Bi-objective Model for Optimal Size and Shape of a Rectangular Facility

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Abstract

This paper presents a bi-objective model for determining the size and shape of a finite size facility. The objectives are to minimize both the closest and barrier distances. The former represents the accessibility of customers, whereas the latter represents the interference to travelers. The total closest and barrier distances are derived for a rectangular facility in a rectangular city where the distance is measured as the rectilinear distance. The analytical expressions for the total closest and barrier distances demonstrate how the size and shape of the facility affect the distances. The model focuses on the tradeoff between the closest and barrier distances, and the tradeoff curve provides alternatives for the size and shape of the facility.

Keywords: Location, Closest distance, Barrier distance, Rectilinear distance, Pareto optimal

1 Introduction

Classical facility location models usually assume that facilities can be represented as points with no area. For some types of facilities, however, the size of facilities cannot be negligible compared to the size of a study region. Examples of such finite size facilities include parks, stadiums, and cemeteries. Finite size facilities may act as barriers to travel if traveling within the facilities is prohibited. The interference to travelers as well as the accessibility of customers should therefore be considered when locating finite size facilities.

The optimal location and shape of finite size facilities have been addressed. Drezner1) considered the Weber problem where both the facility and demands have circular shapes. Carrizosa et al.2) examined a more generalized case where the facility and demands are randomly distributed inside their regions. Carrizosa et al.3) obtained the location and shape of a rectangular facility that minimize the average distance to the demand set. Brimberg & Wesolowsky4)–6) formulated the minisum and minimax location problems where both facilities and demands are represented by areas. Savas et al.7) developed the facility placement problem for finding the optimal location and orientation of a finite size facility in the presence of barriers. The problem was generalized by Wang et al.8) to determine input/output points and Zhang et al.9) to restrict travel on aisles. Kelachankuttu et al.10) and Sarkar et al.11) examined the placement of a rectangular facility in the presence of existing facilities. Kelachankuttu et al.10) regarded facilities as barriers to travel, whereas Sarkar et al.11) regarded facilities as generalized congested regions, where traveling is permitted at extra cost. Date et al.12) proposed an efficient procedure for the placement problem of a rectangular facility. Sarkar et al.13) considered the facility placement problem with the center objective. Miyagawa14) developed a location model of a rectangular facility that considers both the

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accessibility of customers and the interference to travelers. In most of the location models reviewed above, the size of facilities to be located is given and cannot be determined.

In this paper, we present a bi-objective model for determining the size and shape of a finite size facility. The objectives are to minimize both the closest and barrier distances. The former represents the accessibility of customers, whereas the latter represents the interference to travelers. To obtain analytical expressions for the closest and barrier distances, the facility is represented as a rectangle, and the distance is measured as the rectilinear distance. The analytical expressions provide fundamental characteristics of the closest and barrier distances, leading to a better understanding of the effect of the size and shape of the facility. Although Miyagawa\textsuperscript{14) proposed a similar model for locating a rectangular facility, we extend the scope by determining the size of the facility. This extension allows us to determine the size and shape of the facility simultaneously. The present model will thus supplement further location models of a finite size facility.

The remainder of this paper is organized as follows. The next section gives an analytical expression for the total closest distance. The following section obtains the total barrier distance. The penultimate section considers a bi-objective problem where both the closest and barrier distances are minimized. The final section presents concluding remarks.

2 Closest distance

Consider a rectangular city with side lengths $a_1$ and $a_2$, as shown in Fig. 1. A facility is represented as a rectangle with side lengths $b_1$ and $b_2$. The facility is assumed to be located at the center of the city. The reason for this assumption is that if customers of the facility are uniformly distributed within the city, the location that minimizes the sum of distances from customers to the facility is the center of the city, as shown by Miyagawa.\textsuperscript{14) The assumption enables us to focus on the effect of the size and shape of the facility on the closest and barrier distances. The distance is measured as the rectilinear distance. The rectilinear distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is defined as $|x_1 - x_2| + |y_1 - y_2|$. The rectilinear distance is a good approximation for the actual travel distance in cities with a grid road network.\textsuperscript{15)-17) Customers are assumed to be uniformly distributed within the city. The uniform distribution serves as a basis for further analysis with more realistic distribution. Let $N$ be the number of customers and $\rho = N/(a_1a_2 - b_1b_2)$ be the density of customers.

![Figure 1. Calculation of the closest distance.](image-url)
Let $R_a$ be the rectilinear distance from a randomly selected customer in the city to the closest point on the facility. We call $R_a$ the closest distance. In this section, we derive the distribution of the closest distance and the total closest distance.

Let $F(r)$ be the number of customers such that $R_a \leq r$. $F(r)$ is given by

$$F(r) = \rho S(r),$$

where $S(r)$ is the area of the region such that $R_a \leq r$ in the city excluding the facility. The distribution of the closest distance $f(r)$ is defined as

$$f(r) = \frac{dF(r)}{dr} = \rho \frac{dS(r)}{dr}.$$  

The region such that $R_a \leq r$ is depicted in Fig. 1. The area of the region is, if $a_1 - a_2 \leq b_1 - b_2$,

$$S(r) = \begin{cases} 
2r(b_1 + b_2 + r), & 0 < r \leq \frac{a_1}{2} - \frac{b_1}{2}, \\
2a_1r - \frac{1}{2}(a_1 - b_1)(a_1 - b_1 - 2b_2), & \frac{a_1}{2} - \frac{b_1}{2} < r \leq \frac{a_2}{2} - \frac{b_2}{2}, \\
a_1a_2 - b_1b_2 - \frac{1}{2}(a_1 + a_2 - b_1 - b_2 - 2r)^2, & \frac{a_2}{2} - \frac{b_2}{2} < r \leq \frac{a_1}{2} + \frac{a_2}{2} - \frac{b_1}{2} - \frac{b_2}{2}, \end{cases}$$

and if $a_1 - a_2 > b_1 - b_2$,

$$S(r) = \begin{cases} 
2r(b_1 + b_2 + r), & 0 < r \leq \frac{a_2}{2} - \frac{b_2}{2}, \\
2a_2r - \frac{1}{2}(a_2 - b_2)(a_2 - 2b_1 - b_2), & \frac{a_2}{2} - \frac{b_2}{2} < r \leq \frac{a_1}{2} - \frac{b_1}{2}, \\
a_1a_2 - b_1b_2 - \frac{1}{2}(a_1 + a_2 - b_1 - b_2 - 2r)^2, & \frac{a_1}{2} - \frac{b_1}{2} < r \leq \frac{a_1}{2} + \frac{a_2}{2} - \frac{b_1}{2} - \frac{b_2}{2}. \end{cases}$$

Substituting $S(r)$ into Eq. (2), we have, if $a_1 - a_2 \leq b_1 - b_2$,

$$f(r) = \begin{cases} 
2\rho(b_1 + b_2 + 2r), & 0 < r \leq \frac{a_1}{2} - \frac{b_1}{2}, \\
2\rho a_1, & \frac{a_1}{2} - \frac{b_1}{2} < r \leq \frac{a_2}{2} - \frac{b_2}{2}, \\
2\rho(a_1 + a_2 - b_1 - b_2 - 2r), & \frac{a_2}{2} - \frac{b_2}{2} < r \leq \frac{a_1}{2} + \frac{a_2}{2} - \frac{b_1}{2} - \frac{b_2}{2}. \end{cases}$$

and if $a_1 - a_2 > b_1 - b_2$,

$$f(r) = \begin{cases} 
2\rho(b_1 + b_2 + 2r), & 0 < r \leq \frac{a_2}{2} - \frac{b_2}{2}, \\
2\rho a_2, & \frac{a_2}{2} - \frac{b_2}{2} < r \leq \frac{a_1}{2} - \frac{b_1}{2}, \\
2\rho(a_1 + a_2 - b_1 - b_2 - 2r), & \frac{a_1}{2} - \frac{b_1}{2} < r \leq \frac{a_1}{2} + \frac{a_2}{2} - \frac{b_1}{2} - \frac{b_2}{2}. \end{cases}$$

The total closest distance $T_a$ is then given by

$$T_a = \int_{0}^{a_1/2+a_2/2-b_1/2-b_2/2} rf(r) \, dr = \frac{\rho}{4} \{ a_1a_2(a_1 + a_2 - 2b_1 - 2b_2) + a_2b_1^2 + a_1b_2^2 \}. $$

Let $A = b_1b_2$ and $W = b_1/b_2$ be the area and the aspect ratio of the facility, respectively. Fig. 2 shows $T_a$ for $A = 0.1, 0.3, 0.5$ as a function of $W$. The size of the city is (a) $a_1 = a_2 = 1$, (b) $a_1 = \sqrt{2}, a_2 = 1/\sqrt{2}$ and the number of customers is $N = 10$. In a square city (a), as $W$ increases, $T_a$ decreases for $A = 0.1$ but increases for $A = 0.3, 0.5$. It follows that the rectangular (square) facility is better if the size of the facility is small (large). In a rectangular city (b), the rectangular facility is better because $T_a$ decreases with $W$. These findings imply that the size and shape of the facility should be determined simultaneously and that the shape of the city should also be considered when locating a finite size facility.
3 Barrier distance

Consider trips between two randomly selected customers in the city. The facility hinders trips between regions $\Omega_1$ and $\Omega_2$ and between regions $\Omega_3$ and $\Omega_4$ in Fig. 3. Note that other travelers (such as between regions $\Omega_1$ and $\Omega_3$) need not make a detour.

Let $R_b$ be the additional travel distance to avoid a facility, that is, the difference between the shortest distances in the presence and in the absence of the facility. We call $R_b$ the barrier distance. In this section, we derive the distribution of the barrier distance and the total barrier distance.

Let $G(r)$ be the amount of trips such that $R_b \leq r$. $G(r)$ is given by

$$G(r) = V \cdot P\{R_b \leq r\}, \quad \text{(8)}$$

where $V$ is the amount of trips that make a detour. The distribution of the barrier distance $g(r)$ is defined as

$$g(r) = \frac{dG(r)}{dr} = V \frac{dP\{R_b \leq r\}}{dr}. \quad \text{(9)}$$

The distribution for trips between regions $\Omega_1$ and $\Omega_2$, denoted by $g_1(r)$, is obtained as follows. The
amount of trips $V_1$ between $\Omega_1$ and $\Omega_2$ is

$$V_1 = \frac{\rho^2b_2^2}{2}(a_1 - b_1)^2.$$  \hfill (10)$$

The barrier distance between two randomly selected points $P \in \Omega_1$ and $Q \in \Omega_2$ is

$$R_b = \begin{cases} s + t - |s - t|, & s + t \leq b_2, \\ 2b_2 - s - t - |s - t|, & s + t > b_2, \end{cases}$$  \hfill (11)$$

where $s$ and $t$ are the distances shown in Fig. 3. The region such that $R_b \leq r$ on the $s$-$t$ plane is depicted in Fig. 4. Since $P$ and $Q$ are randomly selected in each region, $s$ and $t$ are uniformly distributed over the interval $[0, b_2]$. The probability that $R_b \leq r$ is then

$$P\{R_b \leq r\} = \frac{b_2^2 - (b_2 - r)^2}{b_2^2}, \quad 0 < r \leq b_2.$$  \hfill (12)$$

Substituting $V_1$ and $P\{R_b \leq r\}$ into Eq. (9), we have

$$g_1(r) = \rho^2(a_1 - b_1)^2(b_2 - r), \quad 0 < r \leq b_2.$$  \hfill (13)$$

Similarly, the distribution for trips between regions $\Omega_3$ and $\Omega_4$, denoted by $g_2(r)$, is

$$g_2(r) = \rho^2(a_2 - b_2)^2(b_1 - r), \quad 0 < r \leq b_1.$$  \hfill (14)$$

The total barrier distance $T_b$ is then given by

$$T_b = \int_0^{b_2} r g_1(r) \, dr + \int_0^{b_1} r g_2(r) \, dr = \frac{\rho^2}{6} (b_1^3(a_2 - b_2)^2 + b_2^3(a_1 - b_1)^2).$$  \hfill (15)$$

Fig. 5 shows $T_b$ for $A = 0.1, 0.3, 0.5$ as a function of $W$. The size of the city and the number of customers are the same as those of Fig. 2. In a square city (a), $T_b$ increases with $W$ and thus the square facility is better. In a rectangular city (b), the rectangular facility is better and there exists the optimal aspect ratio between $W = 1$ and $W = 2$. The optimal aspect ratio that minimizes the total barrier distance is $W^* = 1.65$ for $A = 0.1$, $W^* = 1.71$ for $A = 0.3$, and $W^* = 1.77$ for $A = 0.5$. Thus, the optimal shape of the facility depends on the size of the facility as well as the shape of the city.
4 Bi-objective problem

In this section, we consider a bi-objective problem to find the optimal size and shape of the facility that minimize both the closest and barrier distances. We then obtain Pareto optimal solutions for the problem. Pareto optimal solutions are such that no other solution is superior to them. For any two solutions $x$ and $y$, $x$ dominates $y$ if each criterion for $x$ is as good as that for $y$ and at least one criterion for $x$ is strictly better than that for $y$. The solution $x$ is called Pareto optimal if no feasible solution that dominates $x$ exists.

The total closest and barrier distances for $A = 0.1, 0.2, \ldots, 0.5$ and $W = 1, 2$ are shown in Fig. 6. The size of the city and the number of customers are the same as those of Fig. 2. The size and shape of the facility for $A = 0.1, 0.3, 0.5$ and $W = 1, 2$ are depicted in Fig. 7. Observe from Fig. 6 that as the size of the facility becomes larger, the total closest distance decreases but the total barrier distance increases. Thus, there exists a tradeoff between the closest and barrier distances. Observe also that the aspect ratio significantly affects both the closest and barrier distances when the size of the facility is large.
The tradeoff curve in Fig. 6 provides planners with alternatives for the size and shape of the facility. In a square city (a), both the square and rectangular facilities are Pareto optimal if the size of the facility is small. In fact, the square facility minimizes the total barrier distance (e.g., $A = 0.1, W = 1$), whereas the rectangular facility minimizes the total closest distance (e.g., $A = 0.1, W = 2$). In contrast, only the square facility is Pareto optimal if the size of the facility is large. For example, the solution for $A = 0.5, W = 1$ dominates the solution for $A = 0.5, W = 2$.

In a rectangular city (b), the rectangular facility is Pareto optimal irrespective of the size of the facility.

5 Conclusions

This paper has presented a bi-objective model for determining the size and shape of a rectangular facility. The total closest and barrier distances have been derived as criteria of the accessibility of customers and the interference to travelers. The model focuses on the tradeoff between the closest and barrier distances, and the tradeoff curve provides alternatives for the size and shape of the facility.
The proposed model is useful for location models of a finite size facility that consider both the accessibility of customers and the interference to travelers. The analytical expressions for the total closest and barrier distances demonstrate how the size and shape of the facility affect the distances. Note that finding these relationships by using empirical models requires computation of the distances for various combinations of the parameters. These relationships help planners to estimate the size and shape of the facility required to achieve a certain level of service. The estimated size and shape can be used as an input in location models.

We have assumed that customers of the facility and origins and destinations of trips are uniformly distributed within the city. Although the uniform distribution is important as the first approximation, more realistic distribution should be considered in future research.

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