R}^n_+ \) for \( c = 2 \) with the partitioning of the set of nodes \( U(9) = U_1 \cup U_2 \), where \( U_1 = \{u_1, u_2, u_3, u_4, u_5\} \), \( U_2 = \{u_6, u_7, u_8, u_9\} \). Here \( A_1 = (2, 2, 2, 2) \), \( A_2 = (8, 8, 2) \). Applying formulas of the theorem, we obtain 0 \( \leq \delta(u_1, U_2) \leq 6 \), 1 \( \leq \delta(u_1, U_2) \leq 7 \) and 2 \( \leq \delta(u_1, U_2) \leq 14 \). The figure shows two networks from \( \Gamma(A; 2) \).

Here the lower and upper bounds of the capacity of the cut \( \delta(u_1, U_2) \) are attained. It is easy to see that in the first network \( \delta(u_1, U_2) = 6 \) and \( \delta(u_1, U_2) = 7 \) attained the upper constraints and in the second network \( \delta(u_1, U_2) = 0 \) and \( \delta(u_1, U_2) = 1 \) attained lower constraints.

4. References