A New Method for Motion Planning of Rotating Bodies under Multiple Constraints

By Takehiro NISHIYAMA1), Katsuhiko YAMADA2), and Shoji YOSHIKAWA1)

1) Advanced Technology R&D Center, Mitsubishi Electric Corporation, Amagasaki, Japan
2) Department of Aerospace Engineering, Nagoya University, Nagoya, Japan

(Received April 25th, 2008)

This paper presents a new method for generating motion profiles of rotating bodies, for example, the joint angle motion of mechanical systems such as space manipulators and antennas. In the planning of a motion from the given initial conditions to the final conditions, there are many constraints that should be considered, for example, actuator torque limits, rate limits, jerk limits, etc. Furthermore, flexibility of a system cannot be ignored in general. Vibration suppression is of great importance in some applications. For stationary boundary conditions (rest-to-rest cases), if the only constraint is the torque limit, a simple bang-bang type profile is the optimal one. However, if we consider more general boundary conditions, complex constraints, and vibration suppression conditions, the problem becomes considerably complicated. In this study, motion planning for a fixed motion time is considered. The control input or the acceleration of the motion is expressed as a linear superposition of triangle waves whose weight coefficients are determined in order to optimize the motion profile. In this formulation, all the boundary conditions and several constraints, including vibration suppression conditions, are expressed as linear equality and inequality constraints. Therefore, by appropriately setting the performance index, the problem becomes a linear programming problem and can be solved efficiently.

Key Words: Motion Planning, Optimal Control, Vibration Suppression, Linear Programming

Nomenclature

\[ \begin{align*}
    u & : \text{control input} \\
    \theta & : \text{rigid-body mode} \\
    q_i & : i\text{th flexible mode coordinate} \\
    \omega_i & : i\text{th modal frequency} \\
    b_i & : i\text{th modal gain} \\
    n & : \text{number of flexible modes} \\
    T & : \text{control duration (motion time)} \\
    a_k(t) & : k\text{th triangle wave} \\
    w_i & : \text{weight for } k\text{th triangle wave} \\
    \Delta & : \text{half-width of triangle waves} \\
    t_c & : \text{center of } k\text{th triangle wave} \\
    l_i & : N\text{-dimensional vector of unity}
\end{align*} \]

Subscripts

\[ \begin{align*}
    0 & : \text{initial} \\
    f & : \text{final}
\end{align*} \]

1. Introduction

Many applications such as space manipulators and antennas require planning of efficient motion profiles or feed-forward command inputs under multiple constraints. This paper presents new techniques for motion planning for such applications, which is applicable to various linear time-invariant mechanical systems.

In the planning of a motion from the given initial conditions to the final conditions, there are many constraints that should be considered, for example, actuator torque limits, rate limits, etc. Furthermore, in some applications, the flexibility of a system cannot be ignored. Suppression of residual vibrations is of great importance. For the simplest rest-to-rest motion of a rigid body with only the constraint of torque limit, Pontryagin’s maximum principle yields a bang-bang type solution; however, for more general boundary conditions, complex constraints, and vibration suppression conditions, the problem becomes considerably complicated. The purpose of this study is to develop a motion planning method that simultaneously (i) can deal with various boundary conditions and constraints, (ii) can suppress vibrations, and (iii) is easy to compute and can achieve optimality. For property (i), a large number of studies have been carried out in various frameworks, for example, methods that put some assumptions on the motion profiles to obtain the solution algebraically \(^1\), methods that approximate the trajectory using splines or polynomials \(^2\) \(-\(^3\)\), methods based on optimal control theory \(^4\), etc. However, some of these methods cannot deal with multiple constraints or require the application of nonlinear numerical optimization techniques; therefore, there is no guarantee of optimality, i.e., these methods do not satisfy properties (i) and (iii) simultaneously. As for property (ii), i.e., for vibration suppression, various techniques to generate command profiles that suppress vibrations have been developed \(^5\) \(-\(^7\)\). Among these methods, a large number of studies have been carried out in the framework of input shaping \(^7\) \(-\(^8\)\). Input shaping is a technique used to generate a vibration free command profile by convoluting a sequence of impulses with the original command. Several methods have been proposed for increasing robustness to
modeling errors or to deal with various practical constraints \( y_1 \); however, most of them require the application of nonlinear optimization techniques and thus computational difficulties may arise.

In this study, the control input is expressed as a linear superposition of triangle waves with unknown weight coefficients. In this formulation, all the boundary conditions and several constraints, including the torque limits, rate limits, jerk limits, vibration suppression conditions, etc., are expressed as linear equality and inequality constraints with respect to the unknown weight coefficients. Therefore, by appropriately setting the performance index, the problem of determination of the weight coefficients satisfying the constraints becomes a linear programming (LP) problem and can be solved efficiently.

Therefore, the proposed method can deal with various boundary conditions and constraints including vibration suppression, and has no computational difficulty because of its LP formulation, i.e., satisfies all of above three properties (i)–(iii).

The problem is explained in Sec. 2, the proposed method is formulated for a rigid-body mode in Sec. 3, and it is extended to include flexible modes and suppress residual vibrations in Sec. 4. The numerical results obtained to verify the proposed method, including a comparison with other methods, are shown in Sec. 5. The last section summarizes our results.

2. Problem

Consider a dynamic system described by a linear time-invariant system given by

\[
M \ddot{x} + Kx = Gu
\]

where \( x \) is a displacement vector and \( M \), \( K \), and \( G \) are the mass, stiffness, and input distribution matrices, respectively. This system includes, for example, a simple two-mass-spring system, as shown in Fig. 1(a), and a rigid body with a beam-like appendage which has a finite number of flexible modes, as shown in Fig. 1(b).

![Sample model system](image)

**Fig. 1.** Sample model system comprising (a) two-mass-spring system and (b) rigid body with beam-like appendage

In this study, we consider a case with the scalar input \( u(t) \) for simplicity; however the proposed method can easily be generalized to handle multi-input case. For this simple case, Eq. (1) is transformed into the following decoupled modal equations.

\[ \dot{\theta} = u \]  (2)
\[ \ddot{\theta} + \alpha^2 \dot{\theta} = \beta u \]  (3)

For the rigid-body mode \( \theta \), boundary conditions are given as follows:

\[
\theta(0) = \dot{\theta}_o, \quad \dot{\theta}(0) = \ddot{\theta}_o, \quad \ddot{\theta}(0) = \dddot{\theta}_o \]  (4)
\[
\theta(T) = \dot{\theta}_f, \quad \dot{\theta}(T) = \ddot{\theta}_f, \quad \ddot{\theta}(T) = \dddot{\theta}_f \]  (5)

where the control duration \( T \) is fixed. Note that all the above conditions may not be present. In fact, some of them can be ignored according to the applications. We consider the following constraints corresponding to the actuator torque limits, rate limits, and jerk limits.

\[
\dot{\theta}_{\text{max}}(t) \leq \dot{\theta}(t) \leq \dot{\theta}_{\text{max}}(t) \]  (6)
\[
\ddot{\theta}_{\text{max}}(t) \leq \ddot{\theta}(t) \leq \ddot{\theta}_{\text{max}}(t) \]  (7)
\[
\dddot{\theta}_{\text{max}}(t) \leq \dddot{\theta}(t) \leq \dddot{\theta}_{\text{max}}(t) \]  (8)

The boundary conditions for the flexible modes are

\[
q_i(0) = \dot{q}_i(0) = 0 \quad (i = 1, \ldots, n) \]  (9)
\[
q_i(T) = \dot{q}_i(T) = 0 \quad (i = 1, \ldots, n) \]  (10)

Moreover, the residual vibrations should be suppressed considering the errors in the modal frequency \( \omega_0 \). For robustness to modeling errors, additional constraints are considered later on.

The problem is to efficiently determine \( u(t) \) that satisfies the abovementioned multiple constraints. In the following formulation, only \( \theta \) is considered initially; then, the flexible modes are taken into account and the formulation is expanded.

3. Basic Formulation for Rigid-Body Mode

3.1. Triangle waves

In this study, \( u(t) \) is expressed as the weighted sum of the triangle waves \( \alpha_i(t) \) as follows:

\[
u(t) = \sum_{i=1}^{K} w_i \alpha_i(t) \]  (11)

where

\[
\alpha_i(t) = \begin{cases} 
\frac{1}{\alpha_1}(t - (t_a - \Delta t)) & \text{if } t_a - \Delta t \leq t \leq t_a \\
\frac{1}{\alpha_2}(t_a + \Delta t - t) & \text{if } t_a \leq t \leq t_a + \Delta t \\
0 & \text{otherwise}
\end{cases} \]  (12)

Substituting Eq. (11) into Eq. (2) and integrating it, we obtain the time responses of the (angular) acceleration, velocity, position, and jerk as follows:

\[
\ddot{\theta}(t) = \sum_{i=1}^{K} w_i \dddot{\alpha}_i(t) \]  (13)
\[
\dot{\theta}(t) = \sum_{i=1}^{K} w_i \dddot{\alpha}_i(t) + \dddot{\theta}_o \]  (14)
\[
\theta(t) = \sum_{i=1}^{K} w_i \gamma_i(t) + \gamma_f + \theta_o \]  (15)
\[
\dddot{\theta}(t) = \sum_{i=1}^{K} w_i \beta_i(t) \]  (16)

where

\[
\beta_i(t) = \beta_i(t) - \beta_i(0), \quad \gamma_i(t) = \gamma_i(t) - \gamma_i(0) - \beta_i(0)t, \quad \beta_i(t) = \int_{-\infty}^{t} \alpha_i(\tau) d\tau \]  (17)
In vector notation, Eqs. (13)–(16) are expressed as
\[
\begin{align*}
\dot{\theta}(t) &= \alpha(t)^\top w + \dot{\theta}_0 \tag{20} \\
\ddot{\theta}(t) &= \beta(t)^\top w + \ddot{\theta}_0 \tag{21} \\
\dddot{\theta}(t) &= \gamma(t)^\top w + \dddot{\theta}_0 \tag{22} \\
\alpha(t) &= [\alpha(t), ..., \alpha_k(t)]^\top \tag{23} \\
\beta(t) &= [\beta(t), ..., \beta_k(t)]^\top \tag{24} \\
\gamma(t) &= [\gamma(t), ..., \gamma_k(t)]^\top \tag{25} \\
\alpha(t) &= [\alpha(t), ..., \alpha_k(t)] \tag{26} \\
\beta(t) &= [\beta(t), ..., \beta_k(t)] \tag{27} \\
\gamma(t) &= [\gamma(t), ..., \gamma_k(t)] \tag{28}
\end{align*}
\]
where the superscript \(^\top\) represents the transpose of a vector. The responses are linear with respect to the \( K \) unknown parameters \( w \).

### 3.2. Boundary conditions

The boundary conditions (4) and (5) are represented as
\[
\begin{align*}
\alpha(0)^\top w &= \dot{\theta}_0 \tag{29} \\
\beta(0)^\top w + \dot{\theta}_0 &= \theta_0 \tag{30} \\
\alpha(T)^\top w + \dot{\theta}_0 &= \theta_T \tag{31} \\
\beta(T)^\top w + \dot{\theta}_0 &= \dot{\theta}_T \tag{32} \\
\gamma(T)^\top w + \dot{\theta}_0 &= \ddot{\theta}_T \tag{33} \\
\gamma(T)^\top w + \dddot{\theta}_0 &= \dddot{\theta}_T \tag{34}
\end{align*}
\]
where Eqs. (30) and (31) are always satisfied by the definitions of \( \tilde{\theta}(t) \) and \( \tilde{\gamma}(t) \).

Therefore, all the boundary conditions are represented as linear equalities with respect to \( w \).

### 3.3. Constraints

In this section, the constraints (6)–(8) are considered. Since \( u(t) \) is a piecewise linear function, we consider the constraints only at fixed times \( t_i^+ (i = 1, ..., N^+) \), where \( X = \{A, V, J\} \). The superscript \( ^+ \) represents the corresponding constraint, i.e., \( t_i^+ \), \( t_i^+ \), and \( t_i^+ \) indicate the time for the acceleration, velocity, and jerk constraints, respectively. We can set different \( t_i^+ \) for each constraint. For example, the set of \( t_{i_k}, t_{i_k} + \Delta t \) is sufficient for \( t_i^+ \) because the acceleration is linear in the time interval. On the other hand, \( t_i^+ \) should be set closer together. The constraints (6)–(8) are given as follows:
\[
\begin{align*}
\dot{\theta}_{\min}(t_i^+) &\leq \dot{\theta}_{\max}(t_i^+) \quad (i = 1, ..., N^+) \tag{35} \\
\ddot{\theta}_{\min}(t_i^+) &\leq \ddot{\theta}_{\max}(t_i^+) \quad (i = 1, ..., N^+) \tag{36} \\
\dddot{\theta}_{\min}(t_i^+) &\leq \dddot{\theta}_{\max}(t_i^+) \quad (i = 1, ..., N^+) \tag{37}
\end{align*}
\]
In matrix notation, Eqs. (35)–(37) are represented as follows:
\[
\begin{align*}
\begin{bmatrix} \dot{\theta}_{\min} & \dot{\theta}_{\max} & \ddot{\theta}_{\min} & \ddot{\theta}_{\max} & \dddot{\theta}_{\min} & \dddot{\theta}_{\max} \end{bmatrix}^\top &\leq \begin{bmatrix} B & 0 & A & 0 & 0 & 0 \end{bmatrix} \tag{38} \\
\dot{\theta}_{\min} &\leq \dot{\theta}_{\max} \quad &\leq \ddot{\theta}_{\max} &\leq \ddot{\theta}_{\max} \quad &\leq \dddot{\theta}_{\min} &\leq \dddot{\theta}_{\max} \tag{39} \\
\theta_{\min} &\leq \theta_{\max} &\leq \theta_{\max} \quad &\leq \theta_{\max} &\leq \theta_{\max} \tag{40}
\end{align*}
\]
where
\[
A = [\alpha(t_1^+), ..., \alpha(t_{N^+}^+)]^\top \tag{41}
\]
\[
B = [\beta(t_1^+), ..., \beta(t_{N^+}^+)]^\top \tag{42}
\]
\[
D = [\gamma(t_1^+), ..., \gamma(t_{N^+}^+)]^\top \tag{43}
\]
\[
\dot{\theta}_{\min} = [\dot{\theta}_{\min}(t_1^+), ..., \dot{\theta}_{\min}(t_{N^+}^+)]^\top \tag{44}
\]
\[
\dot{\theta}_{\max} = [\dot{\theta}_{\max}(t_1^+), ..., \dot{\theta}_{\max}(t_{N^+}^+)]^\top \tag{45}
\]
\[
\ddot{\theta}_{\min} = [\ddot{\theta}_{\min}(t_1^+), ..., \ddot{\theta}_{\min}(t_{N^+}^+)]^\top \tag{46}
\]
Therefore, all the constraints are represented as linear inequalities with respect to \( w \). Note that the original constraints (6)–(8) may not be satisfied for all \( t \in [0, T] \), but will be satisfied only at the sampled times \( t_i^+ \). However, by appropriately setting each \( t_i^+ \), the constraints will be effective.

### 3.4. Performance index

On the basis of the abovementioned formulation, all the boundary conditions and constraints are represented as linear equalities or inequalities with respect to \( w \). Therefore, by appropriately setting the performance index, the problem becomes an LP problem. For LP problems, in general, the optimal solution is always obtained in relatively short time. In this section, the variations in the performance indices are described depending on the control requirements.

#### 3.4.1. Minimization of maximum

First, consider the minimization of the maximum torque required for an actuator. This is equivalent to minimizing the maximum of the absolute values of acceleration, \( \max_i |\dddot{\theta}(t_i^+)| \). In this case, the performance index is formulated as follows.

Introduce an additional scalar variable \( a_{\text{set}} \) and the following constraint.
\[
- a_{\text{set}} I_{N^+} \leq Aw \leq a_{\text{set}} I_{N^+} \tag{47}
\]

Then, the performance index is set as
\[
L = a_{\text{set}} \rightarrow \min \tag{48}
\]
Due to the constraints (47), \( a_{\text{set}} \) corresponds to \( \max_i |\dddot{\theta}(t_i^+)| \). Therefore, Eq. (48) corresponds to the following equation.
\[
\max_i |\dddot{\theta}(t_i^+)| \rightarrow \min \tag{49}
\]
Note that this procedure is also applicable to jerk or velocity.

#### 3.4.2. Minimization of mean

Next, consider the minimization of the mean or sum of the control inputs. This formulation is useful in cases where the fuel use of a gas-jet thruster should be minimized.

In this case, \( N^x \) additional variables \( a_{\text{set}} (i = 1, ..., N^x) \) are introduced with the following constraint.
\[
- a_{\text{set}} \leq Aw \leq a_{\text{set}} \tag{50}
\]
where \( a_{\text{set}} = [a_{\text{set}}, ..., a_{\text{set}}]^\top \).

Then, the performance index is set as
\[
L = \frac{1}{N^x} \sum_{i=1}^{N^x} a_{\text{set}} \rightarrow \min \tag{51}
\]
3.4.3. Combination of performance indices

The abovementioned formulations can be easily combined to obtain a solution that satisfies multiple requirements. For example, to consider the maximum acceleration and mean acceleration simultaneously, \( N^4 + 1 \) additional variables \( a_{r_{0}} \) and \( a_{r_{i}} (i=1,\ldots,N^4) \) are introduced with the constraints (47) and (50), and the performance index is given as

\[
L = \kappa_1 a_{r_{0}} + \kappa_2 \frac{1}{N^4} \sum_{i=1}^{N^4} a_{r_{i}} \rightarrow \min
\]  

(52)

Similarly, various combinations are possible, e.g., “minimization of the maximum jerk and mean acceleration,” “minimization of the maximum acceleration and mean velocity,” etc. The weights \( \kappa \) are appropriately determined according to the control requirements.

4. Formulation for Vibration Suppression

In this section, one or more flexible modes are taken into account. The approach is basically similar to that followed in the abovementioned formulations. The responses of the flexible modes to the input (11) are expressed as linear functions with respect to \( w \), and some appropriate linear constraints and/or performance indices are formulated.

4.1. Flexible mode response

Substituting Eq. (11) into the flexible mode equation (3) and solving it, we obtain the time responses

\[
q_{i}(t) = b_{i} \frac{1}{\omega_{i}^{2}} \int_{0}^{\tau} \sin \omega_{i}(t-\tau)a(\tau) y(\tau) d\tau
\]

(53)

\[
\dot{q}_{i}(t) = b_{i} \int_{0}^{\tau} \cos \omega_{i}(t-\tau)a(\tau) y(\tau) d\tau
\]

(54)

Let \( \psi_{i}(t, \omega) \) be the response to \( \alpha_{r_{i}} \),

\[
\psi_{i}(t, \omega) = \frac{1}{\omega_{i}^{2}} \int_{0}^{\tau} \sin \omega_{i}(t-\tau)a(\tau) d\tau
\]

(55)

and

\[
\rho_{i}(t, \omega) = \psi_{i}(t, \omega) = \int_{0}^{\tau} \cos \omega_{i}(t-\tau)a(\tau) d\tau
\]

(56)

Then, we obtain the following.

\[
q_{i}(t) = b_{i} \psi_{i}(t, \omega)^{T} w
\]

(57)

\[
\dot{q}_{i}(t) = b_{i} \rho_{i}(t, \omega)^{T} w
\]

(58)

where

\[
\psi(t, \omega) = [\psi_{1}(t, \omega), \ldots, \psi_{K}(t, \omega)]^{T}
\]

(59)

\[
\rho(t, \omega) = [\rho_{1}(t, \omega), \ldots, \rho_{K}(t, \omega)]^{T}
\]

(60)

Therefore, the response of the flexible mode \( q_{i}(t) \) is expressed as a linear superposition of the responses to each triangle wave with \( w \).

4.2. Terminal zero-vibration constraint

If Eqs. (57) and (58) are substituted into the terminal condition (10), the equality constraints for \( w \) are obtained. However, the existence of a solution is not guaranteed for such strict constraints depending on the input resolution \( K \) and \( \omega \). Hence, we slightly relax the condition (10) as

\[
-q_{\text{lim}} \leq b_{i} \psi_{i}(T, \omega_{i})^{T} w \leq q_{\text{lim}}
\]

(61)

\[
-q_{\text{lin}} \leq b_{i} \rho_{i}(T, \omega_{i})^{T} w \leq q_{\text{lin}}
\]

(62)

where \( q_{\text{lim}} \) and \( q_{\text{lin}} \) have extremely small values. Inequalities (61) and (62) are additional constraints for the LP problem obtained by the abovementioned formulations for the rigid-body mode.

4.3. Robustness

The abovementioned terminal constraints often result in a solution that is sensitive to the errors in \( \omega_{i} \). If there is an error in \( \omega_{i} \), the vibration may not be suppressed; on the contrary, it may be amplified. In this section, we describe some approaches for generating control inputs that are robust to the errors in \( \omega_{i} \).

4.3.1. Vibration minimization

The first approach attempts to find inputs that do not excite the flexible modes during the control duration. For that purpose, we consider following methods.

- Minimization of the maximum deflection \( \max |q(t)| \)
- Minimization of the maximum vibration velocity \( \max |\dot{q}(t)| \)

Both these methods can be formulated similar to the formulation described in Sec. 3.4.1. The details of the second method are described as follows.

First, introduce an additional scalar variable \( \dot{q}_{\text{err}} \) and consider the following constraints at \( N^{0} \) fixed \( i^{0} \) \((i=1,\ldots,N^{0})\).

\[
-q_{\text{err}} \leq b_{i} \phi(t^{0}_{i}, \omega_{i})^{T} w \leq q_{\text{err}} \quad (i=1,\ldots,N^{0})
\]

(63)

In matrix notation, the constraints are given as

\[
 Q(\omega_{i}) = [b_{1} \phi(t^{0}_{1}, \omega_{i}), \ldots, b_{N^{0}} \phi(t^{0}_{N^{0}}, \omega_{i})]^{T}
\]

(64)

Then, the performance index is set as

\[
 L = q_{\text{err}} \rightarrow \min
\]

(66)

\( \dot{q}_{\text{err}} \) corresponds to the maximum vibration velocity \( \max |\dot{q}(t^{0})| \); therefore, the resulting solution will minimize the vibration during the control duration.

4.3.2. Direct constraints for modeling errors

Our next approach attempts to directly deal with the errors in \( \omega_{i} \), i.e., for some values of \( \omega_{i} \) around the nominal value, constraints such as Eqs. (61) and (62) are imposed, where \( q_{\text{lim}} \) and \( q_{\text{lin}} \) are nonzero values corresponding to the allowed residual vibrations. For example, by imposing constraints for multiple frequencies between \( 0.8 \times \omega_{i} \) and \( 1.2 \times \omega_{i} \), 20% of uncertainty in \( \omega_{i} \) will be taken into account. This concept has been first introduced in the framework of the input shaping and is called as specified-insensitivity shaper or frequency sampling \( \omega_{i}^{**} \).
5.1. Results for rigid-body mode
First, the results for rigid-body mode are shown. It is assumed that $K = 20$, and the triangle waves are placed at equal intervals, i.e.,

$$t_{i+1} = (k - 1)\Delta t, \quad \Delta t = \frac{T}{K - 1} \quad (k = 1, \ldots, K) \quad (67)$$

The evaluation times for the constraints, $t_i^*$, are given as follows:

$$t_i^* = i\Delta t \quad (i = 1, \ldots, N^i, \quad N^i = K - 2) \quad (68)$$

$$t_i^* = i\frac{\Delta t}{4} \quad (i = 1, \ldots, N^i, \quad N^i = 4K - 5) \quad (69)$$

$$t_i^* = \left(1 - \frac{1}{2}\right)\Delta t \quad (i = 1, \ldots, N^i, \quad N^i = K - 1) \quad (70)$$

The constraints are given as follows:

$$\dot{\theta}_{\text{max}} = -\dot{\theta}_{\text{min}} = 18[\text{deg/s}] \quad (71)$$

$$\ddot{\theta}_{\text{max}} = -\ddot{\theta}_{\text{min}} = 6[\text{deg/s}^2] \quad (72)$$

$$\dddot{\theta}_{\text{max}} = -\dddot{\theta}_{\text{min}} = 10[\text{deg/s}^3] \quad (73)$$

The following two cases are considered for the boundary conditions.

• Case 1 (rest-to-rest)
  \[\begin{align*}
  \theta_0 &= 0[\text{deg}], \quad \dot{\theta}_0 = 0[\text{deg/s}], \quad \ddot{\theta}_0 = 0[\text{deg/s}^2] \\
  \theta_f &= 90[\text{deg}], \quad \dot{\theta}_f = 0[\text{deg/s}], \quad \ddot{\theta}_f = 0[\text{deg/s}^2] \\
  T &= 10[\text{s}] 
  \end{align*}\]  

\[\begin{align*}
  \theta_0 &= 0[\text{deg}], \quad \dot{\theta}_0 = 0[\text{deg/s}], \quad \ddot{\theta}_0 = 0[\text{deg/s}^2] \\
  \theta_f &= 5[\text{deg}], \quad \dot{\theta}_f = 16[\text{deg/s}], \quad \ddot{\theta}_f = 0[\text{deg/s}^2] \\
  T &= 10[\text{s}] 
  \end{align*}\]  

The results for case 1 are shown in Fig. 2 to Fig. 4, where variations in the performance indices are considered, i.e., the minimization of the maximum acceleration (Fig. 2), minimization of the mean acceleration (Fig. 3), and minimization of the maximum jerk (Fig. 4). In these figures, the solid lines are the responses of $\theta(t), \dot{\theta}(t), \ddot{\theta}(t), \dddot{\theta}(t)$ to $u(t)$ obtained by LP, the dots indicate $t_i^*$ and the boundary conditions, the broken lines indicate the constraints (71)–(73), and the dotted lines in the graphs of acceleration indicate the triangle waves multiplied by the obtained weights, $w_i\alpha_i(t)$. Bang-bang and bang-off-bang type solutions are obtained for the minimization of the maximum acceleration and mean acceleration cases, respectively. For the minimization of the maximum jerk case, a smoothly accelerating profile is obtained. It is known that minimizing jerk is effective in suppressing vibrations, and this property is verified in the next section.

Results for the case 2 are shown in Fig. 5, where the maximum acceleration is minimized. Although motion planning for the nonzero terminal rate cases is known to be considerably more difficult than that for the rest-to-rest cases, appropriate inputs have been obtained by our general formulation for the former case.

5.2. Results for rigid-body and single flexible mode
Next, we obtain the results for the case where the rigid-body mode and a single flexible mode exist. Here, we use $\omega = 6$ [rad/s] and the boundary condition case 1 described in the preceding section.
The results are shown in Fig. 9. The third and the fourth graphs in Fig. 9 indicate the responses of \( q(t) \) and \( \dot{q}(t) \) for the nominal value of \( \omega \). The vibration velocity \( \dot{q}(t) \) is suppressed and the jerk is significantly reduced. As a result, the residual vibrations are suppressed more than the jerk minimization case (Fig. 7).

In addition to the aforementioned technique, the technique described in Sec. 4.3.2 is also employed and the results are shown in Fig. 10. The terminal zero-vibration constraints (61) and (62) are imposed for frequencies that are 80\%, 90\%, 100\%, 110\%, and 120\% of the nominal value \( \omega = 6 \text{ [rad/s]} \). The limits are set as \( \varpi_{\text{lim}} = 10^{-4} \text{ [deg]} \) and \( \varpi_{\text{lim}} = 10^{-3} \text{ [deg/s]} \). The residual vibrations are suppressed further in the specified interval \( \pm 20\% \) of the errors. Although the residual vibrations increase outside the specified interval, this technique is valid when the uncertainty of the model is known to some extent.

The abovementioned results for vibration suppressions are summarized in Table 1.
Fig. 7. Responses of flexible mode to the jerk minimization input in Fig. 4

Fig. 8. Minimization of maximum acceleration with terminal zero-vibration constraints

Fig. 9. Minimization of maximum vibration velocity with terminal zero-vibration constraints
5.3. Comparison with other methods

The proposed method is compared with some types of input shapers, the zero-vibration (ZV) shaper, the zero-vibration and derivative (ZVD) shaper, and the specified-insensitivity (SI) shaper, in terms of robustness and computational costs.

The boundary conditions Case 1 (Eq. (74)–(76)) and the model frequency $\omega = 6$ [rad/s] in the previous section are used and the results for the input shapers are compared with the last result by the proposed method, Fig. 10. In the general formulation of the input shaping, an impulse sequence is convolved with some desired input to generate a shaped input. Here, a simple bang-bang command is used as the unshaped input, i.e., the shaped input command is generated in the following way:

1) Generate the impulse sequence for each input shaper using the model frequency. Let $\Delta T$ be the duration of the input shaper which is equal to the time location of the last impulse.

2) Give the unshaped bang-bang command as follows:
   - Motion time: $T - \Delta T$
   - Maximum acceleration: $4 \theta / (T - \Delta T)^2$

3) Generate the shaped command by convolving the bang-bang command in step 2 with the impulse sequence given in step 1.

The acceleration conditions $\ddot{\theta}_i = \ddot{\theta}_f = 0$ and the constraints (71)–(73) are omitted in this procedure, because it is hard to consider all of them in the framework of the input shaping. The impulse sequence for the SI shaper is given by solving a nonlinear programming (NLP) problem in which the residual vibration level is constrained at some sampling frequencies. The number of the impulses is set so that the vibration constraints are satisfied. The same sampling frequencies as Fig. 10 ($\pm 10\%$ and $\pm 20\%$) are used, and the level of the residual vibration is determined by trial and error so that the similar robustness is achieved.

The command profiles generated by the ZV, ZVD, and SI shapers and the responses are shown in Figs. 11–13. Figure 14 compares the sensitivity curves for these methods with the proposed method (Fig. 10). The SI shaper and the proposed method are more robust than the ZV and ZVD shapers. In fact, any amount of robustness can be achieved by the SI shaper and the proposed method by appropriately setting the vibration constraints. As for

Table 1. Comparison of maximum residual vibrations for ±20% and ±40% errors.

<table>
<thead>
<tr>
<th>Method</th>
<th>(Exact)</th>
<th>±20%</th>
<th>±40%</th>
</tr>
</thead>
<tbody>
<tr>
<td>MJ</td>
<td>$&lt;10^{-3}$ deg</td>
<td>0.090 deg</td>
<td>0.11 deg</td>
</tr>
<tr>
<td>MA+TZV</td>
<td>$&lt;10^{-3}$ deg</td>
<td>0.40 deg</td>
<td>0.64 deg</td>
</tr>
<tr>
<td>MVV+TZV</td>
<td>$&lt;10^{-3}$ deg</td>
<td>0.025 deg</td>
<td>0.077 deg</td>
</tr>
<tr>
<td>MVV+TZVSF</td>
<td>$&lt;10^{-3}$ deg</td>
<td>0.0057 deg</td>
<td>0.23 deg</td>
</tr>
</tbody>
</table>

MJ: minimization of the maximum jerk
MA: minimization of the maximum acceleration
TZV: terminal zero-vibration constraints
MVV: minimization of maximum vibration velocity
TZVSF: terminal zero-vibration constraints for specified frequencies

Fig. 10. Minimization of maximum vibration velocity with terminal zero-vibration constraints for specified frequencies
In conclusion, the proposed method has an advantage in the following respects:

- The method can generate a whole command profile considering the rigid-body conditions, constraints, and flexibility simultaneously.
- Any desired level of the robustness can be achieved in a similar way to the SI shaper, while the computational cost is less and the certainty of obtaining the optimal solution is much higher.

Fig. 11. ZV shaper

Fig. 12. ZVD shaper

Fig. 13. SI shaper
6. Conclusion

In this paper, a new method for motion planning of rotating bodies has been presented. In this method, the control input is expressed as a linear superposition of triangle waves and their weight coefficients are optimized. Various conditions and constraints, including the conditions for vibration suppression, are represented as linear equalities and inequalities with respect to unknown parameters. Therefore, the problem is formulated as a linear programming problem. It has been shown numerically that the basic linear formulations work efficiently and can generate appropriate command profiles depending on the rigid-body boundary conditions and constraints. For vibration suppression, on the basis of the linear formulations, several techniques have been presented and verified by numerical examples.

References