Optimal H_∞ PD Controller Design for Flexible Spacecraft by Branch and Bound Algorithm*

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This paper proposes optimization of the PD controller gains by minimizing H_∞ norm for flexible spacecraft attitude control. Solving the problem is known to be difficult because it is described by a bilinear matrix inequality (BMI) that cannot be transformed into linear matrix inequalities (LMIs). In order to solve it within practical computation time, we revise the branch and bound algorithm and incorporate the quantization of the variable domains. The ETS-VI mathematical model is used to demonstrate its ability.

Key Words: Flexible Spacecraft, Optimal H_∞ PD, Bilinear Matrix Inequality, Branch and Bound Algorithm, Quantization

1. Introduction

It is known that the PD controller guarantees the asymptotic stability of the collocated flexible spacecraft closed-loop system, regardless of the model parameter uncertainties.¹⁾ However, controller gains have been heuristically designed using, for example, bode plots, and there has been little discussion on the optimization of the controller gains. Viewing this, this paper describes an optimal design method based on the H_∞ norm. Since the problem is described by a BMI that cannot be reduced to LMIs, the PD gain optimization problem has been regarded as unsolvable. For the direct BMI solution, we have been studying application of the branch and bound algorithm.²⁾ In order to obtain the optimal solution within practical computation time, we proposed some improvements to the algorithm in a separate paper.³⁾ However, it was still insufficient for practical control problems. In this paper, the algorithm is further improved by incorporating the quantization of variable domains. Then, its ability is confirmed through a numerical study using an ETS-VI flexible spacecraft model.⁴⁾

2. Model and Problem Description

The three-axis stabilized spacecraft model is usually described by three SISO systems, and the attitude controller is independently designed for the roll, pitch and yaw. Even when the spacecraft has large flexible solar paddles, its

model can be treated as three SISO systems by the coordinate transform to the paddle-fixed coordinate (i.e., out-of-plane bending, in-plane bending, and torsion which are decoupled with each other). The attitude control system of the ETS-VI is a typical example.⁴⁾ This paper considers controlling only the vibration of out-of-plane bending of an ETS-VI spacecraft model, although controllers for the other motions are designed in the same way.

The unconstrained modal equation with the collocated measurement outputs is described as:

$$\begin{aligned}\ddot{\eta} + 2\zeta\Sigma\dot{\eta} + \Sigma^2\eta &= \Phi^T(u + w) \\ q_1 &= \Phi\eta, \quad q_2 = \Phi\dot{\eta},\end{aligned}\tag{1}$$

where Φ is the modal shape at the rigid body, u is the control input torque, w is the disturbance torque, $\Sigma^2 \geq 0$ is the modal stiffness, and $2\zeta\Sigma \geq 0$ is the modal damping. The measurement outputs q_1 and q_2 correspond to the attitude angle and angular velocity, respectively. This is an unstable, stabilizable and detectable plant. The closed-loop system of Eq.(1) with the PD controller

$$u = -pq_1 - dq_2\tag{2}$$

becomes

$$\begin{aligned}\ddot{\eta} + (2\zeta\Sigma + d\Phi^T\Phi)\dot{\eta} + (\Sigma^2 + p\Phi^T\Phi)\eta &= \Phi^Tw \\ q_1 &= \Phi\eta, \quad q_2 = \Phi\dot{\eta}.\end{aligned}\tag{3}$$

It is known to be asymptotically stable, regardless of the modal parameter errors, if $p > 0$, $d > 0$ when Eq.(1) is stabilizable and detectable.¹⁾ This paper aims to design optimal PD gains by minimizing the H_∞ norm of the closed-loop transfer function from w to the control variable z .

Let's assume that the control variable z consists of the attitude angle q_1 and the control input u . In order to

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fully utilize the frequency shaping capability of the H_∞ problem formulation, frequency weighting functions $W_q(s)$ and $W_u(s)$ are added to yield the control variable z :

$$\begin{aligned} z &= \begin{bmatrix} W_q(s)q_1 \\ W_u(s)u \end{bmatrix} \\ &= \begin{bmatrix} W_p(s) & 0 \\ -pW_u(s) & -dW_u(s) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \end{aligned} \quad (4)$$

where $W_q(s)$ and $W_u(s)$ have the low-pass and high-pass filter characteristics, respectively. With this set-up, the resultant controller satisfies both specifications (i.e., control input small at high frequency and displacement output being small at low frequency). Then the closed-loop generalized plant is constituted by combining the state-space equation of Eq.(3) and the state-space minimal realization of Eq.(4). When this closed-loop generalized plant is described by $G(s) = D_w + C_w(sI - A_w)^{-1}B_w$, the problem is to find the optimal gains p and d that internally stabilize $G(s)$ and minimize γ of $\|G(s)\|_\infty < \gamma$. This is described by the bounded real lemma to minimize γ , subject to

$$\begin{bmatrix} A_w^T X + X A_w & X B_w & C_w^T & 0 \\ B_w^T X & -\gamma I & D_w^T & 0 \\ C_w & D_w & -\gamma I & 0 \\ 0 & 0 & 0 & -X \end{bmatrix} < 0, \quad (5)$$

where I is the unit matrix of the appropriate dimension. This is a BMI for a static output feedback optimization problem, where it is not possible to reduce further to LMIs. Therefore, the convex optimization algorithm cannot be effectively applied to it.

This paper attempts to solve the BMI directly by the branch and bound algorithm. To this end, by describing the variables in Eq.(5) by scalars, the conditions are rewritten as

$$F(x, y) + \gamma F_\gamma < 0, \quad (6)$$

where x corresponds to the elements of the matrix X , $y = (p, d)$ are gains of the PD controller, and $F(x, y)$ is the bilinear matrix function with respect to $(x, y) \in R^{N_x} \times R^{N_y}$:

$$F(x, y) := F_{0,0} + \sum_{i=1}^{N_x} x_i F_{i,0} + \sum_{j=1}^{N_y} y_j F_{0,j} + \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} x_i y_j F_{i,j}. \quad (7)$$

In the sequel, a certain domain $(x, y) \in Q_0$ is defined, and the following optimization problem is considered

$$\phi(Q_0) := \text{minimize } \gamma, \text{ subject to Eq.(6).} \quad (8)$$

3. Branch and Bound Algorithm and Its Revision

Originally, Goh et al. proposed the branch and bound algorithm²⁾ for solving BMI. However, since it is not applicable to practical problems because of the computation time, we have added some improvements to the algorithm.³⁾

3.1. Lower and upper bound computation

The computation of the lower and upper bounds with respect to a certain domain Q is needed for the branch and

Fig. 1. Domain of x_{p_i} and x_{n_i} .

bound algorithm, where

$$\begin{aligned} Q &= \{(x, y) | L_{x_i} \leq x_i \leq U_{x_i} (i = 1 \sim N_x), \\ &\quad L_{y_j} \leq y_j \leq U_{y_j} (j = 1 \sim N_y)\}. \end{aligned} \quad (9)$$

The computation algorithms of a lower bound and an upper bound used hereafter are shown in this subsection.

3.1.1 Lower bound

We rewrite x_i by $x_i = x_{p_i} - x_{n_i}$ ($i = 1 \sim N_x$), and the following matrix function is defined.

$$\begin{aligned} F_L(x_p, x_n) &:= \\ &F_{0,0} + \sum_{i=1}^{N_x} (x_{p_i} - x_{n_i}) F_{i,0} + \sum_{j=1}^{N_y} (L_{y_j} F_{0,j}^+ + U_{y_j} F_{0,j}^-) \\ &+ \sum_{i=1}^{N_x} \{x_{p_i} (L_{y_j} F_{i,j}^+ + U_{y_j} F_{i,j}^-) - x_{n_i} (U_{y_j} F_{i,j}^+ + L_{y_j} F_{i,j}^-)\}, \end{aligned} \quad (10)$$

where

$$0 \leq x_{p_i} \leq U_{x_i}, \quad 0 \leq x_{n_i} \leq -L_{x_i} \quad (11)$$

if

$$U_{x_i} > 0, \quad L_{x_i} \leq 0. \quad (12)$$

Even when Eq.(12) is not satisfied, the change of variables always transforms Eq.(7) so as to satisfy Eq.(12). The eigenvalues of $F_{0,j}^+$ ($j = 1 \sim N_y$), $F_{i,j}^+$ ($i = 1 \sim N_x, j = 1 \sim N_y$) are nonnegative, and those of $F_{0,j}^-$, $F_{i,j}^-$ are non-positive. They satisfy the identity $F_{0,j} = F_{0,j}^+ + F_{0,j}^-$, $F_{i,j} = F_{i,j}^+ + F_{i,j}^-$, and L_{y_j} , U_{y_j} are the minimum or maximum values of y_j . Since

$$F_L(x_p, x_n) - F(x_p, x_n, y) \leq 0 \quad \forall (x = x_p - x_n, y) \in Q, \quad (13)$$

the lower bound is computed as follows:

$$\begin{aligned} \phi_L(Q) &:= \text{minimize } \gamma, \\ &\text{subject to } F_L(x_p, x_n) + \gamma F_\gamma < 0 \\ &\text{and } 0 \leq x_{p_i} \leq U_{x_i}, 0 \leq x_{n_i} \leq -L_{x_i} (i = 1 \sim N_x). \end{aligned} \quad (14)$$

When Eq.(14) is computed, (x_{p_i}, x_{n_i}) are moved freely to the inside including its boundary of the rectangle defined by four points $(0, 0)$, $(U_{x_i}, 0)$, $(U_{x_i}, -L_{x_i})$ and $(0, -L_{x_i})$ as shown in Fig. 1. If the condition of x_{p_i} and x_{n_i} Eq.(12) is further narrowed, the accuracy of the lower bound becomes better, and the computation becomes easier. Viewing this, we next reduce the number of the variables for the computation. When $(x_{p_i}, x_{n_i}) = (U_{x_i}, 0)$,

x_i is the maximum value U_{x_i} . And when $(x_{p_i}, x_{n_i}) = (0, -L_{x_i})$, x_i is the minimum value L_{x_i} . Therefore, all values of $L_{x_i} \leq x_i \leq U_{x_i}$ are expressed by $x_i = x_{p_i} - x_{n_i}$, even if (x_{p_i}, x_{n_i}) moves only on the straight line that connects $(U_{x_i}, 0)$ and $(0, -L_{x_i})$. Hence, this straight line,

$$x_{n_i} = \frac{L_{x_i}}{U_{x_i}} x_{p_i} - L_{x_i}, \quad (15)$$

is considered as the condition of the constraint. Since this constraint is linear, the linearity of Eq.(10) holds. By substituting Eq.(15) into Eq.(10), the following is obtained.

$$F_L(x_p) = \quad (16)$$

$$\begin{aligned} & F_{0,0} + \sum_{i=1}^{N_x} L_{x_i} F_{i,0} + \sum_{j=1}^{N_y} \left\{ L_{y_j} \left(F_{0,j}^+ + \sum_{i=1}^{N_x} L_{x_i} F_{i,j}^- \right) \right. \\ & \left. + U_{y_j} \left(F_{0,j}^- + \sum_{i=1}^{N_x} L_{x_i} F_{i,j}^+ \right) \right\} + \sum_{i=1}^{N_x} x_{p_i} \left[\left(1 - \frac{L_{x_i}}{U_{x_i}} \right) F_{i,0} \right. \\ & \left. + \sum_{j=1}^{N_y} \left\{ (L_{y_j} F_{i,j}^+ + U_{y_j} F_{i,j}^-) - \frac{L_{x_i}}{U_{x_i}} (U_{y_j} F_{i,j}^+ + L_{y_j} F_{i,j}^-) \right\} \right] \end{aligned}$$

In the sequel, the lower bound is computed as follows:

$$\begin{aligned} \phi_L(Q) &= \text{minimize } \gamma, \\ &\text{subject to } F_L(x_p) + \gamma F_\gamma < 0 \\ &\text{and } 0 \leq x_{p_i} \leq U_{x_i} (i = 1 \sim N_x). \end{aligned} \quad (17)$$

The constraint of Eq.(17) is LMIs whose number of variables is $N_x + 1$. In this case, x_i is:

$$\begin{aligned} x_i &= x_{p_i} - x_{n_i} \\ &= \left(1 - \frac{L_{x_i}}{U_{x_i}} \right) x_{p_i} + L_{x_i}. \end{aligned} \quad (18)$$

Since the accuracy of this lower bound is determined only by the domain size of y_j , only y_j are objects in the split operations performed in the branch and bound algorithm.

3.1.2 Upper bound

We use $\hat{x}_i$, the value of x_i obtained by Eq.(18) at the lower bound computation, and $\hat{y}_j$, a certain point of y_j such as the center of the domain, for the computation of the upper bound. Then the upper bound is given as follows:

$$\begin{aligned} \phi_U(Q) &:= \text{minimize } \gamma, \\ &\text{subject to } F(\hat{x}, \hat{y}) + \gamma F_\gamma < 0. \end{aligned} \quad (19)$$

3.2. Quantization of variable domain

As shown later in the simulation results, the aforementioned improvement is not still sufficient to apply the practical controller synthesis problem. Accordingly, we consider adding the quantization of variable domains in this subsection. By the quantization, the unnecessary splits of variable domains other than the splits needed to obtain a solution satisfying the prescribed accuracy required for the controller are excluded. Hence, the quantization is expected to reduce the iteration number. In the branch and bound algorithm using the lower bound Eq.(17), it is attained by quantizing only the variable domain of y_j , by

Fig. 2. Split method.

leaving the variable domain of x_i continuous. It is noted that, in the lower bound computation Eq.(17) and the upper bound computation Eq.(19), y_j is not contained in the variables, and therefore the quantization of the variable domain of y_j is realized by adding the correction only to split operations.

On the other hand, as is easily conjectured, the quantization degrades the optimal value accuracy. Indeed the optimal solution of the discontinuous value computed by the branch and bound algorithm incorporating the quantization is larger than the ϵ -global minimum that is achieved if the variable domain is continuous. However, a certain amount of accuracy degradation is not severe for practical usage, as long as the accuracy of some, say three or four, significant digits of gains are preserved.

The split operation of variable domain is shown in Fig. 2 schematically. Points in the figure show the discrete values. The computation by the required significant digit can be performed by applying this split operation. For example, when the required significant digit is two and the search domain of y_1 is $0 \leq y_1 \leq 10$, the optimal solution is found in:

$$y_1 = \{0, \dots, 0.098, 0.099, 0.10, 0.11, \dots, 0.98, 0.99, 1.0, 1.1, \dots, 9.9, 10\}.$$

When the domain of y_j that has the minimum value of the lower bound includes only one element, or $L_{y_j} = U_{y_j} (j = 1 \sim N_y)$, this value is the optimal solution because of $F_L = F$. Hence, the computation is terminated if the above condition is satisfied.

The branch and bound algorithm incorporating the quantization is as follows.

- Step 1 Give the significant digit of y_j . Set $k := 0$, $S_0 := Q_0$.
- Step 2 Divide matrices $F_{i,j}$ into $F_{i,j}^+$, $F_{i,j}^-$, and $F_{0,j}$ into $F_{0,j}^+$, $F_{0,j}^-$.
- Step 3 Compute $L_0 := \phi_L(Q_0)$ and $U_0 := \phi_U(Q_0)$.
- Step 4 Select $\bar{Q}$ from S_k such that $L_k = \phi_L(\bar{Q})$. $S_{k+1} := S_k - \{\bar{Q}\}$.
- Step 5 If $L_{y_j} = U_{y_j} (j = 1 \sim N_y)$ in $\bar{Q}$, then computation is terminated, else go next step.
- Step 6 Split $\bar{Q}$ along its longest edge of y_j into $\bar{Q}_1$ and $\bar{Q}_2$.

Fig. 3. Bode plots of open-loop.

- Step 7 For $n = 1, 2$, if $\phi_L(\bar{Q}_n) \leq U_k$, then compute $\phi_U(\bar{Q}_n)$ and $S_{k+1} := S_{k+1} \cup \{\bar{Q}_n\}$.
- Step 8 $U_{k+1} := \min_{Q \in S_{k+1}} \phi_U(Q)$.
- Step 9 Pruning: $S_{k+1} := S_{k+1} - \{Q : \phi_L(Q) > U_{k+1}\}$.
- Step 10 $L_{k+1} := \min_{Q \in S_{k+1}} \phi_L(Q)$.
- Step 11 $k = k + 1$. Go to Step 4. □

4. Simulation Results

The availability of the proposed algorithm confirmed through a numerical experiment using an ETS-VI model. The computer used was a Dell Dimension 8400 (Pentium(R)4 CPU 3.20 GHz, 2.00 GB RAM), and the operating system was Windows XP. For solving LMIs, "mincx" which is the command in the Matlab LMI Control Toolbox was used. The controller was designed for a reduced-order model consisting of the rigid and first vibration modes, and the simulation was performed for the full-order model.

The bode plots from w to q_1 of the open-loop full-order model are shown in Fig. 3. The two frequency weighting functions $W_q(s)$ and $W_u(s)$ added to q and u are:

$$W_q(s) = \frac{100}{s + 100}, \quad W_u(s) = \frac{s}{s + 100}. \quad (20)$$

Since both functions are scalar (i.e., the order is one), the order of the resultant generalized plant is six. The search domain Q_0 is set as $0 \leq x_i \leq 10$ for the diagonal elements of X and $-10 \leq x_i \leq 10$ for other elements, and $0 \leq y_j \leq 10^4$. We set the required accuracy to two significant digits. The obtained optimal solutions are $p = 150$, $d = 920$, and $\gamma = 0.011906$. The iteration number was 151903, the elapsed time was 397 [h] 40 [m] 03 [s], and optimization terminated within a reasonable computation time.

The upper and lower bounds, U_k and L_k , were computed using two methods, the branch and bound algorithm incorporating quantization, and the branch and bound algorithm whose variable domain is continuous. Those results

Fig. 4. U_k and L_k of quantization (solid) and continuous (dashed). Fig. 5. Difference of U_k and L_k of quantization (solid) and continuous (dashed).

are shown and compared in Figs. 4,5. Since the convergence of the former is clearly better compared with the latter, it is concluded that the improvement by the quantization is effective.

Furthermore, we computed a more accurate solution based on the above two-digit solution. The search domain of y_j is set as $100 \leq y_1 \leq 200$ and $870 \leq y_2 \leq 970$, that is ± 50 of the above solution, and is narrower than the above computation. The required accuracy is set to three significant digits. The result is $p = 147$, $d = 930$, and $\gamma = 0.011905$. The iteration number was 10200, and the elapsed time was 26 [h] 19 [m] 27 [s].

A simulation with the feedback gain of three significant digits was performed. The gain plot from w to q_1 of the closed-loop system is shown in Fig. 6 compared with that of the open-loop system. The frequency characteristics are apparently improved. The impulse response is shown in Fig. 7. It is asymptotically stable by the definition, and

Fig. 6. Gain plots of closed-loop (solid) and open-loop (dashed).

Fig. 7. Impulse response of displacement (dashed) and velocity (solid).

a good disturbance response was obtained, as expected.

5. Conclusion

In this paper, an optimal PD controller for flexible spacecraft has been designed by minimizing H_∞ norm. Because this problem is described by BMI, we have improved the branch and bound algorithm incorporated with the quantization of variable domains. The availability of this method has been confirmed by applying it to an ETS-VI spacecraft model. Thus, we have shown an example of a BMI problem that is successfully solved directly by the branch and bound algorithm, without using any approximation methods.

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